TMF

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1 The spectrum of the category of spectra

Let $\mathsf{SHC}^{\mathrm{fin}}$ be the stable homotopy category of finite spectra. This is a **tensor triangulated category** with respect to the smash product – that is, it has compatible monoidal and triangulated structures.

Definition 1. Let $\mathcal{T}$ be a tensor triangulated category. An **ideal** $\mathcal{C} \subseteq \mathcal{T}$ is a thick triangulated subcategory such that for all $X \in \mathcal{C}$ and $Y \in \mathcal{T}$, $X \otimes Y \in \mathcal{C}$.

We can view $K(n)_*$ as a functor from $\mathsf{SHC}^{\mathrm{fin}}_{(p)}$ to the category of graded $K(n)_*$-modules, and there's a Künneth isomorphism $K(n)_*(X \wedge Y) = K(n)_* X \otimes K(n)_* Y$.

Theorem 2 (Thick subcategory theorem)**.** *The categories $\mathcal{C}_{p,n}$ of $K(n)$-acyclics are the only thick subcategories in $\mathsf{SHC}^{\mathrm{fin}}_{(p)}$.*

Corollary 3. *The $\mathcal{C}_{p,n}$ are the only prime ideals of $\mathsf{SHC}^{\mathrm{fin}}_{(p)}$.*

Definition 4. Let $\mathcal{P}_{p,n}$ be the preimage of $\mathcal{C}_{p,n}$ under $\mathsf{SHC}^{\mathrm{fin}} \to \mathsf{SHC}^{\mathrm{fin}}_{(p)}$. $\mathrm{Spec}(\mathsf{SHC}^{\mathrm{fin}})$ is the set of prime ideals of $\mathsf{SHC}^{\mathrm{fin}}$, with a topology that's something like the opposite of the Zariski topology that we won't discuss here.

There's an evident map $\mathrm{Spec}(\mathsf{SHC}^{\mathrm{fin}}) \to \mathrm{Spec}(\mathbb{Z})$ sending each $\mathcal{P}_{p,n}$ to (p). The only prime over (0) is $\mathcal{P}_0 = \ker(H\mathbb{Q}) = \mathsf{SHC}^{\mathrm{fin}}_{\mathrm{torsion}}$. We can think of $\mathrm{Spec}(\mathsf{SHC}^{\mathrm{fin}}) = \mathrm{Spec}(\mathbb{Z}) \wedge \{0, \dots, \infty\}$.
[There's a nice picture here of this prime spectrum.]
If we globalize in the chromatic direction, multiplying all the $\mathcal{P}_{p,k}$ for fixed p and $k \leq n$, we get Morava E-theory E_n at p. But we can also globalize in the arithmetic direction, multiplying $\mathcal{P}_{p,n}$ for fixed n and all p. This should give us a homology theory from which we can recover $E_{p,n}$ at all primes.
At level 0, this is easy, since there's only one prime, namely $E_0 = H\mathbb{Q}$.
At level 1, $E_1 = \widehat{KU_p}$, so globalization gives us KU.
At level 2, globalizing gives us *something like TMF*.

2 What is TMF?

TMF is linked to homology theories, but also to elliptic curves.

Definition 5. A **Weierstrass elliptic curve** C over a scheme S is a flat, proper, finitely presented map $p : C \to S$, together with a section called the **zero section**, such that all geometric fibers are either

1. smooth elliptic curves (in the standard sense),

2. singular cubics in $\mathbb{P}^2$ with a node, or

3. singular cubics in $\mathbb{P}^2$ with a cusp.

(A more algebraic way of saying this is that the geometric fibers are given by a **Weierstrass equation**

$$y^2 z + a_1 xyz + a_3 yz^2 = x^3 + a_2 x^2 z + a_4 xz^2 + a_6 z^3.)$$

This is required to satisfy this property that $\omega_C = p_* \Omega^1_C$ is invertible.

Example 6. Over $\operatorname{Spec}\mathbb{Z}$, there's an elliptic curve $y^2 = x^3 + 2x^2 + 6$ with discriminant $-2^6 \cdot 3 \cdot 97$. We can think of this as a curve sitting over each point of $\operatorname{Spec}\mathbb{Z}$, all of which are smooth except for those at 2 (which has a cusp) and 3 and 97 (which have nodes).

Note that elliptic curves are parametrized by $A = \mathbb{Z}[a_1, a_2, a_3, a_4, a_6]$. The only allowed transformations of elliptic curves are those of the form $[x \mapsto x + r, y \mapsto y + sx + t]$ and $[x \mapsto \lambda^{-2}x, y \mapsto \lambda^{-3}y]$. These define a map $\operatorname{Spec}(A) \times \operatorname{Spec}(G) \to \operatorname{Spec}(A)$, where $G = \mathbb{Z}[\lambda^{\pm 1}, r, s, t]$. Thus if $\Gamma = A \otimes G$, we have a Hopf algebroid (A, Γ) which represents the groupoid of Weierstrass elliptic curves and transformations. This is $\mathcal{M}_{\operatorname{Weier}}$, the moduli stack of Weierstrass elliptic curves.

3 Stacks

Generally, elliptic curves over a scheme can be constructed by gluing together elliptic curves over subschemes, and the same applies for other moduli problems. This suggests that sheaves of groupoids are generally the right way to talk about moduli problems. To state this correctly, we have to modify the notion of 'sheaf' slightly, which we do by modifying the notion of 'covering' – this is done by introducing a new **Grothendieck topology**, which is roughly an axiomatic definition of 'open cover' satisfying certain axioms (a pullback of a cover is a cover, a composition of covers is a cover, . . .). Since we're talking about groupoids, we also need a notion of homotopy theory, at least enough to talk about weak equivalences and homotopy limits. Then the sheaf condition should just say that if $\{U_i \hookrightarrow U\}$ is a cover, then for S to be a sheaf on U, we must have

$$S(U) \simeq \operatorname{holim}\left(\textstyle\prod_i S(U_i) \rightrightarrows \prod_{i,j} S(U_i \times_U U_j) \Rrightarrow \cdots\right).$$

This homotopy limit is called the **descent datum** $\operatorname{Desc}(S, U)$.

Definition 7. A **stack** is a sheaf of groupoids.

Definition 8. Let (X_0, X_1) be a presheaf of groupoids. The associated stack $\mathcal{M}_{(X_0, X_1)}$ is given by $\operatorname{colim}_U \operatorname{Desc}((X_0, X_1), U)$.

Example 9. The stack $\mathcal{M}_{\operatorname{Weier}}$ is represented by (A, Γ). The stack $\mathcal{M}_{\operatorname{ell}}$ is represented by $(A[\Delta^{-1}], \Gamma[\Delta^{-1}])$, where Δ is the discriminant. The moduli stack $\mathcal{M}_{FG}$ of formal groups is represented by $(MP_0, MP_*MP) = (L, W)$.

Definition 10. A $\mathcal{D}$-**valued sheaf** over a stack $\mathcal{M}$ on a site $\mathcal{C}$ is a sheaf on the site $\mathcal{C} \downarrow \mathcal{M}$.

Definition 11. For $f : \operatorname{Spec}(R) \to \mathcal{M}_{FG}$, define the pullback functor

$$\mathcal{F}^*_f : \operatorname{Qcoh}(\mathcal{M}_{FG}) \to \operatorname{Qcoh}(\operatorname{Spec} R)$$

by $\mathcal{F}^*_f(S)(U) = S(U \to \operatorname{Spec}(R) \to \mathcal{M}_{FG})$ for $U \subseteq \operatorname{Spec} R$ an open subset.

Proposition 12. *f is flat iff $\mathcal{F}^*_f$ is exact.*

Definition 13. For $f : \operatorname{Spec}(R) \to \mathcal{M}_{FG}$ flat, define a homology theory $E(R, G)$ by $E(R, G)(X) = \mathcal{F}^*_f MP_*(X)$.

Note that $MP_*(X)$ will be a quasicoherent sheaf on $\mathcal{M}_{FG}$, i.e. an (L, W)-comodule, and its pullback is a quasicoherent sheaf on $\operatorname{Spec} R$, i.e. an R-module. If f factors through $\operatorname{Spec}(L)$, which means that we can choose a coordinate, then this R-module will just be $MP_*(X) \otimes_{MP_0} R$.

Theorem 14 (Hopkins-Miller). *The map $\mathcal{M}_{\operatorname{ell}} \to \mathcal{M}_{FG}$ sending a curve C to its formal group law $\widehat{C}$ is flat.*

Corollary 15. *If $C : \mathrm{Spec}(R) \to \mathcal{M}_{\mathrm{ell}}$ is flat, then so is $\widehat{C} : \mathrm{Spec}(R) \to \mathcal{M}_{FG}$, and so we get a homology theory Ell_*^C given by $X \mapsto \mathrm{Ell}_*^C = \mathcal{F}_{\widehat{C}}^* MP_* X$.*

Definition 16. $\mathcal{O}^{\mathrm{hom}}(\mathrm{Spec}\, R \xrightarrow{C} \mathcal{M}_{\mathrm{ell}})$ is the homology theory Ell_*^C.

4   Defining TMF

We've defined a functor $\mathcal{O}^{\mathrm{hom}}$ from the category of étale schemes over $\mathcal{M}_{\mathrm{ell}}$ to the category of homology theories, and we'd like to represent this by a single homology theory. Unfortunately, $\mathcal{M}_{\mathrm{ell}}$ isn't a scheme, so there's no terminal object. Moreover, we can't take colimits in the category of homology theories. However, what we can do is lift these homology theories to honest spectra and take a colimit in the category of spectra.

Theorem 17 (Goerss-Hopkins-Miller)**.** *There exists a sheaf $\mathcal{O}^{\mathrm{top}}$ of E_∞-rings on $\mathcal{M}_{\mathrm{ell}}$ such that $\pi_* \mathcal{O}^{\mathrm{top}} = \pi_* \mathcal{O}^{\mathrm{hom}} = \omega^{\otimes *}$.*

The global sections of this is the spectrum TMF.
There's a spectral sequence with

$$E_2^{s,t} = H^s(\mathcal{M}_{\mathrm{ell}}, \omega^{\otimes t}) \Rightarrow \pi_* TMF.$$

We can do a similar thing over $\mathcal{M}_{\mathrm{Weier}}$, giving tmf and a similar spectral sequence.

5   The homotopy of TMF

If 6 is invertible in R, then the Weierstrass equation simplifies to $y^2 = x^3 + \frac{c_1}{48} x + \frac{c_6}{864}$, and one can check that there are no transformations fixing this. Thus $A[1/6] \cong \Gamma[1/6] = \mathbb{Z}[c_1, c_6]$, the spectral sequence collapses at the E_1-page, and we have

$$H^s(\mathcal{M}_{\mathrm{ell}}, \omega^{\otimes t}) = H^0(\mathcal{M}_{\mathrm{ell}}, \omega^{\otimes t}) = \mathbb{Z}[c_4, c_6, \Delta^{\pm 1}]/(\sim).$$

At the prime 3, where 2 is invertible, then the Weierstrass equation becomes $y^2 = x^3 + \frac{b_2}{2} x^2 + \frac{b_4}{4} x + \frac{b_6}{2}$, and the transformations are of the form $x \mapsto x + r$. Thus we get $A_{(3)} = \mathbb{Z}_{(3)}[b_2, b_4, b_6]$ and $\Gamma_{(3)} = A_{(3)}[r]$. E_2 is given by $\mathbb{Z}_{(3)}[c_4, c_6, \Delta, \alpha, \beta]/(\alpha^2, 3\alpha, 3\beta, \sim)$. There is no differential on Δ^3, so we get a periodicity of degree 72.

[A picture of the spectral sequence is shown.]

At the prime 2, similar phenomena occur. Δ^8 is now a permanent cycle, giving us a 192-periodicity. Thus, globally, TMF is 576-periodic.

As a final note, this doesn't quite give us the thing we want at chromatic level 2, because there are automorphisms at singular points, but it's something close. You get closer if you put a level structure on the elliptic curves involved.