Higher real K-theories

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1 Real K-theory

Real K-theory is represented by the spectrum KO. In general, we care about KO because we care about real vector bundles. As homotopy theorists, we care about KO because we care about $\operatorname{Im} J$, which shows up as the fiber of a map on p-completed KO. This gives us the height 1 information in $\pi_* S$. The only bad prime at height 1 is $p = 2$, which Vitaly avoided during his talk; its badness corresponds to a $\mathbb{Z}_2 \subseteq \mathbb{S}_1$.

An upper bound for $\operatorname{im} J$ is given by the Adams conjecture, which says morally that Ψ^p wants to be a Frobenius map. If this is true, then $\operatorname{im} J$ in dimension k is a quotient of a group whose cardinality is the denominator $m(2k)$ of $\frac{B_{2k}}{4k}$. A lower bound for $\operatorname{im} J$ is given by the theory of cannibalistic classes and Adams modules over KO, and this puts $|\operatorname{im} J| = m(2k)$ in dimension k up to a factor of 2. This is all in Adams's papers *On the groups* $J(X)$, mostly in *II* and *IV*. We can repeat the whole theory of K after replacing it with KO, and we can indeed define $KO = K^{\mathrm{h}C_2}$.

At height n, the analogue of K-theory is E_n, which now has a $\mathbb{S}_n$-action. If p is bad, there's a large maximal finite subgroup $G \subseteq \mathbb{S}_n$, and we define $EO_n = KO^{\mathrm{h}G}$.

2 Stacks

We've been talking a lot about things like 'comodules over a Hopf algebroid,' which make algebraic sense but aren't terribly intuitive. It's time now to get some geometric intuition, inspired by schemes.

Recall that a scheme is a certain kind of functor $\mathsf{Ring} \to \mathsf{Set}$. A **stack** is a certain kind of functor $\mathsf{Ring} \to \mathsf{Gpd}$, the category of groupoids – we've loosened things up a little by replacing 0-types with 1-types. One way of thinking of this is a pair of schemes $Y \underset{t}{\overset{s}{\rightrightarrows}} X$, with structure maps like those of a groupoid. (The two maps displayed are source and target maps.) Of course, to make things work out nicely you'll need technical conditions, as well – the source and target maps need to be fppf, for example.

Another to introduce stacks is to start with a scheme X with a G-action, and formally define the quotient $X//G$. This is a special case of the other definition.

In algebraic topology, the main source of stacks are the Hopf algebroids (E_*, E_*E), where E is a ring spectrum such that E_*E is flat over E_*. Such an object gives us a stack $\operatorname{Spec} E_* // \operatorname{Spec} E_*E$, with a coaugmentation from the scheme $\operatorname{Spec} E_*$. In general, given a Hopf algebroid (A, Γ) with Γ flat over A, we have a pullback of stacks

$$\begin{array}{ccc} \operatorname{Spec}\Gamma & \xrightarrow{\ s\ } & \operatorname{Spec} A \\ {\scriptstyle t}\downarrow & \lrcorner & \downarrow \\ \operatorname{Spec} A & \longrightarrow & \operatorname{Spec} A// \operatorname{Spec}\Gamma =: \operatorname{Spec} A//\Gamma. \end{array}$$

A **quasicoherent sheaf** on a stack can be defined as a sheaf that pulls back to a quasicoherent sheaf on any scheme mapping to that stack. Since the above pullback square is a cover of $\operatorname{Spec} A//\Gamma$, we see that a qc sheaf on $\operatorname{Spec} A//\Gamma$ is just a qc sheaf on $\operatorname{Spec} A$ that pulls back to isomorphic sheaves on $\operatorname{Spec}\Gamma$ along the two unit maps. That is, it's an A-module M with an isomorphism of Γ-modules $M \otimes_A \Gamma \to \Gamma \otimes_A M$.

Of course we can peel off one of these Γs, thus defining a qc sheaf on $\operatorname{Spec} A//\Gamma$ as an A-module M with a map of A-modules

$$M \to \Gamma \otimes_A M$$

satisfying some axioms. If you look at this, you'll realize it's exactly the definition of a Γ-comodule.

(You may be concerned since we haven't distinguished between left and right comodules, but in fact we shouldn't! The conjugation/inverse structure map gives us a canonical equivalence of the two categories.)

Given a qc sheaf (comodule) $\mathcal{F}$, we can take the cohomology $R\mathrm{Hom}^*(\mathcal{O}_{\operatorname{Spec} A//\Gamma}, \mathcal{F})$. In degree zero, this should just be the global sections of $\mathcal{F}$. What are the global sections of $\mathcal{F}$ if it corresponds to the comodule M? Well, they're just maps $A \to M$ that agree whichever way you go around the pullback square – that is, they're $eq(M \rightrightarrows M \otimes_A \Gamma)$. This is the box or cotensor product $A \Box_\Gamma M$, also known as $\mathrm{Hom}_\Gamma(A, M)$.

Letting Sect be the global sections functor, we have $\mathrm{Sect}(\operatorname{Spec} A//\Gamma, -) = \mathrm{Sect}(\operatorname{Spec} A, -) \circ (A\Box_\Gamma -)$. Since $\operatorname{Spec} A$ is affine, it has no higher cohomology, so the Grothendieck spectral sequence for the right derived functors of this composition collapses, giving

$$R\mathrm{Sect}^*(\operatorname{Spec} A//\Gamma, \mathcal{F}) = \mathrm{Cotor}^*_\Gamma(A, M).$$

That is, the mysterious Cotor functor is just sheaf cohomology!

Example 1. Let $E = MU$. By the results of Quillen, the stack $\operatorname{Spec} MU_*//MU_*MU$ is equivalent to the stack (now viewed as a functor from rings to groupoids) FGLs // strict isomorphisms. In fact, we can also write this as $\operatorname{Spec} L//G$, where L is the Lazard ring and G is a group acting on it. The ANSS starts from

$$MU_* \Box^R_{MU_*MU} MU_*X \cong H^*(\operatorname{Spec} L//G, \mathcal{F}) \cong H^*(G, \mathcal{F}(\operatorname{Spec} L)) \cong H^*(G, MU_*X).$$

When $X = S$, this is just $H^*(G, L)$.

Example 2. The classical ASS comes from $\operatorname{Spec} \mathbb{F}_p//\mathcal{A}_*$. But $\operatorname{Spec} \mathbb{F}_p$ is a point, so this is a global group quotient. If we replace $\mathcal{A}_*$ with $\mathcal{A}(1)_*$ or something else of finite rank, we get honest-to-god group cohomology.

3 Computing $E_n^{\mathrm{h}G}$

Consider the ANSS for BP converging to $\pi_* E_n^{\mathrm{h}G}$. We have a commutative square

$$\begin{array}{ccc} \operatorname{Spec} E_{n*} & \longrightarrow & \operatorname{Spec} E_{n*}//G \\ \downarrow & & \downarrow f \\ \operatorname{Spec} BP_* & \longrightarrow & \mathcal{M}_{pFGc}. \end{array}$$

The lower right hand corner is the moduli stack of p-typical formal groups with coordinates. We don't actually get a map to $\operatorname{Spec} BP_*$ since taking orbits gives us a formal group, not a formal group law.

Proposition 3. *$K(n)$-locally, $BP_* E_n^{\mathrm{h}G} \cong f_* \mathcal{O}_{\operatorname{Spec} E_{n*}//G}$ as a sheaf on $\mathcal{M}_{pFGc}$.*

'The Morava change-of-rings theorem should now be obvious.'

As an aside, $\mathcal{M}_{pFGc}$ has a stratification $\mathcal{M}^{=0}_{pFGc} \cup \mathcal{M}^{=1}_{pFGc} \cup \cdots$ by the height of the formal groups it's parametrizing. A sheaf being $K(n)$-local means that it's supported on the nth stratum. If $\iota : \mathcal{M}^{=n} \to \mathcal{M}$ is the inclusion, we have $R\Gamma(\iota_* \mathcal{F}) = R\Gamma(R\iota_* \mathcal{F})$ (since ι is affine), which is $R\Gamma(\mathcal{F})$ (by the Lurie spectral sequence). Also, $\mathcal{M}^{=n}$ is the stack $\operatorname{Spec} K(n)_*//\Sigma(n)_*$, where

$$\Sigma(n)_* = K(n)_* \otimes_{BP_*} BP_*BP \otimes_{BP_*} K(n)_*.$$

There's also an E_n-local version of this, where we look at sheaves supported on $\mathcal{M}^{\leq n}$. The MCOR theorem appears as changing stacks to $\operatorname{Spec} E_{n*}//E_{n*}E_n$.

Now, $\operatorname{Spec} E_{n*}//G \to \mathcal{M}$ is (probably) affine, so the above argument gives us

$$R\Gamma(Rf_* \mathcal{O}_{\operatorname{Spec} E_{n*}//G}) \cong R\Gamma(\operatorname{Spec} E_{n*}//G, \mathcal{O}_{\operatorname{Spec} E_{n*}//G}) \cong H^*(G, E_{n*}),$$

and G is finite, so we can (probably) replace the ANSS with a homotopy fixed point spectral sequence.

Example 4. Let $n = 2$ and $p = 2$. We can show that $G \cong GL_2(\mathbb{F}_3)$, and $\operatorname{Spec} E_{2*}//G$ corresponds to the moduli stack of supersingular height 2 elliptic curves over $\mathbb{F}_4$, of which there's only one. E_{n*} is complete, so $\operatorname{Spec} E_{2*}//G = \operatorname{Spf} E_{2*}//G$. Serre-Tate deformation theory over a perfect field k of positive characteristic p tells us that a deformation of an abelian variety is the same as a deformation of its p-divisible group (the pro-abelian formal group of all its p-power torsion). Thus

$$\operatorname{Spf}(E_{2*}//G) \cong \mathcal{M}_{1,1}^{\mathrm{ss}} \cong \mathcal{M}_{1,1,ss}^{\wedge},$$

the ss denoting 'supersingular,' and this is a sixfold quotient of $\mathcal{M}_{\mathrm{Weierstrass}}$.