Talbot 2012: The Calculus of Functors

Mentored by Gregory Arone and Michael Ching

Notes by Claudia Scheimbauer

Syllabus of Talks

(1) **Introduction and overview**, by Greg Arone (UVA).
(2) **Polynomial and analytic functors**, by Dan Lior (UIUC).
(3) **Constructing the Taylor tower**, by Geoffroy Horel (MIT).
(4) **Homogeneous functors**, by Matthew Pancia (UT Austin).
(5) **First examples**, by Joey Hirsh (CUNY).
(6) **The derivatives of the identity functor**, by Gijs Heuts (Harvard).
(7) **Operad and module structures on derivatives**, by Emily Riehl (Harvard).
(8) **Classification of polynomial functors**, by Michael Ching (Amherst).
(9) **Orthogonal Calculus I: theory**, by Kerstin Baer (Stanford).
(10) **Orthogonal Calculus II: examples**, by Sean Tilson (Wayne State).
(11) **Introduction to embedding calculus**, by Daniel Berwick-Evans (UC Berkeley).
(12) **Multiple disjunction lemmas**, by Greg Arone (UVA).
(13) **Embedding calculus, the little disks operad, and spaces of embeddings**, by Alexander Kupers (Stanford)
(14) **Factorization homology**, by Hiro Lee Tanaka (Northwestern).
(15) **Applications to algebraic K theory I**, by Pedro Brito (Aberdeen)
(16) **Applications to algebraic K theory II**, by Ernest E. Fontes (UT Austin).
(17) **Calculus of functors and chromatic homotopy theory**, by Tobias Barthel (Harvard).
(18) **Taylor tower of the identity functor, part 2**, by Vesna Stojanoska (MIT).
(19) **Where do we go from here?** by Greg Arone.

This PDF is a collection of hand-written notes taken by Claudia Scheimbauer at the 2012 Talbot Workshop. The workshop was mentored by Gregory Arone and Michael Ching, and the topic was the calculus of functors.

The aim of the Talbot Workshop is to encourage collaboration among young researchers, with an emphasis on graduate students. We make these notes available as a resource for the community at large, and more resources can be found on the Talbot website:

http://math.mit.edu/conferences/talbot/

Support for the 2012 Talbot Workshop came from the NSF grant DMS-1007096.

Goodwillie calculus

I. Introduction and overview – Greg Arone

Idea: want to study functors like functions.

Consider X, Y topol. spaces. Study $\mathrm{Maps}(X,Y)$, hard!

$\rightsquigarrow \pi_0 \mathrm{Maps}(X,Y) = [X,Y]$
homotopy classes of maps

Ex: $[S^m, S^n]$ still hard.

Why? This is a very complicated function of Y.
Even if $Y = Y_1 \cup_{Y_0} Y_2$, and know $\mathrm{Maps}(-, Y_i)$, still hard
Ex: $S^2 = D^2 \cup_{S^1} D^2$

Let M, N be smooth mflds, $\mathrm{Emb}(M,N)$ even more difficult!
Why? "function of 2 variables"

Some basic ideas:

1. Some functors deserve to be called polynomial functors.
2. General functors can be approximated with polynomial functors
   (Interpolation polynomials, Taylor polynomials)
   "closeness": $F \longrightarrow P_n F$ or $L_n F \longrightarrow F$ + some prop.
3. The n-th term of a polynomial approximation is determined by the n-th cross-effect or the nth derivative.

What are polynomial functors?

(1) linear functors: for functions: $f(x+y-a) = f(x) + f(y) - f(a)$

for functors:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & X \cup_A Y \end{tikzcd} \longrightarrow \begin{tikzcd} F(A) \arrow[r] \arrow[d] & F(X) \arrow[d] \\ F(Y) \arrow[r] & F(X \cup_A Y) \end{tikzcd}$$

push-out square $\qquad$ push-out

First defn: F linear if it takes homotopy pushout squares to homotopy pushout squares.

F is polynomial of degree n if it takes strongly cocartesian $(n+1)$-cubes to cocartesian $(n+1)$-cubes.

[cross-effects: 2nd cross-effect $f(x+y+z) - f(x+y) - f(y+z) - f(x+z) + f(x) + f(y) + f(z) - f(0)$

$= 0 \iff$ polynomial of degree 2.]

Note: - the 2 pushout squares play different roles.
- If would take strongly coc. $\to$ str. coc., get linear

Examples: linear functors: $F(X) = X$

$F(X) = K \times X$, K fixed space

(constant functors)

quadratic functors: $F(X) = X \times X$

[poly of deg. n $\quad F(X) = X^n$

homot. colimits of poly still poly of same degree]

$F(X) = X \times X \times_{\Sigma_2} E\Sigma_2$ quadratic

Take $\Sigma_2 \circlearrowright K$. $F(X) = (K \times X \times X) \times_{\Sigma_2} E\Sigma_2$ quadratic

These ex. are essentially all.

How to approximate?

F_n = category of finite sets of cardinality $\leq n$.

$G: \mathbf{Top} \longrightarrow$ Top / Spectra (only need that homotopy pushouts exist!)

$L_n G := L^{\mathbf{Top}}_{F_n} G|_{F_n}$

$L_n G(X) = \operatorname*{hocolim}_{n \geq i \to X} G(i) = X^i \otimes_{i \in F_n} G(i)$

$L_n G(X) \longrightarrow G(x)$

interpolation at $0, 1, \ldots, n$

Remark: Could also take $G : \mathrm{Top} \longrightarrow \mathrm{Top}^{op}$

$$G \longrightarrow R_n G(X) = \underset{i \in F_n}{\mathrm{Nat}}\,(X^i, G(i))$$

looks like right Kan extension.

Variant: $\mathcal{M}^d$ = category of d-dim'l manifolds and embeddings
$B^d \subseteq \mathcal{M}^d$ s/category of balls (unions of balls)
$B^d_n \subseteq B^d$ unions of $\leq n$ balls.

$$B^d_n \xrightarrow{\pi_0} F_n$$

$$G : \mathcal{M}^d \longrightarrow \begin{array}{l} \mathrm{Top} \\ \mathrm{Top}^{op} \\ \mathrm{Spectra} \end{array}$$

$$\to L_n G \to L_{n+1} G \to \cdots \to G$$

$$G : \mathcal{M}^d \longrightarrow \text{something}$$

$$L_n G(M) = \mathrm{Emb}\,(i \times \mathbb{R}^d, M) \underset{B^d_n}{\otimes} G(i \times \mathbb{R}^d)$$

If G is contravariant, $R_n G(M) = \underset{B^d_n}{\mathrm{Nat}}(\mathrm{Emb}(i \times \mathbb{R}^d, M), G(i \times \mathbb{R}^d))$

.... embedding calculus

Back to homotopy case:

$$G : \mathrm{Top} \longrightarrow \begin{array}{l} \mathrm{Top} \\ \mathrm{Spectra} \end{array}$$

We have a sequence of approximations

$$L_0 G \to \cdots \to L_n G \to L_{n+1} G \to \cdots \to G.$$

$$L_n G / L_{n+1} G\,(X) = X^n /_{f.d.} \underset{\Sigma_n}{\wedge} cr_n G$$

fat diagonal!

$$\begin{tikzcd} G(n-2) \arrow[r] \arrow[d] & G(n-1) \arrow[d] & \\ G(n-1) \arrow[r] & G(n) \arrow[dr] & \\ & & cr_n G \end{tikzcd}$$

$g: \mathbb{R} \to \mathbb{R} \quad \rightsquigarrow \quad (L_n g - L_{n-1} g)(x) = \binom{x}{n} \, cr_n g$

Rem: · subquotients don't determine the sequence
Need more information ... maps betw. n-th cross-effects
$n \to i \quad cr_n h \to cr_n i$

Taylor approximation

$F: \mathcal{C} \to \mathcal{D}$ is linear if it takes homotopy pushouts to homotopy pullbacks.
... is polynomial of degree n ... similar.

$F: Top \to Top$ linear
$F(X \sqcup Y) \xrightarrow{\simeq} F(X) \times F(Y)$ weak equiv.
For Top, $A \times (B \sqcup C) \cong A \times B \sqcup A \times C$

$$\Omega^\infty \Sigma^\infty X \longleftrightarrow e^{x-1}$$
$$\rotatebox{90}{$\cong$}$$
$$\sum X^{\wedge n}/\Sigma_n \longleftrightarrow \sum \frac{(x-1)^n}{n!}$$

approx. give tower

$$\begin{array}{c} X \\ \downarrow \\ \Omega^\infty \Sigma^\infty X \end{array} \longleftrightarrow \quad x = e^{x-1} e^{-\frac{x-1}{2}} e^{\frac{x-1}{3}} \cdots$$
$$= e^{\ln(x-1+1)} = x.$$

no analogy in topology.

Take Comm (Comm sort of plays the role of the dual of Top)
get dual version.

evening discussion

II. Polynomial (& Analytic Functors) - Dan Lior

What is a polynomial functor of degree n?

$F: \mathcal{C} \to \mathcal{D}$, $\mathcal{C}, \mathcal{D} \in \{Top_*, \text{Spectra}\}$

- $X \xrightarrow{\sim} Y \rightsquigarrow F(X) \xrightarrow{\sim} F(Y)$
- Cubes:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}$$ 2-cube

$$\begin{tikzcd}[row sep=small, column sep=small] & X_{\{3\}} \arrow[rr] & & X_{\{2,3\}} \arrow[dd] \\ X_{\emptyset} \arrow[ur] \arrow[rr] \arrow[dd] & & X_{\{2\}} \arrow[ur] \arrow[dd] & \\ & & & X_{\{1,2,3\}} \\ X_{\{1\}} \arrow[rr] & & X_{\{1,2\}} \arrow[ur] & \end{tikzcd}$$ 3-cube

$\mathcal{P}(n) \xrightarrow{\mathcal{X}} \mathcal{C}$ n-cube

There are natural maps

$$X \xrightarrow{(*)} \operatorname{holim}\left(\begin{tikzcd}[row sep=small] & X \arrow[d] \\ Z \arrow[r] & W \end{tikzcd}\right)$$

$$X_\emptyset \xrightarrow{(*)} \operatorname{holim}\left(\mathcal{P}(n) \setminus \{\emptyset\} \hookrightarrow \mathcal{P}(n) \xrightarrow{\mathcal{X}} \mathcal{C}\right)$$

(same for cohlim, other way.)

Def'n: An n-cube $\mathcal{X}$ is (homotopy) cartesian if $(*)$ is an (weak) equivalence

... k-cartesian if $(*)$ is k-connected.

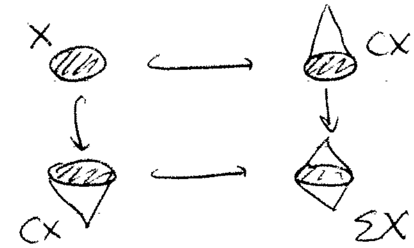
Def'n: $X \xrightarrow{f} Y$ is k-connected if the induced map

$\pi_j X \to \pi_j Y$ is an isomorphism for $j < k$

$\pi_k X \to \pi_k Y$ is surjective

<u>Example:</u>

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$



is coCartesian and

$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] & X^I \arrow[d] \\ X^I \arrow[r] & X \end{tikzcd}$$

is cartesian.

<u>Def'n:</u> F is <u>1-excisive</u> if it takes coCartesian squares to Cartesian squares

<u>Remark:</u> We want to think of this as polynomial of degree 1, and linear means sth a little ~~different~~ more (poly of deg 1 + "reduced")

($\longleftrightarrow$ $\underset{\text{linear}}{f(x)=mx}$ vs. $\underset{\text{1-excisive}}{f(x)=mx+b}$)

<u>Example:</u> $Id:$ Spectra $\to$ Spectra is 1-excisive.

<u>Thm</u> <u>Blakers-Massey</u> If

$$\mathcal{X} = \begin{tikzcd} X_\emptyset \arrow[r] \arrow[d] & X_{\{2\}} \arrow[d] \\ X_{\{1\}} \arrow[r] & X_{\{1,2\}} \end{tikzcd}$$

is coCartesian and $X_\emptyset \to X_{\{i\}}$ is k_i-connected. Then $\mathcal{X}$ is (k_1+k_2-1)-cartesian.

<u>Remark:</u> B-M implies that in Spectra,

coCartesian $\Longleftrightarrow$ Cartesian. ⊛ 120 sec pf see end.

<u>Example:</u> $Id: Top_+ \to Top_+$ is not 1-excisive!

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma S^0 \end{tikzcd}$$

is coCartesian, but not Cartesian

b/c $S^0 \longrightarrow \Omega\Sigma S^0$ is not a weak equivalence.

$\wr \pi_0$

$2 \qquad \mathbb{Z}$

<u>Example</u>: $\mathrm{Top}_+ \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{Spectra}$

Σ^∞ and Ω^∞ are 1-excisive.

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd} \quad \curvearrowright \quad \begin{tikzcd} \Omega^\infty X \arrow[r] \arrow[d] & \Omega^\infty Y \arrow[d] \\ \Omega^\infty Z \arrow[r] & \Omega^\infty W \end{tikzcd}$$

coCartesian

$\underset{\text{Rem.}}{\Leftrightarrow}$ Cartesian, so $\Omega^\infty X \simeq \Omega^\infty \operatorname{holim}\left(\begin{smallmatrix} & Y \\ & \downarrow \\ Z \to & W \end{smallmatrix} \right)$

$\simeq \operatorname{holim}\left(\begin{smallmatrix} & \Omega^\infty Y \\ & \downarrow \\ \Omega^\infty Z \to & \Omega^\infty W \end{smallmatrix} \right)$

so the right square is Cartesian.

Σ^∞ similarly.

<u>Example</u>: $\Omega^\infty\Sigma^\infty : \mathrm{Top}_+ \longrightarrow \mathrm{Top}_+$ is 1-excisive, <u>but</u>
$\Sigma^\infty\Omega^\infty : \mathrm{Spectra} \to \mathrm{Spectra}$ is <u>not</u> 1-excisive.

<u>Example</u>: $G : \mathrm{Top}_+ \longrightarrow \mathrm{Spectra}$
$X \longrightarrow C \wedge \Sigma^\infty X$, where C is a fixed spectrum

One can check that

(i) G is 1-excisive
(ii) $G(*) \simeq *$
} linear

(iii) G satisfies the "<u>colimit axiom</u>"

⌜If X is a filtered colimit of finite CW complexes
$\operatorname{colim}_\alpha (GX_\alpha) \xrightarrow{\sim} G(\operatorname{colim}_\alpha X_\alpha)$⌟

<u>Classification</u>: ~~$\hom(X,Y) \wedge \to \Omega^\infty \hom(GX, GY)$~~

Let $F : \mathrm{Top}_+ \to \mathrm{Spectra}$ be a linear functor with colimit axiom, i.e. (i)-(iii)

$\hom(X,Y) \longrightarrow \Omega^\infty \hom(F(X), F(Y))$

$\Sigma^\infty \hom(X,Y) \wedge F(X) \longrightarrow F(Y)$ Claim: This is an equivalence

<u>Rem</u>: i.e., $LF \to F$ is an equivalence

<u>Example</u>: $X = S^0$: $\quad \Sigma^\infty Y \wedge \underbrace{FS^0}_{=C} \longrightarrow FY$

<u>Idea</u>: Think of this as 2 lines which agree at 2 pts: $0 \; (\hat{=} *)$
$1 \; (\hat{=} S^0)$

- $\underline{Y = S^{n+1}}$

$$\begin{tikzcd} F(S^n) \arrow[r] \arrow[d] & F(*) = * \arrow[d] \\ * = F(*) \arrow[r] & F(S^{n+1}) \end{tikzcd}$$

pushout, i.e. $F(S^{n+1})$ determined by $\downarrow \longrightarrow$

use induction hypothesis.

- <u>attaching cells</u>:

$$\begin{tikzcd} \partial D_n \arrow[r] \arrow[d] & X \arrow[d] \\ * \simeq D_n \arrow[r] & Y \end{tikzcd}$$

and use colimit axiom....

<u>2-excisive</u>

<u>Example</u>:

$$\begin{tikzcd} & X_{\{3\}} \arrow[rr, no head] & & X_{\{2,3\}} \arrow[dd] \\ X_\emptyset \arrow[ur, hook] \arrow[rr, hook] \arrow[dd, hook] & & X_{\{2\}} \arrow[ur] \arrow[dd, hook] & \\ & & & X_{\{1,2,3\}} \\ X_{\{1\}} \arrow[rr, hook] & & X_{\{1,2\}} \arrow[ur] & \end{tikzcd}$$

is coCartesian

<u>Def'n</u> Any n-cube constructed this way is called <u>strongly homotopy coCartesian</u>. (by constructing pushouts)

<u>Example</u>: $X \longrightarrow \Sigma^\infty(X \wedge X)$ ~~[illegible]~~ is 2-excisive, where ... ↓

<u>Def'n</u>: F is <u>n-excisive</u> if it takes strongly coCartesian $(n+1)$-cubes to Cartesian $(n+1)$-cubes.

<u>Thm</u> (<u>Goodwillie</u>)
If $C(X,Y)$ is bilinear, then $C^\Delta(X) := C(X,X)$ is 2-excisive.

<u>Example</u>: $C(X,Y) = \Sigma^\infty(X \wedge Y) = \Sigma^\infty X \wedge \Sigma^\infty Y$ is bilinear,
thm gives $\Sigma^\infty(X \wedge X)$ is 2-excisive.

$$\begin{tikzcd} S^0 \wedge S^0 \arrow[r] \arrow[d] & S^0 \wedge * \arrow[d] \\ * \wedge S^0 \arrow[r] & * \vee * \end{tikzcd}$$

$\underline{2} = S^0 \wedge S^0$, $\underline{1} = S^0 \wedge *$, $\underline{1} = * \wedge S^0$, $* \vee * = \underline{0}$

$\rightsquigarrow G$

$$\begin{tikzcd} \underline{4} \arrow[r] \arrow[d] & \underline{1} \arrow[d] \\ \underline{1} \arrow[r] & \underline{0} \end{tikzcd}$$ is not Cartesian

$\underline{0} \to *$

So this is 2-excisive, but not 1-excisive.

(*) 120 sec pf.

$$\begin{tikzcd} A_n \arrow[r, "n-c"] \arrow[d, "n-c"'] & B_n \arrow[d] \\ C_n \arrow[r] & D_n \end{tikzcd}$$

conn.

homot. coCartesian

BM $\longrightarrow$ $2n-2c$ - cartesian

$$\begin{tikzcd} \Omega^n A_n \arrow[r] \arrow[d] & \Omega^n B_n \arrow[d] \\ \Omega^n C_n \arrow[r] & \Omega^n D_n \end{tikzcd}$$ $n-2c$ cartesian

$\Omega^n A_n \longrightarrow \operatorname{holim}(\quad)$

$\Omega^\infty A_n \xrightarrow{\sim} (\quad)$

III. Construction of the Taylor tower – Geoffroy

$F : Top_* \longrightarrow Top_*$
$(F(*) = *)$

Goal: Construct $\forall n$: $F \to P_n F$ s.t.

- $P_n F$ is n-excisive
- $\forall G$ n-excisive, $G \to F$ $\exists!$ factorization $F \to P_n F \to G$ in the homotopy category of functors
- $\forall$ $F \to G \to H$ fiber sequence is preserved: $\rightsquigarrow P_n F \to P_n G \to P_n H$

n-1 excisive $\Rightarrow$ n excisive, so

$$\begin{tikzcd} & \vdots \arrow[d] \\ & P_3 F \arrow[d] \\ & P_2 F \arrow[d] \\ F \arrow[r] \arrow[ur] \arrow[uur] & P_1 F \end{tikzcd}$$

Taylor tower

Definition: X a pointed space, S a set
$$X * S := \text{cofiber}\left(\bigvee_S X \longrightarrow X\right)$$

$\boxed{\begin{array}{l} P(\underline{n+1}) = \text{poset of subsets of } \underline{n+1} \\ P_0(\underline{n+1}) = \text{poset of non-empty s/sets of } \underline{n+1} \end{array}}$

Let n be an integer. Then
$$P(\underline{n+1}) \longrightarrow Top_*$$
$$S \longmapsto X * S$$
is strongly cocartesian.

Example:
$X * \phi = X$
$X * pt = CX$
$X * \{0,1\} = \Sigma X$

$\leftrightsquigarrow$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd}$$

Def'n. $T_n F(X) := \underset{S \in P_0(\underline{n+1})}{\text{holim}} F(X * S)$

$F \xrightarrow{t_n F} T_n F$ b/c $F \xrightarrow{\simeq} F(X * \phi) \longrightarrow \underset{S \in P_0(\underline{n+1})}{\text{holim}} F(X * S)$

If F is n-excisive, then $F \longrightarrow T_n F$ is an equivalence.

<u>Def'n</u> $P_nF(X) = \operatorname{hocolim}\left(F(X) \xrightarrow{t_nF(X)} T_nF(X) \xrightarrow{t_nT_nF(X)} T_n^2F(X) \to \cdots \right)$

$F \xrightarrow{p_nF} P_nF$

If F is n-excisive, $F \xrightarrow[p_nF]{\sim} P_nF$

<u>Remark:</u> $(X * S) * T = (X * T) * S$

$(\Sigma X) * S = \Sigma(X * S)$

$\Rightarrow T_n(F \circ \Sigma) = (T_nF) \circ F \Rightarrow (P_nF) \circ \Sigma = P_n(F \circ \Sigma)$

(So, functors T_n, P_n only depend on highly connected spaces, i.e. P_nF depends only on the local behaviour of F around $*$.)

<u>Proposition:</u> $P_n : \operatorname{Fun}(Top_+, Top_+) \longrightarrow \operatorname{Fun}(Top_+, Top_+)$

commutes with
- filtered hocolims
- finite holimits

<u>Proof:</u> In Top_+, hocolim commutes with hocolim

holim " holim

filtered hocolim " finite holim

<u>Lemma:</u> Let $\mathcal{X}$ be a strongly cocartesian $(n+1)$-cube. Then $F(\mathcal{X}) \to T_nF(\mathcal{X})$ factors through a cartesian cube.

<u>Pf</u> later.

<u>Thm:</u> P_nF is n-excisive.

<u>Pf:</u> $\mathcal{X}$ strongly cocartesian.

$$F\mathcal{X} \to T_nF\mathcal{X} \to T_n^2F\mathcal{X} \to \cdots \qquad (*)$$

By lemma, $F\mathcal{X} \to C_1 \to T_nF\mathcal{X} \to C_2 \to T_n^2F\mathcal{X} \to C_3 \to \cdots \qquad (**)$

Then $\operatorname{hocolim}(*) = \operatorname{hocolim}(**) = \operatorname{hocolim}(C_1 \to C_2 \to C_3 \to \cdots)$

each C_i is cartesian $\Rightarrow \operatorname{hocolim}(C_1 \to C_2 \to \cdots)$ is cartesian

So $P_nF(\mathcal{X})$ is cartesian $\Rightarrow P_nF$ is n-excisive.

Proof of Lemma:

Existence of the factorization

Let G be n-excisive.

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \arrow[d, "\wr"] \\ P_nF \arrow[r] & P_nG \end{tikzcd} \Rightarrow \text{In the homotopy category, } F \to P_nF \to G.$$

Lemma $P_nF \xrightarrow[P_n t_n F]{} P_n T_n F$ is a weak equivalence.

Pf: Let S be some finite set.

$J_S F(X) := F(X * S)$

$$P_nF \xrightarrow{*} P_n\Big(\operatorname*{holim}_{U \in \mathcal{P}_0(n+1)} J_U F\Big) \xrightarrow{\sim} \operatorname*{holim}_U P_n J_U F \xrightarrow{\sim} \operatorname*{holim}_U J_U P_n F$$

The composition of these maps is t_nP_nF, so ~~enough~~ to show that $*$ is an equivalence, we ~~show~~ need that t_nP_nF is an equivalence

~~Let $X \in Top_{(F)}$ $P_nF(X)$~~

This is true because P_nF is n-excisive. ∎

Corollary: $P_nF \xrightarrow{P_n p_n F} P_n^2 F$ is an equivalence.

Unicity of factorization:

$F \to G$, where G is n-excisive. $F \to P_nF \xrightarrow{v} G$

$$\begin{tikzcd} F \arrow[r, "p_nF"] \arrow[d, "p_nF"'] & P_nF \arrow[r, "v"] \arrow[d, "\wr"', "P_np_nF"] & G \arrow[d, "\wr"] \\ P_nF \arrow[r, "\sim", "P_np_nF"'] & P_n^2F \arrow[r, "P_nv"'] & P_nG \end{tikzcd}$$

v is uniquely determined by P_nv

P_nv ‖ $P_nv \circ P_np_nF = P_n(v \circ p_nF)$

$\Rightarrow v$ is uniquely determined by $v \circ p_nF$.

Lemma: $F\mathcal{X} \longrightarrow T_n F \mathcal{X}$ factors through a cartesian cube if $\mathcal{X}$ is strongly cocartesian.

Proof (Rezk)

$\mathcal{X}$ cube, let $U \in P(\underline{n+1})$

$$\mathcal{X}_U(T) = \operatorname{hocolim}\left(\begin{array}{ccc} \coprod_{s\in U} \mathcal{X}(T) & \longrightarrow & \coprod_{s \in U} \mathcal{X}(T \cup \{s\}) \\ \downarrow & & \\ \mathcal{X}(T) & & \end{array} \right)$$

$$\mathcal{X}_U(T) \longrightarrow \mathcal{X}(T) * U$$

$$F(\mathcal{X}(T)) \longrightarrow \underset{U \in P_0(\underline{n+1})}{\operatorname{holim}} F(\mathcal{X}_U(T)) \longrightarrow \underset{U\in P_0(\underline{n+1})}{\operatorname{holim}} F(\mathcal{X}(T) * U)$$

$t_n F(\mathcal{X}(T))$

Want to show that middle is cartesian if $\mathcal{X}$ str. cocartesian.

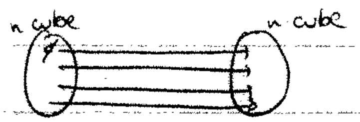
If $\mathcal{X}$ is strongly cocartesian, $\mathcal{X}_U(T) = \mathcal{X}(T \cup U)$

Want to show that each cube in holim is cartesian:

$$\mathcal{X}(T \cup U) \xrightarrow{\sim} \mathcal{X}(T \cup \{s\} \cup U) \quad \text{if } s \in S$$

$$\left(T \longmapsto F(\mathcal{X}_U(T))\right) = \left(T \longmapsto F(\mathcal{X}(U \cup T))\right)$$

If U is nonempty,



A cube of this form is always cartesian.

IV. Homogeneous functors – Matt

$$P_n(f) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n$$

$$D_n(f) = P_n(f) - P_{n-1}(f) = \frac{f^{(n)}(0)}{n!}x^n$$

Nice properties: (1) ~~deg~~ D_n is of deg n

(2) D_n is exactly of deg n, homogeneous of deg n

(3) $\frac{f^{(n)}(0)}{n!}$ determines the function

Functors: $F: Top_* \longrightarrow \begin{cases} Top_* \\ Spectra \end{cases}$

(1) deg $n \longleftrightarrow$ n-excisive

"determined by values on $(n+1)$-points"

$$\begin{array}{ccc} X & \to & Y \\ \downarrow & & \downarrow \\ Z & \to & Z \cup_X Y \end{array} \quad \rightsquigarrow \quad \begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(Y) \arrow[d] \\ F(Z) \arrow[r] & F(Z \cup_X Y) \end{tikzcd}$$

$\simeq \text{holim}$

(Think of Y and Z as perturbations)

(2) $D_{n-1}F = *$

Def'n A functor F is homogeneous of deg n if

(1) F is n-excisive

(2) $D_{n-1}F = *$ (n-reduced)

Example: Layers of the Taylor tower

$$D_nF = \text{hofiber}(P_nF \to P_{n-1}F)$$

D_nF is homogeneous of degree n.

This follows from:
- P_n preserves homotopy fiber sequences
- $P_{n-1}P_n \longrightarrow P_{n-1}P_{n-1}$ is an equivalence
- $F \to G \to H$ fiber sequence of functors, if G, H are n-excisive, so is F.

Then $D_nF \to P_nF \longrightarrow P_{n-1}F$

$\rightsquigarrow P_{n-1}D_nF \to P_{n-1}P_nF \longrightarrow P_{n-1}P_{n-1}F$

<u>Example</u>: $F(X) = X^{\wedge n}$, $F: Sp \to Sp$
$G(X) = \Sigma^\infty(X^{\wedge n})$, $G: Top_+ \to Sp$

<u>Prop</u>: Both are homogeneous of degree n.

<u>Lemma</u>: $L: C^n \to D$ is k_i-excisive in each slot, then the composite functor
$$C \xrightarrow{\Delta} C^n \to D \quad \text{is } \left(\sum_{i=1}^{n} k_i\right) \text{excisive}.$$

<u>Lemma</u>: If $L: C^n \to D$ is reduced in each slot, then
$$C \xrightarrow{\Delta} C^n \xrightarrow{L} D \quad \text{is } n\text{-reduced}.$$

<u>Example</u>: $\hat{F}(X_1, \dots, X_n) = X_1 \wedge \dots \wedge X_n$ is homog. of deg n by lemmas above.

<u>Example</u>: C a fixed spectrum
$$F(X) = C \wedge X^{\wedge n}, \quad G(X) = C \wedge \Sigma^\infty X^{\wedge n}$$
are also n-homogeneous.

Moreover, if C has a Σ_n-action, then
$$F(X) = (C \wedge X^{\wedge n})_{h\Sigma_n} \quad \text{also is } n\text{-homogeneous}.$$
This is the analog of $\frac{f^{(n)}(0)}{n!}$

<u>Nice property</u>:

<u>Thm</u>: if $F: Top_+ \to Top_+$ is homogeneous of deg n, then $F(X)$ is an infinite loop space $\forall X \in Top_+$.

<u>Example</u>: If F is hom of deg 1,
$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} F(X) \arrow[r] \arrow[d] & F(CX) \simeq * \arrow[d] \\ * \simeq F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

$$\Rightarrow F(X) \to * \to F(\Sigma X)$$
$$F(X) \cong \Omega F(\Sigma X)$$

Pf of thm uses the following lemma:

<u>Lemma</u>: If F is reduced, then there is a homogeneous degree n functor R_nF, which fits into a fiber sequence

$$P_nF \longrightarrow P_{n-1}F \longrightarrow R_nF$$

<u>Note</u>: If F is hom of deg $n \rightsquigarrow \quad F \longrightarrow * \longrightarrow R_nF$

<u>Example</u>: $f(x)$ is linear iff $\hat{f}(x_1, x_2) = 0$, where

$$\hat{f}(x_1, x_2) = f(x_1 + x_2) - f(x_1) - f(x_2) + f(0)$$

Suppose $f(x) = ax^2 + bx + c$,

$$\hat{f}(x_1, x_2) = (ax_1x_2) \cdot 2$$

i.e. $\hat{f}$ measures failure to be linear.

Similarly, $\hat{f}(x_1, \ldots, x_n) = f(x_1 + \cdots + x_n) - \sum_i f(x_1 + \cdots + \hat{x}_i + \cdots x_n)$

$$\mp \sum_{i<j} f(x_1 + \cdots + \hat{x}_i + \cdots + \hat{x}_j + \cdots + x_n)$$

$$\vdots$$

$$(-1)^{\cdots} f(0)$$

and $\hat{f}(x_1, \ldots, x_n) = n! \, a \cdot x_1 \cdots x_n$, a ~~highest~~ coeff of x^n in f. (if f is a poly)

<u>Def'n</u> The <u>n-th cross-effect</u> cr_nF is the functor of n variables given by applying F to the cube $\mathcal{X}(\underline{n} \setminus T) = \bigvee_{S \in T} X_S$ and taking the total homotopy fiber

<u>Ex</u>: $n=2$ $\quad cr_2F(X_1, X_2)$ is given by total hofiber of

$$\begin{array}{ccc} X_1 \vee X_2 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ X_1 & \longrightarrow & * \end{array} \quad \rightsquigarrow \quad \begin{array}{ccc} F(X_1 \vee X_2) & \longrightarrow & F(X_2) \\ \downarrow & & \downarrow \\ F(X_1) & \longrightarrow & F(*) \end{array}$$

i.e. "$\big(F(X_1 \vee X_2) - F(X_1)\big) - \big(F(X_2) - F(*)\big)$"

<u>Prop</u>: If F is n-excisive, then $cr_{m+1} F$ is $(n-m)$-excisive in each variable. In particular, if F is n-excisive, then $cr_n F$ is symmetric multilinear (1-exc. in each slot) and if F is $(n-1)$-excisive, then $cr_n F$ is trivial.

So, $cr_n : \begin{Bmatrix} \text{homogeneous functors} \\ \text{of deg } n \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{symmetric} \\ \text{multilinear functors} \end{Bmatrix}$

$C = Top_+$
$D = \text{Spectra}$

If $L : C^n \longrightarrow D$ symmetric multilinear, then

$$C \xrightarrow{\Delta} C^n \xrightarrow{L} D$$

$(L \circ \Delta)_{h\Sigma_n} = (L(X, \dots, X))_{h\Sigma_n}$ is homogeneous of degree n

So $L \xmapsto{\Delta_n} (L \circ \Delta)_{h\Sigma_n}$ sends symm. multilin to hom of deg n

$$\begin{Bmatrix} \text{homogeneous functors} \\ \text{of deg } n \\ Top_+ \longrightarrow \text{Spectra} \end{Bmatrix} \underset{\Delta_n}{\overset{cr_n}{\rightleftarrows}} \begin{Bmatrix} \text{Symmetric multilinear} \\ \text{functors } Top_+^n \longrightarrow \text{Spectra} \end{Bmatrix}$$

Suppose that we have a fixed spectrum C, consider

$$L(X_1 \wedge \dots \wedge X_n) = C \wedge X_1 \wedge \dots \wedge X_n.$$

This is symmetric multilinear.

On the other hand, if L is symm. multilinear, then

$$L(S^0, \dots, S^0) \wedge X_1 \wedge \dots \wedge X_n \longrightarrow L(X_1, \dots, X_n)$$

Layers of the Taylor tower $D_n F$ are homogeneous of deg n. So,

$$(cr_n D_n F)(X_1, \dots, X_n) = C \wedge X_1 \wedge \dots \wedge X_n$$

$$(D_n F)(X) = (\underset{\substack{\uparrow \\ \overset{?}{=}}}{C} \wedge X^{\wedge n})_{h\Sigma_n}$$

$$C = (cr_n D_n F)(S^0, \dots, S^0) =: \partial^{(n)}(F)(*)$$

<u>Def'n</u> $D^{(n)}F = cr_n D_n F$ is the <u>nth differential of F</u>

$$D^{(n)}F(X) = \left(\partial^{(n)}F(*) \wedge X^{\wedge n}\right)_{h\Sigma_n}$$

<u>Thm:</u> The nth differential $D^{(n)}F$ (of any functor) is equivalent to the multilinearization of $cr_n F$, i.e. an symmetric/equivariant version of

$$\operatorname*{hocolim}_{(k_1,\dots,k_n)} \Omega^{k_1+\dots+k_n} cr_n F(\Sigma^{k_1}X_1, \Sigma^{k_2}X_2, \dots, \Sigma^{k_n}X_n)$$

<u>Remark:</u> This allows us to actually compute this.

<u>Notation:</u> $\partial^{(n)} = \partial_n$ in tomorrow's talk, and $\neq \partial^n$

Evening discussion – Monday

Analyticity: $F : Top_* \longrightarrow Top_*$

Recall Taylor tower

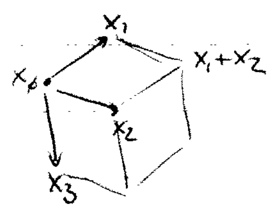
$$\begin{tikzcd} & \vdots \arrow[d] \\ & P_3F \arrow[d] \\ & P_2F \arrow[d] \\ F \arrow[r] \arrow[ur] \arrow[uur] & P_1F \end{tikzcd}$$

$F \to \operatorname{holim}_{\underset{n}{\leftarrow}} P_nF$

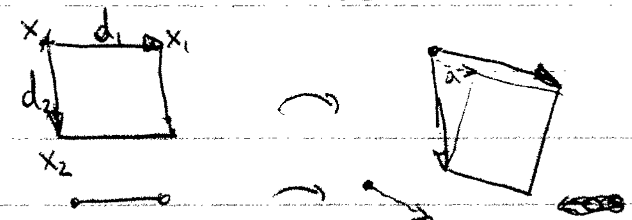
$X \xrightarrow{r} Y$, $\ell(r) = e^{-\mathrm{conn}(r)}$

– length of r

Intuition: $f : \mathbb{R}^k \longrightarrow \mathbb{R}^k$



Fix n vectors. n-th crosseffect
≙ derivative in the n directions of these vectors.



As long as $|x_\phi - x_i| \le e^{-k}$

$|a| \le e^c |x_\phi - x_1|\,|x_\phi - x_2|$ ≙ Lipschitz condition

≙ $E_1(c,k)$ $n=1$

i.e.

for functors $\mathrm{conn}(X_\phi - X_i) \ge k \ \forall i \Rightarrow \mathrm{conn}\ a \ge -c + \sum \mathrm{conn}(X_\phi \to X_i)$

condition $E_n(c,k)$: $|x_\phi - x_i| \le e^{-k} \Rightarrow |a| \le e^c \prod_i |x_\phi - x_i|$ $\forall i = 1,\dots,n$

$\mathrm{conn}(X_\phi - X_i) \ge k \ \forall i = 1,\dots,n+1 \Rightarrow \mathrm{conn}\ a \ge -c + \sum_{i=1}^{n+1} \mathrm{conn}(X_\phi - X_i)$

<u>Def'n</u>: <u>ρ-analicity</u> of F if for some q F has $E_n(n\rho - q, \rho+1)$ ~~where q is any constant~~ $\forall n$.

<u>Again</u>:

Given $\rho \in \mathbb{N}$, F is ρ-<u>analytic</u> if there is a $q \in \mathbb{Z}$ st F has $E_n(n\rho - q, \rho+1)$ $\forall n \geq 1$.

<u>Thm</u>: If F is ρ-analytic and $\operatorname{conn}(X) \geq \rho$ then

$$F(X) \xrightarrow{\sim} \operatorname{holim} P_n F(X)$$

Here $F: Top_+ \longrightarrow Top_+$

$\{X_\emptyset \longrightarrow X_i\}$ $i = 1, \dots, n+1$

$$\begin{tikzcd}
F(X_\emptyset) \arrow[rr] \arrow[dd] \arrow[dr, dashed] & & F(X_2) \arrow[dd] \\
& \operatorname{holim} \arrow[ur] \arrow[dl] & \\
F(X_1) \arrow[rr] & & F(X_1 \cup_{X_\emptyset} X_2)
\end{tikzcd}$$

Given $\begin{matrix} Y \\ \downarrow \\ X \end{matrix}$,

$$\begin{tikzcd}
& \vdots \arrow[d] \\
F(Y) \arrow[r] \arrow[ddr] & P_n^X F(X) \arrow[d] \\
& \vdots \arrow[d] \\
& P_0^X F(X) = F(X)
\end{tikzcd}$$

Taylor tower around X

Today had $X = *$.

<u>Homogeneous functors</u> $F: Top_* \to Top_*$

$$D_n F := \mathrm{hofiber}(P_n F \to P_{n-1} F)$$

n homogeneous

For X a finite CW complex,

$$D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge (\Sigma^\infty X)^{\wedge n})_{h\Sigma_n}$$

$$\begin{tikzcd} Top_* \arrow[r, "D_n F"] \arrow[d, "\Sigma^\infty"'] & Top_* \\ Sp \arrow[r, "D_n F"] & Sp \arrow[u, "\Sigma^\infty"'] \end{tikzcd}$$

factors!

V. First examples – Joey Hirsch

Outline: 1. $id_{S\text{-}Alg}$

2. $Map_*(K,-)$

---

1. $id: S\text{-}Alg \longrightarrow S\text{-}Alg$

Def'n (a) $S\text{-}Mod$ = model category of Spectra

$\wedge$ symmetric monoidal product

S is the unit

(b) $S\text{-}Alg = ComMon(S\text{-}Mod, \wedge)$

Let's compute P_1F, where $F: \mathcal{C} \to \mathcal{C}$, $F(*) = *$

$$P_1F = \operatorname*{hocolim}_{n\to\infty} \Omega^n F \Sigma^n$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} T_1F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & F(*) \arrow[d] \\ F(*) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

$P_1(id_{\mathcal{C}}) = \Omega^\infty \Sigma^\infty$ $\qquad \Sigma(X) = \operatorname{hocolim}\left(\begin{smallmatrix} X & \to & * \\ \downarrow & & \\ * & & \end{smallmatrix} \right)$

$\Omega(X) = \operatorname{holim}\left(\begin{smallmatrix} & & * \\ & & \downarrow \\ * & \to & X \end{smallmatrix} \right)$

$$\mathcal{C} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} Stab(\mathcal{C})$$

$P_1(id_{\mathcal{C}})$ = "stable $\mathcal{C}$-homotopy"

When $\mathcal{C} = R\text{-}Alg$, Basterra-Mandell proved that

$Stab(R\text{-}Alg) \simeq R\text{-}Mod$

via the equivalence ↙ topol. André-Quillen Homology

$\Sigma^\infty \simeq TAQ(\ \ )$ = derived indecomp.

$I(A) := hofib(A \to R)$

$\Rightarrow TAQ(A) =$ "$I(A)/I(A)^2$"

$\Omega^\infty(M) \simeq R \vee M.$

When $R=S$

$$P_1(id_{S\text{-Alg}})(A) = S \vee TAQ(A)$$
$$D_1(id_{S\text{-Alg}})(A) = TAQ(A)$$

Goal: $D_n(id_{S\text{-Alg}})(A) = \text{Multilin}(cr_n(id)) \circ \Delta(A))_{h\Sigma_n}$

Claim: $cr_n(id)(A_1, \dots, A_n) = I(A_1) \wedge \dots \wedge I(A_n)$

(coproducts in symm. mon. category $= \wedge$)

Claim: $\text{Multilin}(cr_n(id))(A_1, \dots, A_n)$
$$= TAQ(A_1) \wedge \dots \wedge TAQ(A_n)$$
$$\Rightarrow D_n(id)(A) = \left(TAQ(A)^{\wedge n}\right)_{h\Sigma_n} = \circledast$$

Fact: If $I(A)$ is 0-connected, then
$$A \xrightarrow{\sim} \operatorname{holim}_{n\to\infty} P_n(id)(A) = P_\infty(A)$$

Corollary: $A \xrightarrow{f} B$ & $I(A), I(B)$ are 0-connected and $TAQ(A) \xrightarrow{\sim} TAQ(B)$.

Then $A \xrightarrow{\sim} B$.

Proof:
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\wr"] & B \arrow[d, "\wr"] \\
P_\infty A \arrow[r, "P_\infty f"'] & P_\infty B
\end{tikzcd}
\qquad
\begin{tikzcd}
D_{n+1}A \arrow[r] \arrow[rr, phantom, "\simeq", bend right=20] & P_{n+1}A \arrow[r] \arrow[d] & P_{n+1}B \arrow[d] & D_{n+1}B \arrow[l] \\
 & P_nA \arrow[r, "\sim", "P_nf"'] & P_nB &
\end{tikzcd}$$

induction

$$\Rightarrow P_{n+1}A \xrightarrow{\sim} P_{n+1}B$$

∎

Remark: $\circledast = {}$"$\Omega^\infty(\partial_n(id) \wedge (\Sigma^\infty A)^{\wedge n})_{h\Sigma_n}$
$$\Rightarrow \partial_n(id) = R \text{ with trivial } \Sigma_n\text{-action}$$

2. $\mathrm{Map}_*(K,-)$

Notation: K based finite CW complex, X a based space

T_X = category of based spaces containing X as a retract

$Y \in T_X$: $X \hookrightarrow Y \xrightarrow{\alpha} X$

K_i^n denotes an equivariant subquotient of $K^{\times n}$

(i.e. $\exists\, K_a^n \subset K_b^n \subset K^n$, $K_i^n \cong K_b^n / K_a^n$)

Def'n: $\overline{\mathrm{Map}}_*(K_i^n, (Y/X)^{\wedge n} \wedge \mathrm{Map}(K,X)_+)$

$= \{ f \in \mathrm{Map}(\text{———''———})$

$\forall\, k_i^n \in K_i^n \setminus f^{-1}\{*\}$,

$P_{\mathrm{Map}(K,X)}(f(k_i^n))(P_{K_i^n,i}(k_i^n)) = \alpha(P_i \circ P_{Y/X}(f(k_i^n)))\ \forall i=1,\dots,n \}$

$P_{K_i^n,i} : K_i^n \setminus \{*\} \to K$ projections.

$$\begin{array}{ccc} K_i^n & \xrightarrow{F} & (Y/X)^{\wedge n} \\ \downarrow P_i & & \downarrow \alpha \circ P_i \\ K & \longrightarrow & X \end{array}$$

Exercise: Using this talk & the def'n of $\overline{\mathrm{Map}}$, figure out $P_n^X Q\mathrm{Map}(K,-)$.

Spoiler Alert: $X = *$

M = category of finite sets w/ surjection

M_n = — " — $\leq n$ — " —

$$\begin{array}{rcccl} Y^{\wedge \cdot}: & M_n^{op} & \longrightarrow & \mathrm{Top}_* & \\ & m & \longmapsto & Y^{\wedge m} & \ni y_{\theta(1)} \wedge \cdots \wedge y_{\theta(m)} \\ & \theta \downarrow & & \uparrow \theta^* & \qquad \uparrow \\ & n & \longmapsto & Y^{\wedge n} & \ni y_1 \wedge \cdots \wedge y_n \end{array}$$

<u>Def'n</u> Fix K, Y. $F_n, G_n : M_n^{op} \longrightarrow Top_*$

$$F_n^K(m) = \Sigma^\infty K^{\wedge m}$$
$$G_n^Y(m) = \Sigma^\infty Y^{\wedge n}$$

<u>Def'n:</u> We give $\underset{M_n}{\mathrm{Nat}}(\Sigma^\infty K^{\wedge \cdot}, \Sigma^\infty Y^{\wedge \cdot})$ the subspace topology wrt

$$\cap$$

$$\prod_{m \in M_n} \mathrm{Map}(\Sigma^\infty K^{\wedge m}, \Sigma^\infty Y^{\wedge n})$$

<u>Why?</u> Observe that

(1) $\begin{array}{c} \mathrm{Nat}(F_n, G_n) \\ \downarrow \text{restr.} \\ \mathrm{Nat}(F_{n-1}, G_{n-1}) \end{array}$ is a fibration

(2) $\underbrace{\Omega^\infty \Sigma^\infty}_{Q =} \mathrm{Map}(K, Y) \xrightarrow{\sigma_n} \mathrm{Nat}(F_n, G_n)$

given by $\mathrm{Map}(K,Y) \longrightarrow \mathrm{Map}(K^{\wedge n}, Y^{\wedge n})$

$$f \longmapsto f^{\wedge n}$$

(3)
$$\begin{tikzcd} & \vdots \arrow[d] \\ & \mathrm{Nat}(F_n, G_n) \arrow[d] \\ {} \arrow[ur, "\sigma_n"] \arrow[r, "\sigma_{n-1}"'] & \mathrm{Nat}(F_{n-1}, G_{n-1}) \arrow[d] \\ & \vdots \end{tikzcd}$$
commutes

<u>Thm:</u>
$$\begin{tikzcd} & \vdots & \vdots \arrow[d] \\ Q\,\mathrm{Map}(K,-) \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{Nat}(F_n, G_n) \arrow[d] \\ & \vdots & \vdots \end{tikzcd}$$

is the Taylor tower for $Q\,\mathrm{Map}(K,-)$, where $Q = \Omega^\infty \Sigma^\infty$.

Moreover, $D_n(Q\,\mathrm{Map}(K,-)) = \mathrm{Map}_*(\Sigma^\infty K^{\wedge n}/\Delta^n K, \Sigma^\infty Y^{\wedge n})^{\Sigma_n}$

fat diagonal

<u>Surprise:</u> $P_\infty(\,Q\mathrm{Map}(K,-)\,) = \underbrace{\mathrm{Nat}(F,G)}$

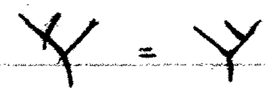
maps of right modules over the commutative operad.

finite sets w/ surj. $=$ Com Operad

<u>Claim:</u> $M \xrightarrow{\mathcal{A}} \mathrm{Top}_+$ which are product preserving are commutative algebras in spaces.

<u>Pf:</u> $\mathcal{A}(1) =: A$

$$\begin{array}{ccc} \mathcal{A}(2) & \longrightarrow & \mathcal{A}(1) \\ \cong & & \| \\ A^{\times 2} & \xrightarrow[\mu]{} & A \end{array}$$

\+ 

∎

VI - Derivatives of the identity functor

We consider $id: Top_+ \longrightarrow Top_+$

Analyticity of id:

Thm (BM, ES, G)

Let $\mathcal{X}$ be a strongly co Cartesian n-cube.
If for $1 \le i \le n$ the maps $X_\phi \to X_i$ are k_i-connected, then $\mathcal{X}$ is $(1-n+\sum k_i)$-Cartesian.

Cor: It satisfies $E_n(n,k)$ $\forall k \in \mathbb{Z}$ $\forall n \ge 1$
$\Rightarrow$ id is 1-analytic.
$\Rightarrow$ Taylor tower converges on simply connected spaces.

Remark: Converges for suitably nilpotent spaces.

Derivatives:

$$\begin{array}{c} \operatorname{colim} \Omega^{k_1+\dots+k_n} cr_n(id)(\Sigma^{k_1}X_1, \dots, \Sigma^{k_n}X_n) \\ \downarrow \\ \Omega^\infty(\boxed{\partial_n(id)} \wedge X_1 \wedge \dots \wedge X_n) \end{array}$$

Construction: Let $\mathcal{X}$ be an n-cube of spaces. For $U \subset \{1, \dots, n\}$, let
$$I^U = \{(t_1, \dots, t_n) \in I^n \mid t_i = 0 \text{ if } i \notin U\}$$
A point in $\operatorname{tfib}(\mathcal{X})$ is a collection $\{\Phi_U\}_{U \subset \underline{n}}$, where
$$\Phi_U : I^U \longrightarrow \mathcal{X}_U$$
satisfying (a)
$$\begin{tikzcd} I^V \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{X}_V \arrow[d] \\ I^U \arrow[r] & \mathcal{X}_U \end{tikzcd} \qquad V \subseteq U$$
(b) if $t_i = 1$ for some i, then $\Phi_U(t_1, \dots, t_n) = *$

Construction of T_n

A point in $cr_n(id)(X_1, \ldots, X_n)$ consists of maps

$$\Phi_U : I^U \longrightarrow \bigvee_{i \notin U} X_i$$

In particular, we get maps

$$I^{n-1} \cong I^{\underline{n} \setminus \{i\}} \longrightarrow X_i$$

Get $T_n' : cr_n(id)(X_1, \ldots, X_n) \to Map_*(I^{n(n-1)}, \prod_{i=1}^n X_i)$

Compose with $\prod_{i=1}^n X_i \to \bigwedge_{i=1}^n X_i$:

$$T_n'' : cr_n(id)(X_1, \ldots, X_n) \to Map_*(I^{n(n-1)}, \bigwedge_{i=1}^n X_i)$$

Make the identification

$$I^{n(n-1)} = \{ (t_{ij})_{1 \le i,j \le n} \mid t_{ii} = 0 \ \forall i \}$$

$$= \left\{ \begin{pmatrix} 0 & & \\ & \ddots & * \\ * & & 0 \end{pmatrix} \begin{matrix} \leftarrow I^{n-1} \\ \leftarrow I^{n-1} \\ \vdots \end{matrix} \right\}$$

Def'n: $Z := \{ t \in I^{n(n-1)} \mid t_{ij} = 1 \text{ for some } ij \}$ ← go to * b/c of (b)

$W_{ij} := \{ t \in I^{n(n-1)} \mid t_{ik} = t_{jk} \ \forall 1 \le k \le n \}$ ← will go to * in smash product

$$K_n := I^{n(n-1)} / (Z \cup \bigcup_{ij} W_{ij})$$

So we get a map

$$T_n : cr_n(id)(X_1, \ldots, X_n) \to Map_*(K_n, X_1 \wedge \cdots \wedge X_n)$$

Claim: This map is Σ_n-equivariant.

Claim: This map becomes an equivalence after multilinearization

Prop: Nonequivariantly, $K_n = \bigvee_{(n-1)!} S^{n-1}$ (Exercise)

$$\Rightarrow Map_*(K_n, X_1 \wedge \cdots \wedge X_n) \cong \prod_{i=1}^{(n-1)!} \Omega^{n-1}(X_1 \wedge \cdots \wedge X_n)$$

"First step": $\Omega^n cr_n(id)(\Sigma X_1, \dots, \Sigma X_n) \to \Omega^n \prod_{i=1}^{(n-1)!} \Omega^{n-1}(\Sigma X_1 \wedge \dots \wedge \Sigma X_n)$

consider only 1 now.

$$\wr$$

$L_n : \underset{cr_n(\Omega\Sigma)(X_1,\dots,X_n)}{\underset{\shortparallel}{\Omega cr_n(id)(\Sigma X_1, \dots, \Sigma X_n)}} \to \prod_{i=1}^{(n-1)!} \Omega^n \Sigma^n (X_1 \wedge \dots \wedge X_n)$

Thm (Hilton-Milnor)

$$\Omega\Sigma(X_1 \vee \dots \vee X_n) \xrightarrow{\sim} \prod \Omega\Sigma(X_1^{\wedge a_1} \wedge \dots \wedge X_n^{\wedge a_n})$$

monomials in a standard basis of $Lie(n)$,
$Lie(n)$ = free Lie algebra on generators $x_1, \dots, x_n$
a_i = number of x_i's in a given monomial.

Cor: $cr_n(\Omega\Sigma)(X_1, \dots, X_n) \simeq \prod_{\substack{\text{st. } a_i \geq 1 \\ \forall i}} \Omega\Sigma(X_1^{\wedge a_1} \wedge \dots \wedge X_n^{\wedge a_n})$

Cor: If all the X_i are k-connected,

$$\pi_m\big(cr_n(\Omega\Sigma)(X_1,\dots,X_n)\big) \to \pi_m\Big(\prod_{i=1}^{(n-1)!} \Omega\Sigma(X_1 \wedge \dots \wedge X_n)\Big)$$

This map is π_m of projection using cor above.

is an iso. if $0 \leq m \leq (n+1)(k+1)-1$

Since $\pi_m(\Omega\Sigma(X_1 \wedge \dots \wedge X_n)) = \pi_m(X_1 \wedge \dots \wedge X_n)$

In this range (BM, Freudenthal), we get

Prop: For $0 \leq m \leq (n+1)(k+1)-1$

$$\pi_m\big(cr_n(\Omega\Sigma)(X_1,\dots,X_n)\big) \simeq \pi_m\Big(\prod_{i=1}^{(n-1)!} X_1 \wedge \dots \wedge X_n\Big)$$
$$\pi_m\Big(\prod_{i=1}^{(n-1)!} \Omega^n\Sigma^n(X_1 \wedge \dots \wedge X_n)\Big) \simeq \pi_m\Big(\prod_{i=1}^{(n-1)!} X_1 \wedge \dots \wedge X_n\Big)$$

and L_n induces iso's on π_m in those degrees.

This gives

$$\Omega^{\ell n} cr_n(id)(\Sigma^\ell X_1, \dots, \Sigma^\ell X_n) \to \Omega^{\ell n} Map_*(K_n, \Sigma^\ell X_1 \wedge \dots \wedge \Sigma^\ell X_n)$$
$$\shortparallel \qquad\qquad\qquad\qquad \shortparallel$$
$$\Omega^{\ell n-1} cr_n(\Omega\Sigma)(\Sigma^{\ell-1}X_1, \dots, \Sigma^{\ell-1}X_n) \to \Omega^{\ell n-1} \prod_{i=1}^{(n-1)!} \Omega^n\Sigma^n(\Sigma^{\ell-1}X_1 \wedge \dots \wedge \Sigma^{\ell-1}X_n)$$

induces iso's on π_m for $0 \leq m \leq (n+1)(k+1)-1-(\ell n-1) = 1 + \text{junk}$

$\Rightarrow T_n$ becomes an equivalence after multilinearizing!

$$\operatorname*{colim}_{l\to\infty} \Omega^{ln} \mathrm{Map}_*(K_n, \Sigma^{ln} X_1 \wedge \dots \wedge X_n)$$
$$\cong$$
$$\mathrm{Map}_*(K_n, Q(X_1 \wedge \dots \wedge X_n))$$

<u>Thm</u>: $\partial_n(\mathrm{id}) = DK_n$

Non-equivariantly, $\partial_n(\mathrm{id}) = \bigvee_{i=1}^{(n-1)!} S^{1-n}$

Arone-Mahowald, Arone-Kankaanrinta

<u>Def'n</u>: $\mathrm{Part}(n)$ = poset of partitions of $\{1, \dots, n\}$

$\mathrm{Part}_{<1}(n)$ = subset of nontrivial partitions $= \mathrm{Part}(n) \setminus \{\underline{1}\}$

$\mathrm{Part}_{>0}(n)$ = subset of proper partitions $= \mathrm{Part}(n) \setminus \{\underline{0}\}$

<u>Prop</u> (Exercise)

$$K_n \simeq \left| N(\mathrm{Part}(n)) / N(\mathrm{Part}_{<1}(n)) \cup N(\mathrm{Part}_{>0}(n)) \right|$$
$$\simeq \Sigma S \left| N(\mathrm{Part}(n) \setminus \{\underline{0}, \underline{1}\}) \right| \quad \text{if } n > 1.$$

VII. Operad & Module Structure on derivatives – Emily Riehl

Reference: Arone–Ching, Operads + chain rules for the calculus of functors

Context: $\mathcal{C}, \mathcal{D}, \mathcal{E} = Top_*$, Spectra = EKMM S-modules

$F: \mathcal{C} \to \mathcal{D}$ ptd, $F(*) = *$, homotopical

$\rightsquigarrow \partial_n F \circlearrowleft \Sigma_n$, $n \in \mathbb{N}$

spectrum

$\partial_* F$ forms a symmetric sequence in Spectra

Σ = finite sets + isos $\quad \partial_* F : \Sigma \to Spec$

Q: What extra structure is present on $\partial_* F$?

Example: $\underset{\underline{\mathbb{1}}}{\partial_* I_{Spectra}} = \begin{cases} \partial I = S \\ \partial_n I = * & n \neq 1 \end{cases}$

$P_1 I = I$

$P_n I = *$

$D_1 I = \mathrm{hofiber}(I \to *) = I$

$D_n I = (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$

$\Rightarrow \partial_1 F = S$

$P_n I = I$ hofiber $= *$

Example: $\partial_* I_{Top_*}$ is an <u>operad</u> = monoid in $(Spec^{\Sigma}, \circ, \mathbb{1})$ (comp. product)

not symmetric monoidal but still have L- or R-modules

Main Thm 1: $\mathcal{C} \xrightarrow{F} \mathcal{D}$ homotopical, then $\partial_* F$ form a $(\partial_* I_{\mathcal{D}}, \partial_* I_{\mathcal{C}})$-bimodule.

Main Thm 2: $\mathcal{C} \xrightarrow{G} \mathcal{D} \xrightarrow{F} \mathcal{E}$ reduced, homotopical + F is finitary then $\partial_*(FG) = \partial_* F \circ_{\partial_* I_{\mathcal{D}}} \partial_* G$ derived comp. product equiv. of $(\partial_* I_{\mathcal{E}}, \partial_* I_{\mathcal{C}})$-bimodule

Remarks: •) $\mathcal{D} = \mathrm{Spec} \Rightarrow \circ =$ usual $\circ$
•) reduced b/c only for derivatives at $*$
•) F finitary necessary
•) derived $\circ$ is via 2-sided bar construction

Context: $(\mathcal{C}, \wedge, S)$ symmetric monoidal category, complete+cocompl

$\mathcal{C}^{\Sigma}$ symm seq. in $\mathcal{C}$

Def'n: $A, B \in \mathcal{C}^{\Sigma}$ $\quad A \circ B(n) = \bigvee_{\text{partitions of } \underline{n}} \underset{\uparrow \\ \#\text{ of sets}}{A(k)} \wedge B(n_1) \wedge \underset{\uparrow \\ \text{sizes (unordered)}}{\ldots} \wedge B(n_k)$

$$= \bigvee_{k=1}^{n} \left(\bigvee_{n \to\!\!\to k} A(k) \wedge B(n_1) \wedge \ldots \wedge B(n_k) \right)_{\Sigma_k}$$

Σ_k-action is evident

Lemma: $\mathcal{C}$ is closed $\Rightarrow (\mathcal{C}^{\Sigma}, \circ, \mathbb{1})$ monoidal category. (Exercise.)

Def'n: An operad P is a monoid $P \circ P \xrightarrow{\mu} P$, $\mathbb{1} \xrightarrow{\eta} P$. assoc.+unital
$S \to P(1)$

μ at n def'd by maps

$$P(k) \wedge P(n_1) \wedge \ldots \wedge P(n_k) \longrightarrow P(\underbrace{n_1 + \ldots + n_k}_{=n})$$

Def'n Right P-module R is a symm. sequence $R \circ P \to R$ assoc.+unital
Left " L " $P \circ L \to L$ "

Example: P is reduced if $\mathbb{1} \to P$ $\quad S \xrightarrow{\simeq} P(1)$ iso in $\mathcal{C}$ $\quad P(1) \longrightarrow S$
$\mathbb{1}$ is a $L+R$-module over P.

F: Spectra $\longrightarrow$ Spectra How to get structure on $\partial_* F$?

Dual derivatives (also in $\mathrm{Spectra}^{\Sigma}$) $\quad \partial^* F = \underset{\uparrow \\ \text{Spectrum}}{\mathrm{Nat}}(FX, X^{\wedge n})$

– tacitly Spectra $\longrightarrow$ Spectra

– EKMM S-modules $\Rightarrow$ all def. fibrant

To get the desired homotopy type need cofibrancy for F:

$QF \to F$ cofibrant ~~[illegible]~~ replacement constructed in $[\mathrm{Spec}, \mathrm{Spec}]_{\mathrm{proj}}$ by SOA

presented cell spectrum: $\rightsquigarrow$ Sub(QF) filtered category of finite subcomplex

$$\partial^n F = \{\mathrm{Nat}(CX, X^{\wedge n})\}_{C \in \mathrm{Sub}(QF)} \quad \text{pro-obj in Spectra}$$

Thm (Christensen + Isaksen)

J. Quillen equivalence

$$\mathbb{D}: \mathrm{Pro}(\mathrm{Spectra})^{op} \underset{\mathrm{Map}(-,S)}{\overset{\mathrm{Map}(-,S)}{\rightleftarrows}} \mathrm{Ind}(\mathrm{Spectra}) \underset{\text{trivial}}{\overset{\mathrm{colim}}{\rightleftarrows}} \mathrm{Spectra}$$

$\mathbb{D}$ defines ~~[illegible]~~ the Spanier Whitehead dual.

$$\partial_* F = \mathbb{D}\{\underbrace{\mathrm{Nat}(CX, X^{\wedge n})}_{\text{termwise cof. replacement}}\}_{C \in \mathrm{Sub}(QF)} = \underset{C \in \mathrm{Sub}(QF)}{\mathrm{hocolim}} \mathrm{Map}(\mathrm{Nat}(X, X^{\wedge n}), S)$$

Claim: This is a model for Goodwillie derivatives

+ furthermore structure on $\partial^* F \rightsquigarrow$ dual structure on $\partial_* F$

Example/Good situation: F homotopical, a comonad, presented cell-functor $\Rightarrow \partial^* F$ are an operad.

Take $\Sigma^\infty \Omega^\infty$ but w/ a modification (replace Top_* w/ $sSet_*$)

$$\Sigma^\infty = S_c \wedge - : sSets_* \rightleftarrows \mathrm{Spectra} : \underline{\mathrm{Spec}}(S_c, -) = \Omega^\infty$$

$S_c \xrightarrow{\sim} S$ cofibrant replacement of S

$\Sigma^\infty \Omega^\infty X = S_c \wedge \underline{\mathrm{Spec}}(S_c, X)$ $\ast \longrightarrow S_c$ is a generating cofibration

$$\partial^n \Sigma^\infty \Omega^\infty = \mathrm{Nat}(S_c \wedge \underline{\mathrm{Spec}}(S_c, X), X^{\wedge n}) \cong \mathrm{Map}(S_c, S_c^{\wedge n})$$
$$\cong \mathrm{Map}(S, S^{\wedge n})$$

internal hom in Spectra

Upshot: dual derivatives of $\Sigma^\infty \Omega^\infty$ are equivalent to S

operad structure coincides w/ coendomorphism operad structure

Thm (Ching)
Turns out that $\partial_* I_{Top_*} \simeq \mathbb{D}B(\mathbb{1}, S, \mathbb{1}) = \mathbb{D}BS \simeq \mathbb{D}(\mathbb{1}, \partial^*(\Sigma^\infty\Omega^\infty), \mathbb{1})$

Slogan: $\partial_* I_{Top_*}$ is Koszul dual to commutative operad

(Pf: $\partial_* I_{Top_*}$ ------)

$\boxed{\text{Bar constructions:}}$

Thm (Ching) P a reduced operad, R a right P-operad, L left P-operad

Then
- $B(\mathbb{1}, P, \mathbb{1})$ is a cooperad BP
- $B(\mathbb{1}, P, L)$ is a left BP-comonad
- $B(R, P, \mathbb{1})$ — " — right — " —

Def'ns:

$(\mathcal{C}, \wedge, S)$ symm. monoidal $\Rightarrow$ $(\mathcal{C}^{op}, \wedge, S)$ symm. mon. category (likely not closed)

composition product applied to $\mathcal{C}^{op}$
$\rightsquigarrow$ dual composition product

$(\mathcal{C}^\Sigma, \hat{\circ}, \mathbb{1})$ "monoidal" product assoc.

$$\underset{\substack{\uparrow \\ \text{not assoc.}}}{A \hat{\circ} B}(n) = \prod_{k=1}^{n} \left(\prod_{n \to k} A(k) \wedge B(n_1) \wedge \dots \wedge B(n_k) \right)_{\Sigma_k}$$

Def'n: Q is a cooperad iff it is a "comonoid" wrt $\hat{\circ}$ in $\mathcal{C}^\Sigma$

$$\begin{tikzcd} & Q \hat{\circ} Q \arrow[r, "\delta \hat{\circ} 1"] & (Q \hat{\circ} Q) \hat{\circ} Q \arrow[dr] & \\ Q \arrow[ur, "\delta"] \arrow[dr] & & & Q \hat{\circ} Q \hat{\circ} Q \\ & Q \hat{\circ} Q \arrow[r, "1 \hat{\circ} \delta"'] & Q \hat{\circ} (Q \hat{\circ} Q) \arrow[ur] & \end{tikzcd}$$

right and left comodules $Q \circ L \leftarrow L$, $R \circ Q \leftarrow R$

Def'n: P operad, L a left P-mod, R a right P-mod.

$B_\bullet(R, P, L) : \Delta^{op} \longrightarrow \mathcal{C}^\Sigma$

$B_k(R, P, L) = R \circ \underbrace{P \circ \dots \circ P}_{k \text{ terms}} \circ L$, $B(R, P, L)$ take termwise geom. realiz. $|\cdot|$

Remark: $P = \mathbb{1} \Rightarrow B(R, \mathbb{1}, L) = R \circ L$

<u>Lemma:</u> $R \xrightarrow{\sim} R'$, $L \xrightarrow{\sim} L'$, $P \xrightarrow{\sim} P'$ a termwise cofibrant

$\Rightarrow B(R,P,L) \xrightarrow{\sim} B(R',P',L')$ is a weak equivalence.

<u>Main ingredient:</u> $B(R,P,L) \longrightarrow B(R,P,\mathbb{1}) \circ B(\mathbb{1},P,L)$

<u>Connect back to ~~Top$_+$~~ sSets$_+$ and chain rules</u>

$\mathcal{C} \xrightarrow{G} sSet_+ \xrightarrow{F} \mathcal{D}$ $\quad F\Omega^\infty$ is a right $\Sigma^\infty\Omega^\infty$-comodule

$\Sigma^\infty G$ " left "

$\rightsquigarrow \partial^*(\Sigma^\infty G)$ and $\partial^*(F\Omega^\infty)$ are left and right modules for $\partial^*(\Sigma^\infty\Omega^\infty)$-module.

<u>Thm:</u> F, G pointed, simp., homotopical + F is finitary

$\Rightarrow P_n(FG) \xrightarrow[\eta]{\sim} \widetilde{\mathrm{tot}}(P_n(F\Omega^\infty(\Sigma^\infty\Omega^\infty)^k\Sigma^\infty G)$

also true for D_n and ∂_n.

<u>Unifying fact:</u> $\partial^*(FG) = \partial^*(F\Omega^\infty) \circ_{\partial^*(\Sigma^\infty\Omega^\infty)} \partial^*(\Sigma^\infty G)$

$F = G = I_{Top_+} \Rightarrow \partial^*(I_{Top_+}) \simeq B(\mathbb{1}, \partial^*(\Sigma^\infty\Omega^\infty), \mathbb{1})$

$\Rightarrow I_{Top_+}$ is Koszul dual to $\partial^*(\Sigma^\infty\Omega^\infty)$

VIII. Classification of polynomial functors – Michael Ching

Story so far: $F : Top_* \to Top_*$

$$\begin{tikzcd} & \vdots \arrow[d] & \\ & P_n F \arrow[d] & D_n F \arrow[l] \\ F \arrow[ur] \arrow[r] & P_{n-1} F \arrow[d] & \\ & \vdots & \end{tikzcd}$$

$D_n F$ fiber

$D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$

$\partial_n F$: spectrum with Σ_n-action

$\partial_* F$: symmetric sequence of spectra

We had that $\partial_* I_{Top_*}$ is an operad

$\partial_* F$ is an $\partial_* I_{Top_*}$-bimodule

Main Question: How can we describe the information needed to reconstruct the tower from the derivatives $\partial_* F$?

General framework for answering questions of the form

Given a functor $\mathcal{A} \xrightarrow{L} \mathcal{B}$,

Can we recover $A \in \mathcal{A}$ from $L(A)$ together with (which) extra information?

.... descent theory

Suppose that L has a right adjoint

$$\mathcal{A} \underset{R}{\overset{L}{\rightleftarrows}} \mathcal{B}$$

We will apply this framework to

$$[Top_*^{fin}, Top] \underset{\Phi}{\overset{\partial_*}{\rightleftarrows}} \partial_* I_{Top_*}\text{-bimodules}$$

We have maps $1_{\mathcal{A}} \xrightarrow{\eta} RL$

$LR \xrightarrow{\varepsilon} 1_{\mathcal{B}}$

and ~~hence~~ $LR \xrightarrow{L\eta R} LRLR$, $LR \xrightarrow{\varepsilon} 1_{\mathcal{B}}$ } $K = LR$ is a <u>comonad</u> on $\mathcal{B}$

$$\begin{tikzcd} K \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & KK \arrow[d, "K\Delta"] \\ KK \arrow[r, "\Delta K"'] & KKK \end{tikzcd} \quad \& \quad \begin{tikzcd} K \arrow[r, "\Delta"] \arrow[dr] & KK \arrow[d, shift right, "K\varepsilon"'] \arrow[d, shift left, "\varepsilon K"] \\ & K \end{tikzcd}$$

and, for any $A \in \mathcal{A}$,

$$LA \xrightarrow{L\eta_A} LRLA,$$

which makes $B = LA$ into a <u>K-coalgebra</u>.

$$\begin{tikzcd} B \arrow[r, "\theta"] \arrow[d, "\theta"'] & KB \arrow[d, "K\theta"] \\ KB \arrow[r, "\Delta"'] & KKB \end{tikzcd} \quad \& \quad \begin{tikzcd} B \arrow[r, "\theta"] \arrow[dr] & KB \arrow[d, "\varepsilon"] \\ & B \end{tikzcd}$$

This [redacted] is our "extra information" on LA.

We try to recover A from LA using a <u>cobar construction</u>:

for any K-coalgebra B we have

$$RB \underset{\eta_{RB}}{\overset{R\theta}{\rightrightarrows}} RLRB \rightrightarrows RLRLRB \cdots$$

The <u>cobar construction</u> B is

$$\mathrm{cobar}(R, LR, B) = \mathrm{Tot}(\qquad)$$

If $B = LA$, $\quad A \longrightarrow \mathrm{cobar}(R, LR, LA)$

<u>Question in general</u>: Find conditions that make this equivalence.

Sp=Spectra

<u>Example</u>: $\mathrm{Top}_+ \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{Spectra}$

$$X \longrightarrow \mathrm{cobar}(\Omega^\infty, \Sigma^\infty\Omega^\infty, \Sigma^\infty X)$$

is an equivalence if X is nilpotent. (Barsfield-Kan)

<u>Thm</u> The following functors have right adjoints:

pointed simplicially enriched functor $\rightsquigarrow$ $[Sp^{fin}, Sp] \xrightarrow{\partial_*} Sp^\Sigma$ ← symmetric sequences

$[\mathrm{Top}_+^{fin}, Sp] \xrightarrow{\partial_*}$ right $\partial_* I_{\mathrm{Top}_+}$-modules $\xrightarrow{\text{forget}} Sp^\Sigma$

$[Sp^{fin}, \mathrm{Top}_+] \xrightarrow{\partial_*}$ left $\partial_* I_{\mathrm{Top}_+}$-modules

$[\mathrm{Top}_+^{fin}, \mathrm{Top}_+] \xrightarrow{\partial_*}$ $\partial_* I_{\mathrm{Top}_+}$-bimodules $\xrightarrow{\text{forget}}$ left $\partial_* I_{\mathrm{Top}_+}$-modules

---

<u>Corollary</u>: Let $[\mathcal{C}^{fin}, \mathcal{D}] \underset{\Phi}{\overset{\partial_*}{\rightleftarrows}} \mathcal{M}$ be one of the above with right adjoint Φ.

Then $K = \partial_*\Phi : \mathcal{M} \longrightarrow \mathcal{M}$ is a comonad, and for any $F \in [\mathcal{C}^{fin}, \mathcal{D}]$, $\partial_* F$ has a K-coalgebra structure.

---

from above.

<u>Theorem</u> ① If F is N-excisive*) for some N (or if $F \xrightarrow{\sim} \mathrm{holim}\, P_n F$), then $F \xrightarrow{\sim} \mathrm{cobar}(\Phi, \partial_*\Phi, \partial_* F)$, ie.

F <u>can</u> be recovered from $\partial_* F$ with its K-coalgebra structure

② More generally, for <u>any</u> $F \in [\mathcal{C}^{fin}, \mathcal{D}]$

$$\begin{array}{ccc} P_n F & \xrightarrow{\sim} & \mathrm{cobar}(\Phi, \partial_*\Phi, \partial_{\le n} F) \\ \downarrow & & \downarrow \\ P_{n-1} F & \xrightarrow[\sim]{} & \mathrm{cobar}(\Phi, \partial_*\Phi, \partial_{\le (n-1)} F) \end{array}$$

*) If ρ-analytic, can recover if Taylor tower converges.

(3) $\mathrm{Nat}(F,G) \longrightarrow \mathrm{Map}_K(\partial_* F, \partial_* G)$
is an equivalence if $G \xrightarrow{\sim} \mathrm{holim}\, P_n G$.

spectrum or space of natural transformations $F \to G$

spectrum or space of derived K-coalgebra maps $\partial_* F \to \partial_* G$

(4) There is an equivalence of homotopy theories (simplicial A_∞-categories)
$$\left\{\begin{matrix} N\text{-excisive} \\ F: \mathcal{C}^{fin} \to \mathcal{D} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} N\text{-truncated} \\ K\text{-coalgebras} \end{matrix}\right\}$$

Remark: This gives the classification of polynomial functors.

Proof of (1) By induction on the Taylor tower.

$$\begin{tikzcd}
P_n F \arrow[r] \arrow[d] & \mathrm{Tot}\, P_n(\Phi(\partial_*\Phi)^\bullet \partial_* F) \arrow[d] \\
P_{n-1} F \arrow[r] \arrow[d] & \mathrm{Tot}\, P_{n-1}(\Phi(\partial_*\Phi)^\bullet \partial_* F) \arrow[d] \\
\Omega^{-1} D_n F \arrow[r, "(*)"] & \mathrm{Tot}\, \Omega^{-1} D_n(\Phi(\partial_*\Phi)^\bullet \partial_* F)
\end{tikzcd}$$

Claim: For any n, $(*)$ is an equivalence.

Pf: $D_n = \Psi \partial_*$, where $\Psi_n(A) = \Omega^\infty (A_n \wedge X^{\wedge n})_{h\Sigma_n}$ (symm. sequence)

So, $(*)$ is
$$\Omega^{-1}\Psi_n \partial_* F \xrightarrow{\sim} \mathrm{Tot}(\Omega^{-1}\Psi_n(\partial_*\Phi)(\partial_*\Phi)^\bullet \partial_* F)$$
using an extra codegeneracy. ⌟

Now,
$$\begin{tikzcd}
F \arrow[r] \arrow[d, "\wr"] & \mathrm{Tot}(\Phi(\partial_*\Phi)^\bullet \partial_* F) \arrow[d, "?"] \\
P_N F \arrow[r] & \mathrm{Tot}(P_N(\Phi(\partial_*\Phi)^\bullet \partial_* F))
\end{tikzcd}$$
follows from the following lemma, which concludes the proof of (1) ∎

<u>Lemma</u>: If A is N-truncated symmetric sequence, then ΦA is N-excisive.

Can we be more explicit about what the K-coalgebra structure actually is?

We need an explicit description of Φ, the right adjoint to ∂_*.

$$[Sp^{fin}, Sp] \xrightarrow{\partial_*} Sp^{\Sigma}$$

<u>Idea</u>: P_n preserves hocolim of spectrum-valued functors.

$\Rightarrow$ D_n —"—

$\Rightarrow$ ∂_n —"—

$\Rightarrow$ ∂_* —"—

<u>Remark</u>: $[Top_+^{fin}, Top_*] \xrightarrow{\partial_*} \partial_* I_{Top_*}$-bimodules

preserves hocolim

F

$\downarrow$

$\Sigma^\infty F \Omega^\infty$

$\downarrow$

$\partial_*(\Sigma^\infty F \Omega^\infty)$ bi-comodule over $\partial_*(\Sigma^\infty \Omega^\infty)$

"Koszul dual" $\downarrow$

$\partial_* F = \mathrm{cobar}(\mathbb{1}, \partial_*(\Sigma^\infty \Omega^\infty), \partial_*(\Sigma^\infty F \Omega^\infty), \partial_*(\Sigma^\infty \Omega^\infty), \mathbb{1})$

cobar constr. on both sides of bi-comodule

<u>Conj</u>: equivalence between $\partial_*(\Sigma^\infty \Omega^\infty)$-bicomodule & $\partial_*(I_{Top_*})$-bimodule

(Can get around this using other stuff.)

Back to spectra.

$$[Sp^{fin}, Sp] \xrightarrow{\partial_*} Sp^{\Sigma}$$

Define ∂_* by left Kan extending from representable functors

<u>Def'n</u>: $X \in Sp^{fin}$ $\quad R_X : Sp^{fin} \longrightarrow Sp$, $R_X(-) = \Sigma^\infty \mathrm{Hom}(X,-)$

(<u>Dual</u>) <u>Yoneda lemma</u>:

$$R_X(-) \wedge_{X \in Sp^{fin}} F(X) \xrightarrow{\simeq} F(-)$$

So we define $\partial_* : [Sp^{fin}, Sp] \longrightarrow Sp^{\Sigma}$ by

$$\partial_*(F) := (\partial_* R_X) \wedge_{X \in Sp^{fin}} F(X)$$

which has the right adjoint $\Phi : Sp^{\Sigma} \longrightarrow [Sp^{fin}, Sp]$,

$$\begin{aligned} \Phi(A) : X \longrightarrow\ & \mathrm{Map}_{Sp^{\Sigma}}(\partial_* R_X, A) \\ & \qquad \| \\ & \prod_{n \geq 1} \mathrm{Map}(\partial_n R_X, A)^{\Sigma_n} \\ & \qquad \| \\ & \prod_{n \geq 1} (A_n \wedge X^{\wedge n})^{h\Sigma_n} \end{aligned}$$

$$\partial_n R_X \simeq D(X^{\wedge n})$$

IX. Orthogonal Calculus 1: Theory - Kerstin Baer

Now we're interested in continuous functors

$$E : \mathcal{J} \longrightarrow Top$$

$\mathcal{J}$ = {finite dim'l inner product spaces of $\mathbb{R}^\infty$},
$mor(V,W) = O(V,W)$

continuous: $mor(V,W) \times E(V) \longrightarrow E(W)$

Examples: $E(V) = $
- $O(V)$
- $BO(V)$
- $Conf(n,V)$
- $Emb(M,V)$
- $\Omega^\infty(V^c \wedge \Theta)$ V^c ... 1pt compactification, Θ some spectrum
- $\Omega^\infty((nV)^c \wedge \Theta)_{hO(n)}$ $\quad nV := \mathbb{R}^n \otimes V$

$\mathcal{E} := Cat(\mathcal{J} \to Top)$

Def'n $E \in \mathcal{E}$ is polynomial of degree n, if

$$E(V) \xrightarrow{\sim} \underbrace{\operatorname{holim}_{0 \neq U \in \mathbb{R}^{n+1}} E(U \oplus V)}_{=: \tau_n E(V)}$$

is a homotopy equivalence $\forall V \in \mathcal{J}$

Taylor polynomials $T_n E = \operatorname{hocolim}(E \to \tau_n E \to \tau_n^2 E \to \cdots)$

$\rightsquigarrow T_n : \mathcal{E} \to \mathcal{E}$

+ natural transf. $\eta_n : 1 \to T_n$

Remarks:
(a) polynomial of deg n-1 $\Rightarrow$ polynomial of deg n
(b) $T_n E$ is a polynomial of degree n
(c) if E is polynomial of degree n, $\eta_n : E \xrightarrow{\sim} T_n E$
(d) $T_n(\eta_n) : T_n E \xrightarrow{\sim} T_n T_n E$

Universality of T_n:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & P_n \\ T_nE \arrow[ur, dashed, "\exists!\text{ up to homotopy}"'] & \end{tikzcd}$$

Pf: **Existence:**

$$\begin{tikzcd} E \arrow[r] \arrow[d] & P_n \arrow[d, "\wr"] \\ T_nE \arrow[r] \arrow[ur, dashed] & T_nP_n \end{tikzcd}$$

$\leftarrow$ invert this map

Uniqueness:

$$\begin{tikzcd} E \arrow[r, "\eta_n"] \arrow[d, "\eta_n"'] \arrow[rr, bend left] & T_nE \arrow[r, "f"] \arrow[d, "\wr\ \eta_n(\eta_n)"] & P_n \arrow[d, "\wr"] \\ T_nE \arrow[r, "\sim", "T_n(\eta_n)"'] \arrow[rr, bend right] & T_nT_nE \arrow[r, "g"] & T_nP_n \end{tikzcd}$$

f uniquely determined by g

Corollary:

$$\begin{tikzcd} & \vdots \arrow[d, dashed] & \\ & T_2E \arrow[d] & \\ E \arrow[ur] \arrow[r] \arrow[dr] & T_1E \arrow[d] & D_1E \arrow[l] \\ & T_0E & \end{tikzcd}$$

Def'n: E is <u>homogeneous of degree n</u> if $E \xrightarrow{\sim} \tau_nE$ and $T_{n-1}E \simeq *$.

Thm: If $E \in \mathcal{E}_0$ is based, then

$$D_nE \simeq \Omega^\infty\Big(\big((nV)^c \wedge \theta^{(n)}\big)_{hO(n)}\Big), \quad \vartheta^{(n)} = \vartheta E^{(n)}$$

$$\hat{=}\, x^n \qquad \hat{=}\, f^{(n)} \qquad \hat{=}\, \tfrac{1}{n!}$$

$$\boxed{\left(\vartheta E^{(n)}\right)_{kn} := E^{(n)}(\mathbb{R}^k)}$$

Def'n: $E^{(n+1)}(V) := \operatorname{hofib}\big(E(V) \longrightarrow \tau_nE(V)\big)$

$$E^{(1)}(V) = \operatorname{hofib}\big(E(V) \longrightarrow E(\mathbb{R}\oplus V)\big)$$

Example: $E(V) = BO(V)$

$$E^{(1)}(V) = \operatorname{hofib}\Big(\underset{\displaystyle \frac{EO(\mathbb{R}+V)}{O(n)}}{\underset{\shortparallel}{BO(V)}} \longrightarrow \underset{\displaystyle \frac{EO(\mathbb{R}\oplus V)}{O(n+1)}}{\underset{\shortparallel}{BO(\mathbb{R}\oplus V)}}\Big) = \frac{O(n+1)}{O(n)} = V^c$$

Moreover (w/o pf), $\vartheta E^{(1)} = S^{(0)}$.

<u>Def'n</u>: $\mathrm{mor}_n(V,W) := \mathrm{Thom}\{(f,x) : f \in \mathrm{mor}(V,W),\ x \in n\cdot \mathrm{coker} f\}$

$\mathcal{J}_{(n)} := \mathcal{J}$ with mor_n , $\mathcal{E}_{(n)} = \{\mathcal{J}_{(n)} \to \mathrm{Top}_*\}$

continuous and <u>pointed</u>

$$\begin{cases} \mathrm{mor}_n(V,W) \longrightarrow \mathrm{Map}_*(E(V), E(W)) \\ \quad * \longrightarrow \mathrm{const.}\, * \end{cases}$$

If $m<n$,

$\mathcal{J}_{(m)} \hookrightarrow \mathcal{J}_{(n)}$.

$\Rightarrow \mathrm{res}^n_m : \mathcal{E}_{(n)} \longrightarrow \mathcal{E}_{(m)}$

res^n_m has a right adjoint ind^n_m

Then,

$$E^{(n)} = \mathrm{ind}^n_0 E$$
$$= \mathrm{nat}_0(\mathrm{mor}_n(V,-), E(-))$$

<u>Remarks</u>: • $O(n)$ acts on $\mathrm{mor}_n(V,-)$

• $E \in \mathcal{E}_0 \Longrightarrow E^{(n)} \in \mathcal{E}_n$

$\mathrm{mor}_n(V,W) \wedge E^{(n)}(V) \longrightarrow E^{(n)}(W)$

$W = \mathbb{R}\oplus V \rightsquigarrow \underset{\substack{\uparrow \\ (\mathrm{incl}, X)}}{\mathrm{mor}_n(V, \mathbb{R}\oplus V)} \wedge E^{(n)}(V) \longrightarrow E^{(n)}(V \oplus \mathbb{R})$

$\cong$

$S^n \wedge E^{(n)}(V) \longrightarrow E^{(n)}(\mathbb{R}\oplus V)$

$$(\vartheta E^{(n)})_{kn} := E^{(n)}(\mathbb{R}^k)$$

$$\begin{tikzcd} E^{(1)}(V) \arrow[r] \arrow[d] & E(V) \arrow[r] \arrow[d] & E(\mathbb{R}\oplus V) \arrow[d] \\ E^{(1)}(\mathbb{R}\oplus V) \arrow[r] & E(\mathbb{R}\oplus V) \arrow[r, "i_1"'] & E(\mathbb{R}^2\oplus V) \end{tikzcd}$$

$\leftarrow$ S^1 of choices for this map.

$E(V) = \Omega^\infty\left(((nV)^c \wedge \vartheta)_{hO(n)}\right)$, every homog. f. looks like this.

<u>Claim</u>: $\vartheta E^{(k)}(V) = \begin{cases} * & k \neq n \\ \vartheta & k = n \end{cases}$

Orthog. spectra

$U \subset V \subset \mathbb{R}^\infty$

$$S^{V-U} \wedge E(U) \longrightarrow E(V)$$

$$\wr$$

$$S^{V-U} \longrightarrow \mathrm{Maps}(E(U), E(V)).$$

$$\mathrm{Mor}(U,V) = \mathrm{Thom}\{(f,x) : f : U \hookrightarrow V,\ x \in V - U\}.$$

a bundle over $[U,V]$, fiber "$V - f(U)$"

fibers become $(V - f(U))^c$.

$$\mathrm{Mor}(U,V) \xrightarrow[\wr]{} \mathrm{Maps}(E(U), E(V)).$$

X. Orthogonal Calculus 2. – Sean Tilson – Examples

Overview: – Derivatives of $BO(V)$ & $BU(V)$
– Derivatives of $\Sigma^\infty C(k,V), \Sigma^\infty \bar{C}(k,N)$
– Derivatives of $QMor(V_0, V)_*$

Thm (Arone)
The nth derivative of $BAut(V)$ is $Map_*(L_n^{\mathbb{F}}, \Sigma^\infty S^{Ad_n})$
(Here, $S^{Ad_n} = (Ad_{Aut(\mathbb{F}^n)})^c$, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$)

Def'n: $\mathcal{L}_n^{\mathbb{F}}$ = poset of direct sum decompositions of $\mathbb{F}^n$, where $\Lambda \leq \Lambda'$ if every summand of Λ is a subspace of a summand of Λ'.

Def'n: $O_{k,n}$ decreasing chains in $\mathcal{L}_n$ $(\lambda_0, \dots, \lambda_k)$, $\lambda_k = \mathbb{F}^n = \mathbb{1}$. with a basept in*
$s_i :=$ repeat ith guy
$d_i :=$ omit ith guy if $i \neq k$

$$d_k(\bar{\lambda}) = \begin{cases} * & \text{if } \lambda_{k-1} \neq 1 \\ (\lambda_0, \dots, \lambda_{k-1}) & \text{if } \lambda_{k-1} = 1 \end{cases}$$

$\Rightarrow |O_{\cdot,n}| = L_n^{\mathbb{F}}$, $\mathbb{L}_n^{\mathbb{F}} = \Sigma |\mathcal{L}_n^{\mathbb{F}} \setminus \text{final object}|$

Thm*): 1) $\exists$ $O(n-1)$-equivariant weak equivalence

$$Map_*(L_n^{\mathbb{R}}, \Sigma^\infty S^{Ad_n^{\mathbb{R}}}) \simeq Map_*(S^1 \wedge K_n, \Sigma^\infty S^0) \underset{\Sigma_n}{\wedge} O(n-1)_*$$

2) $\exists$ $U(n-1)$-equiv. weak equivalence

$$Map_*(L_n^{\mathbb{C}}, \Sigma^\infty S^{Ad_n^{\mathbb{C}}}) \simeq Map_*(S^1 \wedge K_n, \Sigma^\infty S^n) \underset{\Sigma_n}{\wedge} U(n-1)_*$$

Here $K_n = |Part(n) \setminus \text{initial \& final objects}|$.

$\Sigma^\infty C(k,V)$: $\quad C(k,V) = Emb(\underline{k}, V) = V^k \setminus \Delta^k V$,

$$\Delta^k V = \{\bar{x} \in V^k \mid x_i = x_j \text{ for some } i \neq j\} = \bigcup_{\Lambda \in P_k^0} V^{c(\Lambda)} \quad (c(\Lambda) = k/\Lambda)$$

$$\rightsquigarrow C(k,V) = \bigcap_{\Lambda \in P_k^0} V^k \setminus V^{c(\Lambda)} = \lim_{\Lambda \in P_k^0} V^k \setminus V^{c(\Lambda)}$$

*) So, the derivatives are closely related to the derivatives of the identity!

<u>Prop</u>: $\Sigma^\infty C(k,V) \xrightarrow{\sim} \underset{\Lambda\in P_k^\circ}{\mathrm{holim}}\, \Sigma^\infty V^k \setminus V^{c(\Lambda)}$

<u>Pf</u> (sketch): Change the indexing category.

$S = 2^{\binom{k}{2}}$ = collections of pairs of distinct elts in k.

= graphs w/ k (labelled) vertices.

$S \longrightarrow P_k$

$U \longmapsto \Lambda(U)$ = path components of U

$S^1 = S\setminus\phi \longrightarrow P_k^\circ$

For $F: P_k^\circ \longrightarrow$ Spectra, get $\tilde{F}: S^1 \longrightarrow P_k^\circ \xrightarrow{F}$ Spectra.

Then $\underset{P_k^\circ}{\mathrm{holim}}\, F \longrightarrow \underset{S^1}{\mathrm{holim}}\, \tilde{F}$ is a weak equivalence.

Then $\chi: S^1 \longrightarrow$ Spectra. is an n-cube (punctured)

$U \longmapsto \tilde{F}(U) = F(\Lambda(U)) = \Sigma^\infty V^k \setminus V^{c(\Lambda(U))}$ (We want $F(\Lambda) = \Sigma^\infty V^k \setminus V^{c(\Lambda)}$)

Want to show that the last corner of cube is holim of punctured cube.

$\chi: S \longrightarrow \mathrm{Top}$

$U \longmapsto \begin{cases} C(k,V) & U=\phi \\ V^k \setminus V^{c(\Lambda(U))} & \text{else.}\end{cases}$

<u>Facts</u>: 1) $\forall U \subset \binom{k}{2}$ $\chi(U) \overset{\text{open}}{\subset} \chi(\binom{k}{2}) = V^k \setminus V$

2) $\forall U \subset \binom{k}{2}$ nonempty, $\chi(U) = \bigcup_{x\in U} \chi(\{x\})$

3) $\chi(\phi) = \bigcap_{x\in\binom{k}{2}} \chi(\{x\})$

<u>Lemma</u>: If X is a space w/ $X_1, \ldots, X_n \overset{\text{open}}{\subset} X$, $X = \bigcup X_i$, $X_0 = \bigcap X_i$.

Defining $\chi(U) := \bigcup_{i\in U} X_i$, then χ is a homotopy pushout.

So χ is a homotopy pushout, so $\Sigma^\infty\chi$ is as well

$\therefore \Sigma^\infty \chi$ is a homotopy pullback.

So $\Sigma^\infty \chi(\phi) \xrightarrow{\sim} \underset{S^1}{\mathrm{holim}}\, \Sigma^\infty \chi(U) \xrightarrow{\sim} \underset{P_k^\circ}{\mathrm{holim}}\, \Sigma^\infty V \setminus V^{c(\Lambda)}$. ∎

$$T_n \Sigma^\infty C(k,V) = T_n \operatorname{holim}_{\Lambda \in P_k^\circ} \Sigma^\infty V^k \setminus V^{c(\Lambda)}$$

$$= \operatorname{holim}_{\Lambda \in P_k^\circ} T_n \Sigma^\infty V^k \setminus V^{c(\Lambda)}$$

$$V^k \cong V^{e(\Lambda)} \oplus V^{c(\Lambda)}, \quad V^k \setminus V^{c(\Lambda)} \cong V^{e(\Lambda)} \setminus \{0\}$$

$$e(\Lambda) = k - c(\Lambda) \qquad \simeq S^{e(\Lambda)\dim V - 1}$$

$$\Rightarrow \Sigma^\infty V^k \setminus V^{c(\Lambda)} \simeq \Sigma^{-1} \Sigma^\infty S^{e(\Lambda)\dim V}$$

$$\Rightarrow T_k \Sigma^\infty V^k \setminus V^{c(\Lambda)} = \begin{cases} * & \text{, if } e(\Lambda) > n \\ \Sigma^{-1}\Sigma^\infty S^{e(\Lambda)\dim V} & , e(\Lambda) < n \end{cases}$$

$$\Rightarrow T_k \Sigma^\infty C(k,V) = \operatorname{holim}_{\substack{\Lambda \in P_k^\circ \\ e(\Lambda) \leq n}} \Sigma^{-1} \Sigma^\infty S^{e(\Lambda)\dim V}$$

$$\Rightarrow D_k \Sigma^\infty C(k,V) = \operatorname{holim}_{\substack{\Lambda \in P_k^\circ \\ e(\Lambda) \leq n}} F(\Lambda) = \begin{cases} * & e(\lambda) < n \\ \Sigma^{-1}\Sigma^\infty S^{e(\lambda)\dim V} & e(\lambda) = n \end{cases}$$

XI. Introduction to the Embedding Calculus – Dan Berwick-Evans

Goal: Understand $\underbrace{\ }_{\text{GOOD}}$ $F: \mathcal{O}(M)^{op} \longrightarrow \mathrm{Top}$

Examples: ① $\mathrm{Emb}(-,N)$, $m \le n$

② $\mathrm{Imm}(-,N)$

Category of "good" functors $= \mathcal{F}$

Thm $T_1 \mathrm{Emb}(-,N) \simeq \mathrm{Imm}(-,N)$ (if $n=m$)

Thm The following are analytic

① $\mathrm{Emb}(-,N)$ $n-m \ge 3$

② $\mathrm{Imm}(-,N)$ (if $n=m$ need that M has no compact component)

Remarks: - philosophical view: Here we consider more like "interpolation functors"

- We can understand $T_k \mathrm{Emb}(-,N)$ as sheafifications.

(homotopy) (J_k-) Sheaves: presheaf $F: \mathcal{O}(M)^{op} \longrightarrow \mathcal{C}$

... is a (J_k-) sheaf if for a (J_k-) covering $\{U_i \to U\}$

$$F(U) \xrightarrow{\sim} \mathrm{holim}\left(\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_{ij}) \mathrel{\substack{\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow\\[-0.6ex]\textstyle\rightarrow}} \cdots \right)$$

Def'n: $\{U_i \to U\}_{i\in I}$ is a J_k-covering if $\forall S \subset M$, $|S| \le k$, $\exists i: S \subset U_i$

($\rightsquigarrow$ this defines a Grothendieck topology ...)

We can sheafify w.r.t. J_k-coverings (G J_k-sheaf, F any presheaf $\Rightarrow \exists (F)_k$ J_k-sheaf)

$$\begin{tikzcd} F \arrow[r] \arrow[d] & G \\ (F)_k \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

Here,

$$(F)_k(U) = \underset{\mathcal{U} \in J_k}{\mathrm{hocolim}}\ \mathrm{holim}\left(\prod_i F(U_i) \rightrightarrows \cdots\right)$$

Moreover, $\mathrm{Sh}_{J_k}((F_k), G) \simeq \mathrm{Pre}(F, u(G))$

<u>Polynomial Functors:</u>

Let $A_0, \dots, A_k \underset{\text{closed}}{\in} U$, $A_i \cap A_j = \emptyset$, $i \neq j$

$$\chi : \begin{cases} P_{k+1} \longrightarrow \mathrm{Top} \\ S \longmapsto (U \setminus \bigcup_{i \in S} A_i) \end{cases}$$

<u>Remark:</u> This is strongly cocartesian.

<u>Def'n:</u> F is <u>polynomial of degree k</u> if these cubes go to cartesian ones under F.

<u>Thm:</u> F is a sheaf w.r.t. $\mathcal{J}_k$-coverings iff F is polynomial of deg $\leq k$.

<u>Corollary:</u> $T_k F \simeq (F)_k$

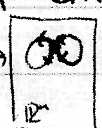
kth polyn. approx. sheafification w.r.t. $\mathcal{J}_k$

<u>Idea of pf:</u> $T_k F(M) \simeq \underset{U \in \mathcal{O}_k(M)}{\mathrm{holim}} F(U)$, where $\mathcal{O}_k(M) := \{ \coprod_j \mathbb{R}^m \xrightarrow[\text{emb}]{} M \mid j \leq k \}$

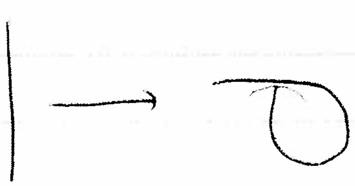
$\cong (F)_k(M)$

$\mathcal{J}_k$-sheaves are determined by values on $\mathcal{O}_k(M)$

<u>Some pictures:</u> Looking at Emb$(-, N)$

① $M = S^1$, $N = \mathbb{R}^n$  $\longrightarrow$  $\in T_1 \mathrm{Emb}(M,N)$, $\notin T_2 \mathrm{Emb}(M,N)$

② $M = \mathbb{R}$, $N = \mathbb{R}^n$



$\in T_k \mathrm{Emb}(M,N)$

$\notin T_\infty \mathrm{Emb}(M,N) \simeq \mathrm{Emb}(M,N)$

(Maybe this is not tame & should be excluded...?)

Layers of the Taylor tower:

Choose a point in $F(M)$ (for Emb, choose $M \subset N$)

$$L_k F(U) := \mathrm{hofib}(T_k F(U) \longrightarrow T_{k-1} F(U))$$

Example: $L_k \mathrm{Emb}(M,N) = \Gamma_0 \left(\begin{array}{c} E_k \\ \downarrow \pi_k \\ \binom{M}{k} \end{array} \right)$

section vanishes near fat diagonal

conf of k pts in M $= C(k,M)$

For $S \in \binom{M}{k}$, $\pi_k^{-1}(S) = \mathrm{thofib}(\chi)$, where $\chi : P_{|S|} \longrightarrow \mathrm{Top}_*$

$$R \longmapsto \mathrm{emb}(R, N), \quad R \subset S$$

fibers of bundle, bundle structure more involved.

Convergence: $\mathrm{Emb}(M,N) \longrightarrow T_k \mathrm{Emb}(M,N)$ is $(3-n+k(n-m-2))$-connected.

So, for $n-m \geq 3$, $\mathrm{Emb}(M,N) \xrightarrow{\sim} T_\infty \mathrm{Emb}(M,N)$

XII. Multiple Disjunction Lemmas - Greg Arone

$X_\bullet$ ... a k-dim'l cubical diagram

Thm: If $X_\bullet$ is strongly cocartesian and the maps $X_0 \to X_i$ are k_i-connected for $i=1,\dots,k$, then $X_\bullet$ is $1-k+\sum_{i=1}^{k} k_i = 1+\sum(k_i-1)$ - cartesian.

Consequence: If X is d-connected, then the map $X \to P_n(\mathrm{Id})(x)$ is $(n+1)d+1$ - connected.

Let $L_1,\dots,L_k, M$ be manifolds, let $L_\bullet$ be the k-dim'l cube

$$S \longmapsto \coprod_{i \notin S} L_i$$

Consider the cubical diagram $\mathrm{Emb}(L_\bullet, N)$.

n ... dim N, ℓ_i ... handle index of L_i

Hard multiple disjunction Lemma

Thm: The cube $\mathrm{Emb}(L_\bullet, N)$ is $3-n+\sum_{i=1}^{n}(n-\ell_i-2)$ - cartesian.

Corollary: The map $\mathrm{Emb}(M,N) \longrightarrow T_k\mathrm{Emb}(M,N)$ is $3-n+(k+1)(n-m-2)$ - connected.

"Easy" multiple disjunction: $\mathrm{Emb}(L_\bullet, N)$ is $3-n+\sum_{i=1}^{n}(n-2\ell_i-2)$ - cartesian

Remark: Greg thinks that easy disjunction holds with "cartesian" replaced with "cocartesian".
Need this for convergence of $\Sigma^\infty \mathrm{Emb}(M,N)$.

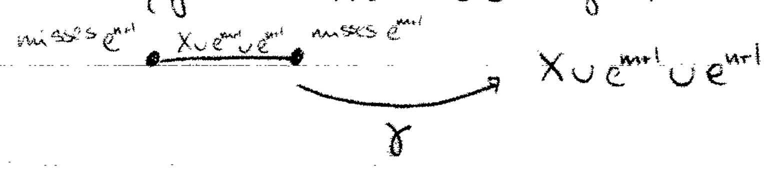
Blakers-Massey thm - proof:

$$\begin{tikzcd}
X \arrow[rr, hook, "m\text{-conn.}"] \arrow[dd, hook, "n\text{-conn.}"'] \arrow[dr] & & X \cup e^{m+1} \arrow[dd, hook] \\
& P \arrow[ur] \arrow[dl] & \\
X \cup e^{n+1} \arrow[rr, hook] & & X \cup e^{m+1} \cup e^{n+1}
\end{tikzcd}$$

$(m+n-1)$ - cartesian

(P,X) is $(m+n-1)$-connected, in other words $\pi_i(P,X)=0$, $i \le m+n-1$.

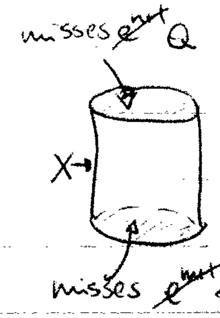
$$P = \{\gamma : I \longrightarrow X\cup e^{m+1} \cup e^{n+1} \mid \gamma(0) \in e^{m+1}, \gamma(1) \in e^{n+1}\}$$



What represents an element in $\pi_i(P,X)$?

... a map $\eta : D^i \times I \longrightarrow X \cup e^{m+1} \cup e^{n+1}$

$$\partial D^i \times I \longrightarrow X$$
$$D^i \times \{0\} \longrightarrow X \cup e^{m+1}$$
$$D^i \times \{1\} \longrightarrow X \cup e^{n+1}$$

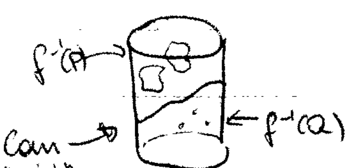


$$\xrightarrow{\ \eta\ } X \cup \underset{\overset{\Psi}{P}}{e^{m+1}} \cup \underset{\overset{\Psi}{Q}}{e^{n+1}}$$

wlog, we can assume that $P \in e^{m+1}$, $Q \in e^{n+1}$ s.t. η is smooth on η^{-1}(small neighborhood of P, Q).

$\eta^{-1}(P)$ is $(i-m)$-dim'l, $\eta^{-1}(Q)$ is $(i-n)$-dim'l

$\Rightarrow \dim(\eta^{-1}(P)) + \dim(\eta^{-1}(Q)) = 2i - m - n \leq i-1$



Can "split" by a fct, move away from forbidden boundary..!

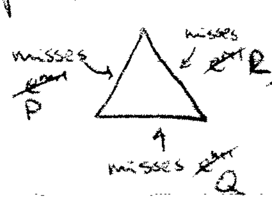
<u>$k=3$</u>:

$$\begin{tikzcd}
 & X\cup e^{n+1} \arrow[rr] \arrow[dd] & & X\cup e^{n+1}\cup e^{m+1} \arrow[dd] \\
X \arrow[ur, "n"] \arrow[rr, "m"] \arrow[dd, "r"] & & X\cup e^{m+1} \arrow[ur] \arrow[dd] & \\
 & X\cup e\cup e \arrow[rr] & & X\cup e^{n+1}\cup e^{m+1}\cup e^{r+1} \\
X\cup e^{r+1} \arrow[ur] \arrow[rr] & & X\cup e\cup e \arrow[ur] &
\end{tikzcd}$$

$(m+n+r-2)$-cartesian

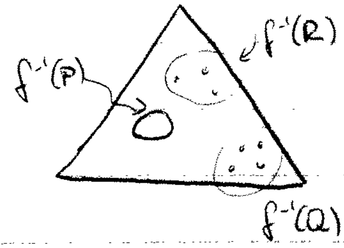
In particular, it is 0-cartesian if $m+n+r \geq 2$.

A point in the homotopy pullback?

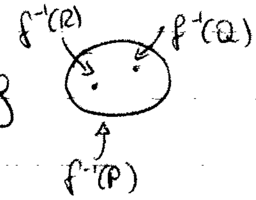


$$\xrightarrow{\ \eta\ } X \cup \underset{\overset{\Psi}{P}}{e^{m+1}} \cup \underset{\overset{\Psi}{Q}}{e^{n+1}} \cup \underset{\overset{\Psi}{R}}{e^{r+1}}$$

Again, smoothness assumption $\Longrightarrow$
$$\begin{aligned} \eta^{-1}(P) &= (1-m)\text{-dim'l} \\ \eta^{-1}(Q) &= (1-n)\text{-dim'l} \\ \eta^{-1}(R) &= (1-r)\text{-dim'l} \\ \hline \textstyle\sum \text{dim's} &= 3-(m+n+r) \leq 1 \end{aligned}$$



How to move out e.g



....? Works..!

Multiple disjunctions

$$\begin{array}{ccccc} \mathrm{Emb}(L_2, N-L_1) & \longrightarrow & \mathrm{Emb}(L_1 \sqcup L_2, N) & \longrightarrow & \mathrm{Emb}(L_1, N) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Emb}(L_2, N) & \longrightarrow & \mathrm{Emb}(L_2, N) & \longrightarrow & * \end{array}$$

Equivalent statement to "easy" disj.:
Let $M, L_1, \ldots, L_n$ be manifolds, $L_i \subseteq N$ disjoint.
Then the cube $\mathrm{Emb}(M, N \setminus L_\bullet)$ is $\left(1+\sum (n-m-\ell_i-2)\right)$-cartesian.

Pf: Claim/Fact 1: The cube $\mathrm{Emb}(M, N\setminus L_\bullet)$ is ~~strongly cocartesian~~ "cocartesian enough"...

$$\begin{array}{ccc} \mathrm{Emb}(M, N\setminus(L_1\cup L_2)) & \xrightarrow{(*)} & \mathrm{Emb}(M, N\setminus L_1) \\ {\scriptstyle (**)}\downarrow & & \downarrow \\ \mathrm{Emb}(M, N\setminus L_2) & \longrightarrow & \mathrm{Emb}(M, N) \end{array}$$

Claim 2: $(*)$ is $(n-m-l_2-1)$-connected, and $(**)$ is $(n-m-\ell_1-1)$-conn.
+ easy exercise in B-M-thm.

Strong disjunction

Tom Goodwillie's thesis: True except on π_0.
Klein-Goodwillie: proof for π_0 using surgery, Poincaré embeddings

<u>analogy</u>

$\mathrm{Emb}(M,N) \longleftrightarrow \mathrm{Map}(K,X)$ ← linear ✓

$\Sigma^\infty \mathrm{Emb}(M,N) \longleftrightarrow \Sigma^\infty \mathrm{Map}(K,X)$

↖ conv. for $\dim K \leq \mathrm{conn}(X)$

$$\begin{tikzcd} \mathrm{Emb}(M,N) \arrow[r] \arrow[dr] & T_{k+1}\mathrm{Emb}(M,N) \arrow[d] \\ & T_k \mathrm{Emb}(M,N) \end{tikzcd}$$

← can check this connectivity

<u>Mult. disj.</u> $\mathrm{Emb}(\hat{\coprod}_{i=1} D^m_i, N) \longrightarrow$ --

↓

--

--

XIII. Formality, the little discs operad and embedding spaces — Sander Kupers

Want to compute $H_*(\mathrm{Emb}(\mathbb{RP}^{2n}, \mathbb{R}^n);\mathbb{Q})$ as a first application.

embedding calculus: $F: \mathcal{O}(M)^{op} \longrightarrow \mathrm{Top}$ good isotopy functor

Recall: $T_k F(M) = (F)_k(M) = \underset{U\in\mathcal{O}_k(M)}{\mathrm{holim}}\ F(U)$

← poset of opens in M homeomorphic to a disjoint union of $\leq k$ balls.

Def'n: little discs operad $\quad \mathbb{B}_n(k) = s\mathrm{Emb}(\coprod_{i=1}^{k} D^n, D^n)$

← translation + dilation on each D^n

Outline:

(1) If M^m is an open s/mfld of $\mathbb{R}^m$, F "context-free", then find general expression for $T_k F(M^m)$ in terms of module maps over $\mathbb{B}_m$

(2) $T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}$, where $\overline{\mathrm{Emb}} = \mathrm{hofib}(\mathrm{Emb} \longrightarrow \mathrm{Imm})$

(1) M open s/mfld of $\mathbb{R}^m$

Def'n A standard ball in M is a ball in $\mathbb{R}^m$ contained in M.

$$\mathcal{O}_k^s(M) \hookrightarrow \mathcal{O}_k(M)$$

Thm: This inclusion induces a homotopy equivalence

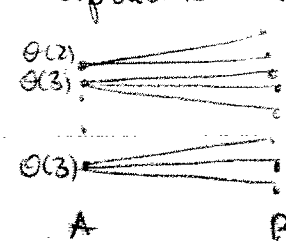
$$\underset{U\in\mathcal{O}_k(M)}{\mathrm{holim}}\ F(U) \xrightarrow{\ \sim\ } \underset{U\in\mathcal{O}_k^s(M)}{\mathrm{holim}}\ F(U)$$

Some operad theory

$\mathcal{O}$ operad $\rightsquigarrow F(\mathcal{O})$ "$\mathcal{O}$-labelled forests"

Def'n: $F(\mathcal{O})$ has objects finite sets A

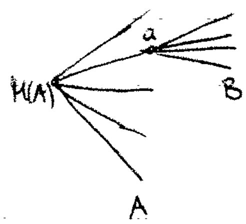
morphisms $\hom_{F(\mathcal{O})}(A,B) = \coprod_{f: B\to A} \bigotimes_{a\in A} \mathcal{O}(f^{-1}(a))$



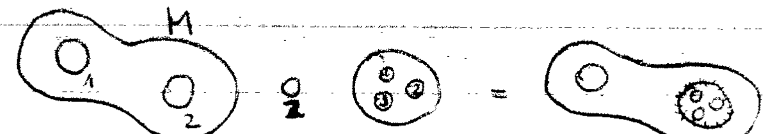
Rem: remembers (a lot? of) structure/info. of the operad...!

Def'n A (weak) right module over $\mathcal{O}$ is a symmetric sequence M with composition maps

$$- \circ_a - : M(A) \otimes \mathcal{O}(B) \longrightarrow M(A \cup_a B)$$



Examples: - Every operad is a right module over itself
- $M(A) = s\mathrm{Emb}(A \times D^n, M)$ module over $\mathbb{B}_n$



Lemma: There is an equivalence of categories

$$\left\{\begin{array}{l} \text{right modules} \\ \text{over } \mathcal{O} \\ + \text{morphisms} \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} \text{contravariant functors} \\ \mathcal{M} : F(\mathcal{O})^{op} \longrightarrow \mathcal{D} \\ + \text{natural transf} \end{array}\right\}$$

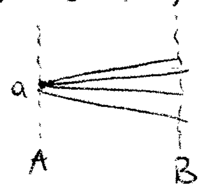
Pf: "$\rightarrow$" $M \rightsquigarrow \mathcal{M}(A) := M(A)$

$$M(B) \otimes \coprod_{f: A \to B} \bigotimes_{b \in B} \mathcal{O}(f^{-1}(b)) \longrightarrow M(A)$$

given by using $- \circ_b -$ repeatedly

"$\leftarrow$" $\mathcal{M} \rightsquigarrow M(A) := \mathcal{M}(A)$

$- \circ_a -$:



Def'n <u>Grothendieck construction</u>

Let $\mathcal{C}$ be any category, $F: \mathcal{C}^{op} \longrightarrow Set$

Then we define $\mathcal{C} \ltimes F$ w/ objects $\coprod_{c \in Ob(\mathcal{C})} F(c)$

morphisms $\coprod_{c,c' \in Ob(\mathcal{C})} Mor_{\mathcal{C}}(c,c') \times F(c')$

<u>Lemma</u>: $F(\mathbb{B}_m^\delta) \ltimes sEmb^\delta(-,M) \xrightarrow{ev^\delta} \mathcal{O}_\infty^s(M)$

is an equivalence of categories, where

$$ev^\delta: \begin{cases} ev^\delta(c,x) \longmapsto im(x) \\ ev^\delta(c,c',\alpha,x) \longmapsto \left(im(x \circ \alpha) \hookrightarrow im(x)\right) \end{cases}$$

Therefore, $T_k F(M) = \underset{U \in \mathcal{O}_k^s(M)}{\operatorname{holim}} F(U) = \underset{F(\mathbb{B}_m^\delta)_{\leq k} \ltimes sEmb^\delta(-,M)}{\operatorname{holim}} F(im(x))$

Def'n $F: \mathcal{O}_\infty^s(M)^{op} \longrightarrow \mathcal{D}$ is called <u>context-free</u>, if

$$\begin{tikzcd} \left(F(\mathbb{B}_m^\delta) \ltimes sEmb^\delta(-,M)\right)^{op} \arrow[r, "ev^\delta"] \arrow[d, "\pi"] & \mathcal{O}_\infty^s(M)^{op} \arrow[r, "F"] & \mathcal{D} \\ F(\mathbb{B}_m^s)^{op} \arrow[urr, dashed, "\exists F'"'] & & \end{tikzcd}$$

<u>Lemma</u>: $\lim_{\mathcal{C} \ltimes F} G \circ \pi \cong Nat_{\mathcal{C}}(F,G)$, where $\mathcal{C}^{op} \underset{G}{\overset{F}{\rightrightarrows}} Set$, $\begin{matrix} \mathcal{C}\ltimes F \\ \downarrow \pi \\ \mathcal{C} \end{matrix}$.

and $\underset{\mathcal{C} \ltimes F}{\operatorname{holim}} G \circ \pi \simeq hNat_{\mathcal{C}}(F,G)$ if F, G land in model category.

So, $T_k F(M) = hNat_{F(\mathbb{B}_m^\delta)_{\leq k}} \left(sEmb^\delta(-,M), F(-)\right)$

$= hRmod_{\leq k}^{\mathbb{B}_m^\delta} \left(sEmb^\delta(-,M), F(-)\right)$

$\boxed{(2)\ H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(-,V)}$

euclidean

$\overline{\mathrm{Emb}}(M,V) := \mathrm{hofib}\left(\mathrm{Emb}(M,V) \longrightarrow \mathrm{Imm}(M,V) \right)$

Example: $M = \coprod D^n_k$, $V = \mathbb{R}^m$

$\mathrm{Imm}(M,V) \cong \prod_k GL_n(\mathbb{R}) \times V^k$

$\mathrm{Emb}(M,V) \simeq \prod_k GL_n(\mathbb{R}) \times C(k,V)$

$\Rightarrow \overline{\mathrm{Emb}}(M,V) \simeq C(k,V)$.

$H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(-,V) : \mathcal{O}(M)^{op} \longrightarrow \mathrm{Spectra}$

Convergence: $\frac{n-1}{2}$ - analytic $\Rightarrow$ Taylor tower converges if $2m+1 < n$

$\left.\begin{array}{l}\text{embedding tower} \\ + \\ \text{Kontsevich formality} \\ 2m+1<n\end{array}\right\} \Longrightarrow$ collapse of orthogonal tower/ reduction to Comm-modules

(2a) $T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq h\mathrm{Rmod}^{\mathrm{comm}}_{\leq k}\left(C_*(M^{(-)}, \mathbb{Q}), H_*(B_n; \mathbb{Q}) \right)$ "blank"

Can conclude that

$H_*\left(\overline{\mathrm{Emb}}(\mathbb{R}P^{2n}, \mathbb{R}^k), \mathbb{Q}\right) = \begin{cases} \mathbb{Q} & ,0 \\ 0 & ,\text{otherwise} \end{cases}$

(2b) Simultaneously expand in V using orthogonal calculus

$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq \prod_{i \geq 0} D_i \underbrace{T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V)}_{\text{Greg calculated that this only depends on } H\mathbb{Q} \wedge M.}$

orthogonal tower

XIV. Factorization Homology & Manifold Calulus - Hiro Tanaka

Recall: $F : \mathrm{Open}(M)^{op} \longrightarrow \mathcal{C}$, sends isotopic to homotopic

$\Downarrow$

$F : \mathrm{Mfld}_{/M}^{op} \longrightarrow \mathcal{C}$

Def'n: Mfld is a Top-enriched category w/ objects n-mflds (+ assump)
+ morphisms $\mathrm{Hom}(X,Y) = \mathrm{Emb}(X,Y)$

$\mathrm{Mfld}_{/M}$ $\quad X \hookrightarrow M$, $\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & M \end{tikzcd}$

$\Downarrow$

$F : \mathrm{Mfld}^{op} \longrightarrow \mathcal{C}$

Factorization Homology: $\quad A : \mathrm{Mfld}^{o} \longrightarrow \mathcal{C} \quad$ $\mathcal{C}$ tensored over spaces

Def'n: $\mathrm{Disk} \subset \mathrm{Mfld}$ is the full subcategory whose objects are of the form $\coprod_j \mathbb{R}^n$ for $0 \leq j < \infty$.

Def'n: $A : \mathrm{Mfld} \longrightarrow \mathcal{C}$, the (ho-) left Kan extension of $A|_{\mathrm{Disk}_{\leq j}}$ along the inclusion $\mathrm{Disk}_{\leq j} \hookrightarrow \mathrm{Mfld}$ will be denoted $T_j A$ and is the jth polynomial approx. to A.

$$\begin{tikzcd} \mathrm{Mfld} \\ \mathrm{Disk} \arrow[u, hook] \\ \mathrm{Disk}_{\leq j} \arrow[u, hook] \\ \vdots \arrow[u, hook] \\ \emptyset \simeq \mathrm{Disk}_{\leq 0} \arrow[u, hook] \end{tikzcd}$$

$$\mathrm{hocolim}\left(T_0 A \longrightarrow T_1 A \longrightarrow T_2 A \longrightarrow \cdots\right) =: T_\infty A$$

$$\begin{tikzcd} T_0 A \arrow[d] & T_1 A \arrow[dl] & T_2 A \arrow[dll] & \cdots \\ A \end{tikzcd}$$

Remark: This is the same as taking (ho-) left Kan extension of $A|_{\mathrm{Disk}}$ along $\mathrm{Disk} \hookrightarrow \mathrm{Mfld}$.

Def'n analytic = $T_\infty A \xrightarrow{\sim} A$ is weak equivalence.

Def'n:

We say $T_\infty A(M)$ is the factorization homology of M with coefficients in A, denoted by $\int_M A$.

Example 1: $A: \mathrm{Mfld}^{fr} \longrightarrow \mathrm{Spaces}$ (framed mflds)

$$M \longmapsto M$$
$$(f: M \to N) \longmapsto (f: M \to N) \quad \text{is analytic.}$$

Pf: This is corepresented by $\mathrm{Hom}_{\mathrm{Mfld}^{fr}}(\mathbb{R}^n, -)$

$$T_\infty A(M) = \mathrm{Emb}^{fr}(-, M) \otimes_{\mathrm{Disk}} \mathrm{Emb}^{fr}(\mathbb{R}^n, -)$$

Co Yoneda lemma $\rightarrow \simeq \mathrm{Emb}^{fr}(\mathbb{R}^n, M)$

$$\simeq M$$

Note that same works for $T_1 A(M) \simeq M$, so A is linear.

" $T_j A(M) \simeq C(M, j)$

Example 2: Let $U = \mathbb{R}^n \setminus \{0\}$. Then $\mathrm{Emb}^{fr}(U, -)$ is NOT analytic.

~~Pf: $\mathrm{Emb}^{fr}(U, U) \neq \mathrm{Emb}^{fr}(\mathbb{R}^n, U)$~~

Let's restrict to symm. mon. $\mathcal{C}$ and

$A: (\mathrm{Mfld}, \amalg) \longrightarrow (\mathcal{C}, \otimes)$ symm. mon. functor

e.g.:
- $(\mathrm{Spaces}, \times)$
- $(\mathrm{Chain}_k, \oplus)$
- $(\mathrm{Chain}_k, \otimes)$
- $(\mathrm{Spectra}, \wedge)$

Observation: $A|_{\mathrm{Disk}}$ ~~(defines a)~~ is the same as E_n^{fr}-algebra

and $A|_{\mathrm{Disk}^{fr}}$ " E_n-algebra

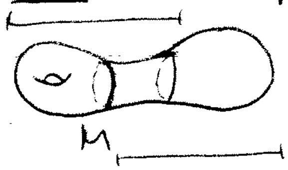
Ex: $n=2$: $\mathbb{R}^n \sqcup \mathbb{R}^n \longrightarrow \mathbb{R}^n$

$\rightsquigarrow A(\mathbb{R}^n) \otimes A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$

$n=1$: $A|_{\mathrm{Disk}^{fr}}$ defines an A_∞-algebra.

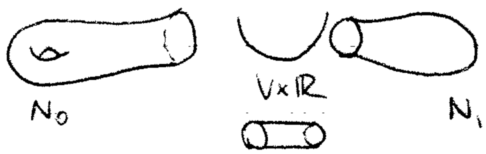
(Francis, Ayala-Francis-T.)

Thm (excision) Let $A : (\mathrm{Mflds}, \sqcup) \to (\mathcal{C}, \otimes)$ be symm. monoidal.



Given $M = N_0 \cup_{V\times\mathbb{R}} N_1$, then

$$\int_M A \simeq \int_{N_0} A \overset{L}{\underset{\int_{V\times\mathbb{R}} A}{\otimes}} \int_{N_1} A$$



Note that this is a bar construction.

Example $M = S^1$, $n=1$ $\quad S^1 = \subset \cup \supset$

"horseshoe" "cohorseshoe" *)

$$\int_{S^1} A = \int_{\mathbb{R}} A \underset{\int_{\mathbb{R}\sqcup\mathbb{R}} A}{\otimes} \int_{\mathbb{R}} A \quad \leftarrow \text{Hochschild homology}$$

$$= \int_{\mathbb{R}} A \otimes \left(\int_{\mathbb{R}} A\right)^{op}$$

justification for factorization hom. ⇝ "higher Hochschild homology".

Def'n: A homology theory for n-mflds is a functor
$$H : \mathrm{Mfld} \longrightarrow (\mathcal{C}, \otimes)$$
s.t. (1) continuous functor
(2) $\sqcup \longrightarrow \otimes$
(3) H satisfies excision

Thm: (Francis) $\mathbb{E}_n^{fr}\text{-alg}(\mathcal{C}) \xrightarrow{\sim} \left\{\begin{array}{l}\text{homology theories}\\ \text{for } n\text{-mflds}\end{array}\right\}$

$\mathbb{E}_n\text{-alg}(\mathcal{C}) \xrightarrow{\sim} \left\{\begin{array}{l}\text{homology theories}\\ \text{for framed } n\text{-mflds}\end{array}\right\}$

Remark: This gives a classification of analytic functors
analytic $\hat{=}$ excision!

*) "What is a cohorse ? " A: "Something that maunts you."

Manifold Calculus

$$F : \mathrm{Mfld}^{op} \longrightarrow \mathcal{C}$$

$$\begin{array}{c} \mathrm{Mfld}^{op} \\ \uparrow \\ \mathrm{Disk}^{op} \\ \vdots \\ \uparrow \\ \mathrm{Disk}^{op}_{\leq j} \\ \uparrow \\ \vdots \end{array}$$

Thm/Def'n: $T_j F :=$ right Kan extension of $F|_{\mathrm{Disk}_{\leq j}}$ along $\mathrm{Disk}_{\leq j} \hookrightarrow \mathrm{Mfld}$

--- j-th polynomial approximation to F.

Thm: $T_\infty F(M) \simeq \mathrm{Hom}_{R\text{-}\mathrm{Mod}_{\mathrm{Disk}}}(\mathrm{Emb}(-,M), F)$

Conj./Thm: $\underbrace{}_{\text{Assume } F \text{ symm. monoidal.}}$ Given $M = N_0 \bigcup_{V \times \mathbb{R}} N_1$, then

"excision"
co-

$$T_\infty F(M) = \mathrm{cobar}\left(T_\infty F(N_0), T_\infty F(V \times \mathbb{R}), T_\infty F(N_1)\right)$$

Def'n: "homology co-theory": $H : \mathrm{Mfld}^{op} \longrightarrow (\mathcal{C}, \otimes)$

s.t. (1) cont.

(2) $\sqcup \longrightarrow \otimes$

(3) (co-) excision

Conj/Thm $\quad \mathbb{E}_n^{fr}\text{-coalg}(\mathcal{C}) \simeq \left\{ \begin{array}{l} \text{homology cotheories} \\ \text{for } n\text{-mflds} \end{array} \right\}$

$\mathbb{E}_n\text{-coalg}(\mathcal{C}) \simeq \left\{ \qquad \text{framed } n\text{-mflds} \right\}$

ANALOGY

Factorization homology

$A : \mathrm{Mfld} \longrightarrow \mathcal{C}$

Left Kan ext.

$T_\infty(A)(M) = \mathrm{Tor}_{\mathrm{Disk}}(\mathrm{Emb}(-,M), A)$

Manifold calculus

$F : \mathrm{Mfld}^{op} \longrightarrow \mathcal{C}$

Right Kan ext.

$T_\infty(F)(M) = \mathrm{Ext}_{\mathrm{Disk\text{-}Mod}}(\mathrm{Emb}(-,M), F)$

XV. Applications to algebraic K-theory I. – Pedro Brito

$A(X)$ – Waldhausen A-theory

$A:$ Spaces $\longrightarrow$ Spectra

$A(X) = K(\underbrace{\mathbb{S}[\Omega X]}_{= \mathbb{S}\wedge(\Omega X)_+})$

Waldhausen: (i) $A(X) \simeq \Sigma^\infty_+ X \ltimes Wh^{Diff}(X)$

(ii) $\Omega^2\Omega^\infty Wh^{Diff}(X) \simeq \mathcal{C}(X)$ "stable concordance/pseudo-isotropy"

Main idea: $A(X) \xrightarrow{\text{trace map}} \mathcal{L}(X) := \Sigma^\infty_+ Map(S^1, X)$

1) derivatives of the free loop space

$F(K) = \Sigma^\infty Map(K, X)$

For $F: Top^{op} \longrightarrow$ Spectra,

$\rightsquigarrow T_k F(K) =$ homotopy sheafification of F wrt J_k

$= \mathbb{R}Hom_{\Gamma_k}(Map(-,K), F)$

$\overset{\text{finite}}{\underset{\text{complex}}{\simeq}} Hom_{\Gamma_k}(Map(-,K), F) = ⊛$

where Γ_k is the category of finite sets of cardinality $\leq k$

$\Omega_k \hookrightarrow \Gamma_k$, $F = \Sigma^\infty Map(-, X)$

finite sets + surjections $\Rightarrow ⊛ \simeq Hom_{\Omega_k}(K^{\wedge -}, \Sigma^\infty X^{\wedge -})$

Consider ~~this~~ F as a functor in X, then ~~this~~ is $P_k F$.

(in the sense of Goodwillie)

relative case.

$x \in X$, $\partial_x \mathcal{L}(X) =$ coefficient of the linearization

$$X \vee_x S^0$$

$$\begin{array}{c} Z \longmapsto \text{hofiber}(F(Z) \longrightarrow F(X)) \\ \downarrow \\ X \end{array}$$

$\partial_x \mathcal{L}(X) \simeq \Gamma(\Sigma^\infty_+ W \longrightarrow S^1)$, where

$W := \{(f,k) \in X^{S^1} \times S^1 : f(k) = x\}$

fibered spectrum over S^1.

Goodwillie's notation for $\Gamma(\ldots)$ is

$$\int^{k \in X} \Sigma^\infty_+ \mathrm{Map}((S^1,k),(X,x))\, dk$$

$$\Gamma(\Sigma^\infty W \longrightarrow S^1) \simeq \mathrm{Map}_*(S^1_+, \Sigma^\infty \Omega_x X)$$

2) <u>A-theory:</u>

$$\mathcal{C}(X) := \operatorname*{hocolim}_k \underbrace{C(X \times I^k)}$$

$C(Z) := \mathrm{Diff}(Z \times I\ ?$, relative to $Z \times \{0\} \cup \partial Z \times I)$

... concordance space of Z

(<u>Ex:</u> Z = disc $\rightsquigarrow C(Z)$ = diff of cylinder fixing the open can.)

$C(Z) \longrightarrow C(Z \times I) \rightarrow \cdots$

<u>Igusa:</u> is $(\dim Z/3)$-connected.

Want to take derivatives of this (homotopy calculus setting)

$\mathcal{C}(-)$ is a homotopy functor (at least on compact mflds)

$\partial_x \mathcal{C}(X)$ = coefficient of the linearization of

$$Z \longrightarrow \mathrm{hofiber}(\mathcal{C}(Z) \longrightarrow \mathcal{C}(X))$$

$$= \operatorname*{hocolim}_n \Omega^n\, \mathrm{hofiber}(\mathcal{C}(X \vee_x S^n) \longrightarrow \mathcal{C}(X))$$

It suffices to look at $\mathrm{hofiber}(C(X \vee_x S^n) \rightarrow C(X))$

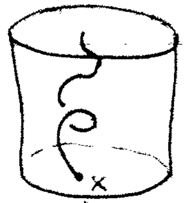
unstable concordance space

Let X' be the manifold modelling $X \vee_x S^n \simeq X'$

$$\begin{matrix} X' \\ \cong \\ C(X \vee_x S^n) \end{matrix} \longrightarrow C(X) \longrightarrow CE_x(I),$$

where $CE_x(I) := \mathrm{Emb}(* \times I, X \times I$; rel $\sqcup$ and $* \times \{0\} \mapsto x \times 1)$

... concordance embeddings



$\in CE_x(I)$

fixed

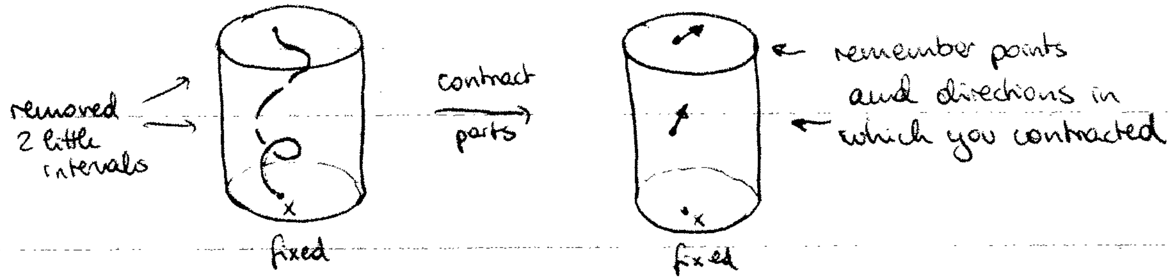
$$CE_x(I) \longrightarrow T_2\, CE_x(I)$$

↑ by Blakers-Massey $2n$-connected where $n = \dim X$

$$\begin{tikzcd} & T_2\, CE_x(I) \arrow[d] & L_2\, CE_x(I) \arrow[l] \\ CE_x(I) \arrow[ur] \arrow[r] & T_1\, CE_x(I) & \end{tikzcd}$$

$\underbrace{L_2\, CE_x(I)} \hookrightarrow \Gamma(E \longrightarrow I^{(2)})$, where E = total fiber of

$$\begin{tikzcd} CE(\text{E}--\exists) \arrow[r] \arrow[d] & CE(\text{E}-\exists) \arrow[d] \\ CE(\text{E}-\exists) \arrow[r] & X \end{tikzcd}$$



$$CE(2) := CE(\text{E}--\exists) \simeq X \times \big(\underbrace{(X\times I)^2 \setminus \Delta}_{\text{conf. space of 2 pts in } X\times I.}\big)$$

Compute hofiber of

$\rightsquigarrow$ diagram
$$\begin{tikzcd} (X\times I)^2 \setminus \Delta \arrow[r] \arrow[d] & X\times I \arrow[d] \\ X\times I \arrow[r] & * \end{tikzcd}$$

$$E \simeq \text{hofiber}\big(\underbrace{(X\times I)\setminus pt}_{\simeq X\vee S^n} \longrightarrow \underbrace{X\times I}_{\simeq X}\big) \simeq \Sigma^n \Omega_x X$$

$$(\dots) \Rightarrow T_2\, CE_x(I) \simeq \Omega^2 \Sigma^n \Omega_x X$$

$$CE_x(I) \longrightarrow \Omega^2 \Sigma^n \Omega_x X$$

$$\Rightarrow \partial_x \mathcal{C}(X) \simeq \Omega^2 \Sigma^\infty_+ \Omega_x X$$
$$\Rightarrow \partial_x A(X) \simeq \Sigma^\infty_+ \Omega_x X.$$

Recap: $A(X) \longrightarrow \mathcal{L}(X)$

We saw that $\partial_x \mathcal{L}(X) = \mathrm{Map}_*(S^1, \Sigma^\infty_+ \Omega_x X)$
$$\Rightarrow (\partial_x \mathcal{L}(X))^{hS^1} \simeq \Sigma^\infty_+ \Omega_x X$$

$$\Rightarrow \partial_x A(X) \xrightarrow{\simeq} (\partial_x \mathcal{L}(X))^{hS^1}$$

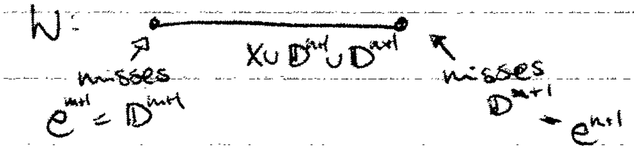
Thm (Goodwillie) $F \to G$ homotopy functors + ρ-analytic and reduced + $\partial_x F(X) \xrightarrow{\simeq} \partial_x G(X)$ $\forall (X,x)$, $x \in X$, (X ρ-connected?)
$$\Rightarrow F(X) \xrightarrow{\simeq} G(X) \quad \forall\ \rho\text{-connected } X.$$

evening discussion

Blakers-Massey pf revisited

$$\begin{tikzcd}
U\setminus\{P,Q\} \simeq X \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "N"] & X \cup \overset{P \atop \cap}{D^{n+1}} \simeq U \setminus P \arrow[d, hook] \\
U\setminus\{Q\} \simeq X \cup \underset{\Psi \atop Q}{D^{m+1}} \arrow[r, hook] & X \cup D^{m+1} \cup D^{n+1} =: U
\end{tikzcd}$$

Want: $\pi_i(W,X) = 0$ for $i \leq n+m-1$



or just P.

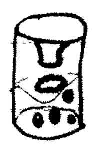
$\pi_i(W,X)$ consists of

$$D^i \times I \longrightarrow X \cup e^{n+1} \cup e^{m+1}$$

$$\left|\begin{array}{l} \partial D^i \times I \longrightarrow X \\ D^i \times \{0\} \longrightarrow X \cup e^{m+1} \ (\cong U \setminus \{P\}) \\ D^i \times \{1\} \longrightarrow X \cup e^{n+1} \ (\cong U \setminus \{Q\}) \end{array}\right.$$

Need: a homotopy which is constant on $\partial D^i \times I$, missing P on $D^i \times \{0\}$, Q on $D^i \times \{1\}$ and ends in $U \setminus \{P,Q\}$

$f^{-1}(P)$ —
$f^{-1}(Q)$ —



If there is the graph of a function st the graph separates $f^{-1}(P)$ and $f^{-1}(Q)$, we are done
By "wiggling" this is always possible.

<u>k-coalgebras</u>

$$\mathrm{Top}_* \longrightarrow \mathrm{Spectra}$$

$$[\mathrm{Top}_*^{fin}, \mathrm{Spectra}] \underset{\Phi}{\overset{\partial_*}{\rightleftarrows}} \mathrm{Spectra}^{\Sigma}$$

$K = \partial_* \Phi : Sp^{\Sigma} \longrightarrow Sp^{\Sigma}$ comonad

$$\begin{pmatrix} n\text{-excisive} \\ F : \mathrm{Top}_*^{fin} \to Sp \end{pmatrix} \longleftrightarrow \begin{pmatrix} n\text{-truncated} \\ k\text{-coalgebras} \end{pmatrix}$$

$$F \longmapsto \partial_* F$$

$R_X : \mathrm{Top}_*^{fin} \longrightarrow Sp$, $\quad X \in \mathrm{Top}_*^{fin}$

$R_X(-) := \Sigma^\infty \mathrm{Hom}(X, -)$

$\partial_n(R_X) = \mathbb{D}(X^{\wedge n}/\Delta^n X)$

Spanier-Whitehead dual

Define $\partial_*(F) := \partial_*(R_X) \underset{X \in \mathrm{Top}_*^{fin}}{\wedge} F(X)$

$$\Phi(A)(X) = \mathrm{Map}_{Sp^\Sigma}(\partial_*(R_X), A)$$
$$\cong$$
$$\prod_{n \geq 1} \mathrm{Map}(\partial_n(R_X), A_n)^{h\Sigma_n}$$
$$\simeq \prod_{n\geq 1} (A_n \wedge X^{\wedge n}/\Delta^n X)^{h\Sigma_n} \xleftarrow[\sim]{N} \prod_{n \geq 1} (A_n \wedge X^{\wedge n}/\Delta^n X)_{h\Sigma_n}$$

(Claim.)

$K = \partial_* \Phi : Sp^\Sigma \longrightarrow Sp^\Sigma$

If A is N truncated, then

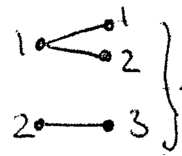
$$K(A) = \partial_* \prod_{n=1}^{N} (A_n \wedge X^{\wedge n}/\Delta^n X)_{h\Sigma_n} \simeq \prod_{n=1}^{N} \left[\partial_*(A_n \wedge X^{\wedge n}/\Delta^n X)\right]_{h\Sigma_n}$$

$X^{\wedge n}/\Delta^n X \simeq B(X^{\wedge *}, \mathrm{Com}, 1)(n)$, where

Com = comm. non-unital operad in Top_+, $\mathrm{Com}(n) = S^0$ for all n

$X^{\wedge *}$ = right Com-module (equiv. $\mathrm{Epi}^{op} \longrightarrow \mathrm{Top}_+$)

$$X^{\wedge k}\wedge S^0 \wedge \cdots \wedge S^0 \xrightarrow{\ \Delta\ } X^{\wedge n} \quad \text{for } \underline{n} \longrightarrow \underline{k}$$

 $\rightsquigarrow \begin{cases} X\wedge X \longrightarrow X\wedge X\wedge X \\ (x,y) \longmapsto (x,x,y) \end{cases}$

Remarks: ① $B(R,P,1) = R \circ_P 1$

② $B(X^{\wedge *}, \mathrm{Com}, 1)$: right comodule over cooperad $B(1,\mathrm{Com},1)$

↖ partition poset complexes

$$\mathbb{D}B(X^{\wedge *}, \mathrm{Com}, 1) \simeq \mathbb{D}(X^{\wedge n}/\Delta^n X) \simeq \partial_* R_X$$

... right module over $\partial_* I$

$$K_r(A) = \prod_{n=1}^{N} \left[\partial_r (A_n \wedge \Sigma^\infty X^{\wedge n}/\Delta^n X)\right]_{h\Sigma_n}$$

$$= \prod_{n=1}^{N} \left[A_n \wedge B(\partial_r(\Sigma^\infty X^{\wedge \bullet}), \mathrm{Com}, 1)(n)\right]_{h\Sigma_n}$$

$$\simeq \prod_{n=1}^{N} \left[A_n \wedge \underbrace{B\left(\boxed{\begin{smallmatrix}\vdots\\ *\\ \Sigma_r\\ *\\ \vdots\\ *\end{smallmatrix}}, \mathrm{Com}, 1\right)(n)}\right]_{h\Sigma_n}$$

Claim: $\simeq \coprod_{\substack{n=n_1+\cdots+n_r \\ \text{unordered} \\ \text{partitions of } n \\ \text{into } r \text{ pieces}}} \underbrace{B(1,\mathrm{Com},1)(n_1)\wedge\cdots\wedge B(1,\mathrm{Com},1)(n_r)}_{[\ \text{———}\ ]}$

$$\Rightarrow K_r(A) = \prod_{n=1}^{N} \Big(\prod_{n=n_1+\cdots+n_r} A_n \wedge [\ \text{———}\ ]\Big)_{h\Sigma_n}$$

$$\simeq \prod_{n=1}^{N} \Big[\prod_{n=n_1+\cdots+n_r} \mathrm{Map}(\partial_{n_1} I \wedge \cdots \wedge \partial_{n_r} I, A_n)\Big]_{h\Sigma_n}$$

Def'n A <u>K-coalgebra structure</u> on A consists of

$$A \longrightarrow KA$$

$$A_r \longrightarrow \Big[\prod_{n=n_1+\cdots+n_r} \mathrm{Map}(\partial_{n_1} I \wedge \cdots \wedge \partial_{n_r} I, A_n)\Big]_{h\Sigma_n}$$

$$\big\downarrow N$$

$$\Big[\qquad\qquad\qquad\qquad\Big]^{h\Sigma_n}$$

This composite gives Σ_n-equivariant maps

$$\underset{\text{triv. } \Sigma_n\text{-action}}{A_r} \longrightarrow \prod_{\substack{n=n_1+\cdots+n_r \\ \updownarrow \\ \underline{n} \twoheadrightarrow \underline{r}}} \mathrm{Map}(\partial_{n_1} I \wedge \cdots \wedge \partial_{n_r} I, A_n)$$

i.e., $A_r \wedge \partial_{n_1} I \wedge \cdots \wedge \partial_{n_r} I \longrightarrow A_n$

i.e., a right $\partial_* I$-module structure.

We refer to a K-coalgebra as a <u>divided power</u> right $\partial_* I$-module.

XVI. Applications to alg. K-theory II – Ernest E. Fontes

Goal: $D_1 K(R;M) \xrightarrow{\sim} THH(R;M)$

K-theory:

Def'n $\mathcal{C}$ Waldhausen category: pointed, equipped w/ cof $\mathcal{C}$, $w\mathcal{C}$ subcategory

(1) $\operatorname{iso}\mathcal{C} \subset \operatorname{cof}\mathcal{C}, w\mathcal{C}$

(2)
$$\begin{tikzcd} \cdot \arrow[r, tail] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, tail] & \cdot \end{tikzcd}$$

(3)
$$\begin{tikzcd} \cdot \arrow[r, tail] \arrow[d] & \cdot & & \cdot \arrow[r, tail] \arrow[d] & \cdot \\ \cdot & \cdot \arrow[rr, bend right, "\eta"'] & & \cdot & \cdot \end{tikzcd} \quad h$$

pushouts along w. equiv are homotopy invariant

(4) $* \rightarrowtail a \quad \forall a \in$

Example: R ring, let $\mathcal{P}_R$ be the cat. of f.g. proj R-modules
$w\mathcal{C}$ = isos
$a \rightarrowtail b \iff$ injection with quotient in $\mathcal{P}_R$

Remark:
$$\begin{tikzcd} a \arrow[r, tail] \arrow[d] & b \arrow[d] \\ * \arrow[r, tail] & c \end{tikzcd}$$
quotient

Def'n: $S_n\mathcal{C}$ = category of $* \rightarrowtail a_{00} \rightarrowtail a_{01} \rightarrowtail \cdots \rightarrowtail a_{0n}$
w/ quotients $\forall\, 0 \le i \le j \le n$ $\quad a_{0i} \rightarrowtail a_{0j} \twoheadrightarrow a_{ij}$, $a_{ii} = *$

$$\begin{tikzcd} * = a_{00} \arrow[r, tail] & a_{01} \arrow[r, tail] \arrow[d] & a_{02} \arrow[r, tail] \arrow[d] & a_{03} \arrow[r, tail] \arrow[d] & \cdots \arrow[r, tail] & a_{0n} \arrow[d] \\ & * = a_{11} \arrow[r, tail] & a_{12} \arrow[r, tail] \arrow[d] & a_{13} \arrow[r, tail] & \cdots \arrow[r, tail] & a_{1n} \arrow[d] \\ & & * = a_{22} \arrow[r] & a_{23} \arrow[r, tail] & \cdots \arrow[r, tail] & a_{2n} \arrow[d] \\ & & & & & a_{n-1\,n} \end{tikzcd}$$

$\boxed{d_i = \text{drop } i\text{th row \& col.}}$
⇝ simplicial object
maps = levelwise

<u>Def'n</u> $\mathcal{C}$ Waldhausen category

$$K(\mathcal{C}) := \Omega |N_\cdot w S_\cdot \mathcal{C}|$$

iterate $S_\cdot$ n-times

<u>Remark</u>: ① $|N_\cdot w S_\cdot^{(n-1)} \mathcal{C}| \xrightarrow{\sim} \Omega |N_\cdot w S_\cdot^{(n)} \mathcal{C}|$

$\Rightarrow K(\mathcal{C})$ is a spectrum

② $|N_\cdot w S_\cdot \mathcal{C}| \xrightarrow{\sim} |N_\cdot w S_\cdot \mathcal{C}|$ if $\mathcal{C}$ is additive

$\Rightarrow K(\mathcal{C}) \simeq \Omega |\mathrm{ob}\, S_\cdot \mathcal{C}|$

③ $|N_\cdot w S_\cdot^{(p)} \mathcal{C}|$ is $(p-1)$-connected

<u>THH $\leftarrow$ Topological Hochschild homology</u>

<u>Idea</u>: For Hochschild homology,

$$[n] \longmapsto \underbrace{R \otimes R \otimes \cdots \otimes R}_{(n+1)\text{-times}}$$

picture as $R \otimes R \otimes R \otimes \cdots$ (in a circle)

$$HH = \{\pi_* |R \otimes R^{\otimes n}|$$

Want to do same for spectra

<u>Def'n</u> Let $\mathcal{C}$ be a spectral category i.e. enriched over spectra

$$N_n^{cyc}(\mathcal{C}) = \bigvee_{c_0, \ldots, c_n} \mathcal{C}(c_0, c_1) \wedge \cdots \wedge \mathcal{C}(c_{n-1}, c_n) \wedge \mathcal{C}(c_n, c_0)$$

Motivationally, $THH(\mathcal{C})$ "=" $|N_\cdot^{cyc} \mathcal{C}|$ (doesn't always work)

<u>Def'n</u>: $THH(R) := THH(\mathcal{P}_R) = \Omega |N_\cdot^{cyc} w S_\cdot \mathcal{P}_R|$

$$\begin{array}{ccc} K(R) = K(\mathcal{P}_R) & \simeq & \Omega |\mathrm{ob}\ S_\cdot \mathcal{P}_R| \\ \overset{\Psi}{X} & & \downarrow \mathrm{tr} \\ \downarrow & & \\ X \circlearrowleft^{id_X} & & \Omega |N_\cdot^{cyc} S_\cdot \mathcal{P}_R| = THH(R) \end{array}$$

Topological cyclic homology – TC

$THH(\mathcal{C})$ as an S^1-spectrum

Fix p

$F, R: THH(\mathcal{C})^{C_{p^n}} \longrightarrow THH(\mathcal{C})^{C_{p^{n-1}}}$ ← cyclic gp of order p^{n-1}

Def'n: $TC(\mathcal{C}, p) := \operatorname{holim} THH(\mathcal{C})^{C_{p^n}}$

Remark: Frobenius b/c π_0 is Witt vectors, there ⇝ Frob

Thm (Dundas–McCarthy)

$R \longrightarrow S$ of simplicial rings, and

On π_0 has nilpotent kernel

Then,

$$\begin{array}{ccc} K(R) & \longrightarrow & TC(R) \\ \downarrow & & \downarrow \\ K(S) & \longrightarrow & TC(S) \end{array}$$

is homotopy cartesian

Rem: $K(\Sigma^\infty \Omega X) = A(X)$

Def'n: Let R be a ring, M an R-bimodule, simplicial

$K(R, M) := K(R \oplus M)$ ↖ square-zero extension

Remark: $M \otimes R$-modules = R-modules P w/ $P \to P \otimes M$

X a finite simplicial set, $M \wedge X := M \otimes X / M \otimes *$

$K(R; M; X) := K(R \oplus (M \wedge X))$, $THH(R; M; X)$ sim.

$K(R; M; X) \xrightarrow{f} K(R) = K(R; 0; *)$

Def'n: $\tilde{K}(R; M; X) := \operatorname{hofiber}(f)$.

Consider $X \longmapsto K(R, M; X)$.

Then, $D_1 K(R; M; X) = \operatorname{hocolim} \Omega^n \tilde{K}(R, M; \Sigma^n X)$.

$K(R; M) = \Omega \left| \coprod_{c \in S^\cdot \mathcal{P}_R} \operatorname{Hom}(c, c \otimes M) \right|$ (Suppressing X...)

$K(R, M) = K(R) \times \tilde{K}(R, M)$

$$THH(R;M) = \bigoplus_{c_0,\dots,c_n \in \mathcal{P}_R} \mathrm{Hom}(c_0,c_1) \otimes \cdots \otimes \mathrm{Hom}(c_{n-1},c_n) \otimes \mathrm{Hom}(c_n, c_0 \otimes m)$$

$$= \Omega \left| \bigoplus_{c \in S_\cdot \mathcal{P}_R} \mathrm{Hom}(c, c\otimes M) \right|$$

<u>Thm</u> If $m = \mathrm{conn}(M)$, tr is $2m$-connected, then for

$$\begin{tikzcd} \tilde{K}(R,M) \arrow[r, "tr"] \arrow[d] & THH(R;M) \arrow[d, "\wr"] \\ D_1 K(R,M) \arrow[r, "\circledast"] & D_1 THH(R;M), \end{tikzcd}$$

$\circledast$ is a weak equivalence.

<u>Pf:</u> $A_p := \left| \coprod_{c \in S^{(p)}_\cdot \mathcal{P}_R} \mathrm{Hom}(c, c\otimes m) \right|$

$$\begin{tikzcd} & C_p \arrow[d, "0"] & C_p \arrow[d, equal] \\ fib_p \arrow[r] \arrow[d, "\alpha"] & A_p \arrow[r, "pr"] \arrow[d] & C_p \arrow[d] \\ cof_p \arrow[r] & cofib_p \arrow[r] & * \end{tikzcd}$$

$B_p := \left| \bigoplus_{c \in S^{(p)}_\cdot \mathcal{P}_R} \mathrm{Hom}(c, c\otimes m) \right|$, $C_p := \left| S^{(p)}_\cdot \mathcal{P}_R \right|$

$$\begin{tikzcd} \tilde{K}(R;M) \arrow[rr, "tr"] \arrow[d, "\wr"] & & THH(R;M) \arrow[d, "\wr"] \\ \Omega^p fib_p \arrow[r, "\Omega^p\alpha"] & \Omega^p cof_p \arrow[r, "\Omega^p\beta"] & \Omega^p B_p \end{tikzcd}$$

$\left(cof_p = \left| \bigvee_{S^{(p)}_\cdot \mathcal{P}_R} \mathrm{Hom}(c, c\otimes m) \right| \right)$

A_p, C_p, cof_p all $(p-1)$-connected, so BM $\Rightarrow$ tfib $\left(\begin{tikzcd} A_p \arrow[r] \arrow[d] & B_p \arrow[d] \\ cof_p \arrow[r] & * \end{tikzcd} \right)$ is $(p-1+p-1-1)$-connected.

$\Rightarrow$ fib$(fib_p \to cof_p)$ is $\underset{2p-3}{\underset{\shortparallel}{(p-1+p-1-1)}}$-connected

$\Rightarrow \Omega^p\alpha$ is $p-3$ connected.

$\beta : cof_p \to B_p$

$$\begin{tikzcd} \bigvee_{fin} \text{ m-conn. spaces} \arrow[d] \\ \prod_{fin} \text{ —"—} \end{tikzcd}$$

is $(2m+1)$-conn. by BM:

$$\begin{tikzcd} * \arrow[r] \arrow[d, "(2m+1)\text{-conn.}"'] \arrow[dr, "X\times Y"] & X \arrow[d] \\ Y \arrow[r] & X \cup Y \end{tikzcd}$$

Let $Z_{\bullet\bullet} = \text{hofib}\big(V(\dots) \longrightarrow \bigoplus(\dots)\big)$

$Z_{q\bullet}$ are all $(2m+1)$-connected

$Z_{q\bullet} \simeq *$ $q < p$

$\rightsquigarrow$ spectral sequence $\rightsquigarrow$ ---

$\Rightarrow \Omega^p \beta$ is $(2m+p-p)$-connected.

This concludes the pf. ∎

XVII. Calculus of functors & chromatic homotopy theory - Tobias Barthel

Intro to chromotopy

Adams: $\Sigma^8 S/2 \xrightarrow{\alpha} S/2$ s.t. $K(\alpha)$ is isom. $\alpha^k \neq 0$

$$S^{8k} \longrightarrow \Sigma^{8k} S/2 \xrightarrow{\alpha^k} S/2 \longrightarrow S^1$$

$$\rightsquigarrow \underset{\neq 0}{\alpha_k} \in \pi_{8k-1} S \quad \forall k$$

Find similar $\beta, \gamma, \ldots$ $\gamma_1 \neq 0$?

We will see a big machine which gives us all at once!

$\boxed{\text{p-local:}}$ (Always work p-locally.) "fields" in the stable ho. cat.

Morava K-theory: $K(n)$ $\forall n \geq 0$ with the following properties

- $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$, $|v_n| = 2(p^n - 1)$
- homotopy comm. ring spectra for $p \neq 2$, complex oriented
- Künneth formula (this essentially characterizes them)

Def'n: X finite spectrum. X has type n if $K(n)_* X \neq 0$, $K(n-1)_* X = 0$

$f: \Sigma^d X \longrightarrow X$, $d > 0$ is a v_n-self map if $K(m)_* f = \begin{cases} \text{isom} & m = n \\ \text{nilpotent} & m \neq n \end{cases}$

Thm (Periodicity thm) Let X be a finite spectrum.
If X has type n, then it admits a v_n-self map $f: \Sigma^d X \to X$.
Moreover, if (X, f, d) and (Y, g, e) are such maps ($X + Y$ of type n), and there is a $\varphi: X \to Y$, then $\exists r, s$ st $dr = es$ and

$$\begin{tikzcd} \Sigma^{dr} X \arrow[r, "\Sigma^{dr}\varphi"] \arrow[d, "f^r"'] \arrow[dr, phantom, "\circlearrowleft"] & \Sigma^{es} Y \arrow[d, "g^s"] \\ X \arrow[r, "\varphi"'] & Y \end{tikzcd}$$

<u>telescopes:</u> If X is finite of type n $\rightsquigarrow$ $\exists$ v_n-self map f of X

$$T(n) = T(X) = T(X,f) = \operatorname{hocolim}\left(X \xrightarrow{f} \Sigma^{-d} X \xrightarrow{\Sigma^{-d} f} \cdots\right)$$

telescopes are Bousfield equivalent

Corollary ("Resolutions", Kuhn)
$\exists$ finite spectra $X(i)$ of type n

$$\begin{tikzcd} X(0) \arrow[r] \arrow[d] & X(1) \arrow[r] \arrow[dl] & \cdots \\ S^0 & \cdots & \end{tikzcd}$$

s.t. $\operatorname{hocolim} X(i) \longrightarrow S^0$ is a $T(m)$-equivalence $\forall m \geq n$.

<u>Localization & Bousfield-Kuhn functors</u>

<u>Def'n</u>: E, X spectra.

- X is <u>E-acyclic</u>, if $X \wedge E \sim *$ $\rightsquigarrow$ category $\mathcal{C}_E$ of E-acyclics
- Y is called <u>E-local</u> if $[X,Y] = 0$ $\forall X \in \mathcal{C}_E$

<u>Thm</u> (Bousfield)
$\forall E$ fixed spectrum, $\exists$ idempotent functor $L_E : Sp \to Sp$ w/ a natural transformation $\eta_E : Id \longrightarrow L_E$ st.

(1) $L_E X$ is E-local $\forall X$

(2) $X \longrightarrow L_E X$ is an E-equivalence

<u>Example</u>: $L_{T(n)}$ --- telescopic localization

$L_{K(n)}$ --- localization at Morava K-theory

$\Rightarrow \mathcal{C}_{T(n)} \subseteq \mathcal{C}_{K(n)} \Rightarrow L_{K(n)} = L_{K(n)} L_{T(n)}$

<u>Telescope conjecture</u>: $\mathcal{C}_{T(n)} = \mathcal{C}_{K(n)}$

(true for $n=1$, false sometimes for $n=2$, unknown otherwise)

<u>Example</u>: $L_{T(1)} X = \operatorname{holim}_n \operatorname{hocolim}\left(X/p^n \xrightarrow{\alpha} \Sigma X/p^n \xrightarrow{\alpha} \cdots\right)$

telesc. conj. for $n=1$ $\rightsquigarrow$ $\|$

$L_{K(1)} X$

<u>Thm</u> (Bousfield, Kuhn)

$\forall n$ $\exists$ functors $\Phi_n : Top_+ \longrightarrow Sp$ s.t.

$$\begin{tikzcd} Sp \arrow[r, "L_{T(n)}"] \arrow[d, "\Omega^\infty"'] \arrow[dr, phantom, "\circlearrowleft"] & Sp \\ Top \arrow[ur, "\Phi_n"'] & \end{tikzcd}$$

($\forall Z \in Top$, $\Phi_n(Z)$ is $T(n)$-local.)

<u>Corollary</u>: $\Sigma^\infty\Omega^\infty X \xrightarrow{\varepsilon} X$ admits a section after $T(n)$-localization.

<u>Pf</u>:
$$\begin{tikzcd}[column sep=small] & \Omega^\infty\Sigma^\infty\Omega^\infty X \arrow[dr, "\Omega^\infty\varepsilon"] & & & & L_{T(n)}\Sigma^\infty\Omega^\infty X \arrow[dr, "L_{T(n)}\varepsilon"] & \\ \Omega^\infty X \arrow[ur, "\eta(\Omega^\infty X)"] \arrow[rr, "id"'] & & \Omega^\infty X & \overset{\Phi_n}{\rightsquigarrow} & L_{T(n)}X \arrow[ur, "\eta_n"] \arrow[rr, "id"'] & & L_{T(n)}X \end{tikzcd}$$

($\eta : Id \to \Omega^\infty\Sigma^\infty$)

<u>Localized Goodwillie tower</u> (Kuhn)

<u>Thm</u> $F : Sp \longrightarrow Sp$. Homotopy calculus gives

$$D_d F(X) \longrightarrow P_d F(X) \longrightarrow P_{d-1} F(X)$$

and this splits $T(n)$-locally.

<u>Corollary</u>: $\operatorname{holim}_d L_{T(n)} P_d F(X) = \prod_{i=0}^{\infty} L_{T(n)} D_i F(X)$

<u>Recap</u>: Tate spectra

Y spectrum with finite G-action $\qquad 1 \quad x \overset{N_G}{\longmapsto} \sum_{g\in G} g\cdot x$

$$Y_{hG} \xrightarrow{N_G} Y^{hG} \xrightarrow{\text{cofiber}} Y^{tG}$$

"Tate spectrum"

Klein: The norm map N_G is uniquely characterized by being an equivalence if Y is finite free.

"dual calculus" for proof

Prop (McCarthy)

$\exists$ homotopy pullback square

$$\begin{tikzcd} D_d F(X) \arrow[r, "\sim"] \arrow[d] & (\Delta_d F(X))_{h\Sigma_d} \arrow[d, "N_{\Sigma_d}"] \\ P_d F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & (\Delta_d F(X))^{h\Sigma_d} \arrow[d] \\ P_{d-1} F(X) \arrow[r] & (\Delta_d F(X))^{t\Sigma_d} \end{tikzcd}$$

Remark: Thus, the following data are equivalent:

$$\{F \text{ } d\text{-excisive functor}\} \longleftrightarrow \left\{ \begin{array}{l} G \text{ } (d-1)\text{-excisive functor} \\ H \text{ } d\text{-homogeneous functor} \\ G \longrightarrow (\delta\, cr_d H)^{t\Sigma_d} \end{array} \right\}$$

$$F \longmapsto \left(\begin{array}{l} G = P_{d-1}F, \; H = \Delta_d F \\ P_{d-1}F(X) \longrightarrow (\Delta_d F(X))^{t\Sigma_d} \end{array} \right.$$

Pf: $\alpha : F(X) \longrightarrow (\Delta_d F(X))^{h\Sigma_d}$

Proof of Thm above:

equiv. after local.

$$\begin{tikzcd} D_d F(X) \arrow[r, "\sim_{L_{T(n)}}"] \arrow[d] & D_d L_{T(n)} F(X) \arrow[r, "\sim"] \arrow[d] & (\Delta_d L_{T(n)} F(X))_{h\Sigma_d} \arrow[d] \\ P_d F(X) \arrow[r] \arrow[d] & P_d L_{T(n)} F(X) \arrow[r] \arrow[d] & (\Delta_d L_{T(n)} F(X))^{h\Sigma_d} \arrow[d] \arrow[u, dashed, bend right] \\ P_{d-1} F(X) \arrow[r] & P_{d-1} L_{T(n)} F(X) \arrow[r] & (\Delta_d L_{T(n)} F(X))^{t\Sigma_d} \end{tikzcd}$$

$\Sigma^{L_{T(n)}}_{*}$

Thm ~~Claim~~: $\sim$ $L_{T(n)}$-locally

$L_{T(n)}(L_{T(n)} S^0)^{t\mathbb{Z}/p} \simeq *$

XVIII. The Taylor tower of the identity, part 2 – Vesna

Study $\partial_* Id$, $Id : Top_* \longrightarrow Top_*$

$$\begin{tikzcd} & P_n X & D_n X \arrow[l] \\ X \arrow[ur] \arrow[dr] & & \\ & P_{n-1} X \arrow[from=uu] & \end{tikzcd}$$

$\downarrow$

$\vdots$

fibers are stable

Call $\mathbb{D}_n X = \Omega^\infty D_n X = \Omega^\infty (\partial_n \wedge X^{\wedge n})_{h\Sigma_n}$, $\partial_n = \partial_n(Id)(S^0)$

$X \xrightarrow{\sim} \operatorname{holim}_n P_n X$

Goodwille spectral sequence: $E_1 = \pi_*^s D_n X \Rightarrow \underbrace{\pi_* X}_{\text{unstable}}$ (stable)

is even more complicated!

How can we compute $E_1(X)$, $E_1(S^k)$?

$H_*(D_n S^k, \mathbb{F}_p) = H_*(\partial_n) \otimes H_*(S^{kn})$

$H_*(\Sigma_n, H_*(D_n S^k, \mathbb{F}_p)) = H_*(\Sigma_n, H_*(\partial_n) \otimes \underbrace{H_*(S^{kn})}_{(\mathbb{F}_p^{\pm})^{\otimes k}})$

$\boxed{\text{Arone-Mahowald, Arone-Dwyer}} \rightarrow$ $\Downarrow$ s.s.

$H_*((\partial_n \wedge S^{kn})_{h\Sigma_n})$ as a comodule over dual Steenrod algebra ← Arone-Mahowald

$S^k \wedge \dots \wedge S^k \circlearrowleft \Sigma_n$ acts, need sign...

input in a spectral sequence which

$\Downarrow$

$E_1(S^k)$

Behrens: EHP & Goodwillie

Chromatic approach $\boxed{p\text{-local}}$

↳ decomposing into frequencies $p, v_1, v_2, v_3, \dots$

Type m complexes know about v_m-periodicity

Want to decompose $\pi_* S^k$ into v_m-periodic parts

Thm: $D_n(S^k) = \begin{cases} * & n \neq p^i \\ D_{p^i}(S^k) & \text{has type } n \end{cases}$ $\Rightarrow$ knows about v_i-periodic homotopy in S^k

k odd

$\mathrm{Id} \xrightarrow{E} \Omega\Sigma \xrightarrow{H} \Omega\Sigma Sq$

$Sq: X \longmapsto X \wedge X$

$\tilde{H}: \Sigma\Omega\Sigma X \xrightarrow{\sim} \Sigma \bigvee_i X^{\wedge i} \longrightarrow \Sigma(X \wedge X)$, H is adjoint to $\tilde{H}$

p=2, 2-locally: $\exists$ fiber sequences

spaces: $P_n S^k \xrightarrow{E} P_n(\Omega\Sigma)(S^k) \xrightarrow{H} P_n(\Omega\Sigma Sq)(S^k)$

$P_n(\Omega\Sigma)(S^k) = \Omega P_n(S^{k+1})$, $\quad P_n(\Omega\Sigma Sq)(S^k) \overset{\text{Lemma}}{=} \Omega P_{[\frac{n}{2}]}(S^{2k+1})$

spectra: $D_n S^k \xrightarrow{E} D_n(\Omega\Sigma)(S^k) \xrightarrow{H} D_n(\Omega\Sigma Sq)(S^k)$

$D_n(\Omega\Sigma)(S^k) = \Omega D_n(S^{k+1})$, $\quad D_n(\Omega\Sigma Sq)(S^k) \overset{\text{Lemma}}{=} \begin{cases} \Omega D_{n'}(S^{2k+1}) & n = 2n' \\ * & \text{else} \end{cases}$

Lemma: $F: Top_* \longrightarrow Top_*$ reduced ~~finitary~~ homotopy functor ~~stably i-excisive~~ $\forall i$, then

$$P_n(F\,Sq) \simeq P_{[\frac{n}{2}]}(F)(Sq)$$

$$D_n(F\,Sq) \simeq \begin{cases} D_{\frac{n}{2}}(F)(Sq) & n \text{ even} \\ * & n \text{ odd} \end{cases}$$

Pf: chain rule

Want to show $\boxed{D_n(S^k) \simeq *}$ if $n \neq 2^j$, $j \geq 0$.

n odd: $D_n S^k \xrightarrow{\sim} \Omega D_n(S^{k+1}) \xrightarrow{\sim} \Omega^2 D_n(S^{k+2}) \xrightarrow{\sim} \cdots$ ($\Rightarrow E$ is $\simeq$)

$\cdots \longrightarrow \Omega^\infty D_n(\Sigma^\infty S^k) = D_1(D_n S^k) \simeq *$

n ~~even~~ general: By induction on j, $n = s 2^j$, s odd. $j=0$ = ind. base case.

For $2n = s\,2^{j+1}$:

$D_{2n} S^k \xrightarrow{\sim} \Omega D_{2n}(S^{k+1}) \longrightarrow \Omega D_n(S^{2k+1}) \simeq *$ (ind. hyp.)

$\Rightarrow D_{2n} S^k \xrightarrow{\sim} D_1(\Omega D_{2n})(S^{k+1}) \simeq *$ ∎

$$\boxed{\partial_n = \left(\Sigma S K_n\right)^\vee}$$

partition complex

$\underline{n} = \{1, \dots, n\}$ $\mathcal{K}$ = poset of nontriv. partitions (>1 & $<n$ sets in a partition)

$K_n = |\mathcal{K}|$

∂_n is $\simeq \left(\bigvee S^{n-1}\right)^\vee$, $K_n = \bigvee S^{n-3}$

$\underline{n=2}$: $K_n = \phi \Rightarrow \partial_2 = S^{-1}$, trivial Σ_2-action.

Goal: Find a smaller complex B_k s.t. $K_{p^k} \sim B_k$ and along the way, show that $(K_n) \simeq *$, $n \neq p^k$.

Def'n: B_k ... Bruhat-Tits building for $GL_k(\mathbb{Z}/p)$.

= simplicial set of flags in $(\mathbb{F}_p)^k$

$0 \subset V_1 \subset V_2 \subset \dots \subset V_s \subseteq \mathbb{F}_p^k$

subspaces

Idea: $\mathbb{F}_p^k \simeq \underline{p}^k$, $B_k \longrightarrow K_{p^k}$, think of flags as giving a partition

general n: $\mathcal{K} \xrightarrow[\text{order-preserving}]{\text{bijective}} \mathcal{S}$ ... poset of stabilizers

$\lambda \longmapsto H_\lambda \subset \Sigma_n$

stabilizes each subset of λ up to conjugacy

$H_\lambda \simeq \Sigma_{n_1} \times \dots \times \Sigma_{n_j}$ $0 < n_i < n$ & $\exists i_0 : n_{i_0} > 1$

$K_n \simeq |\mathcal{S}|$

$\mathcal{C}$: collection of subgps $\leq G$ (closed under conjugation)

$X \circlearrowleft G$, $X \longmapsto \mathrm{Iso}(X) \ni H$ which stabilize simplices of X

Def'n: X has $\mathcal{C}$-isotropy if $\mathrm{Iso}(X) \subseteq \mathcal{C}$

$X \longrightarrow Y$ is $\mathcal{C}$-equiv if $\forall H \in \mathcal{C}$, $X^H \xrightarrow{\sim} Y^H$

Prop: $\exists$ functorial unique $\mathcal{C}$-approximations s.t. it is a $\mathcal{C}$-equiv. & $X_\mathcal{C}$ has $\mathcal{C}$-isotropy.

Example: $\mathcal{C}$ = all s/gps $X_\mathcal{C} = X$

$\mathcal{C} = \{G\}$ $X_\mathcal{C} = X^G$

$\mathcal{C} = \{\{e\}\}$ $X_\mathcal{C} = EG \times X$

Def'n: $E\mathcal{C} = (*)_\mathcal{C}$

$\mathcal{C}$ is a poset $\Rightarrow$ $E\mathcal{C} \longrightarrow |\mathcal{C}|$ not an equivariant equivalence

$(E\mathcal{C})^H = |H \downarrow \mathcal{C}| = \{H' \in \mathcal{C} \mid H \subset H'\}$

If $H \in \mathcal{C}$, this is $\simeq *$.

$\mathcal{F} :=$ collection of non-transitive, non-trivial subgroups of Σ_n

$\mathcal{S} \subset \mathcal{F}$ & it turns out that $E\mathcal{S} \xrightarrow[\Sigma_n]{\sim} E\mathcal{F}$

This means that $K_n \simeq |\mathcal{F}|$.

$\mathcal{E}$: non-trivial elementary abelian subgroups $\subset \Sigma_n$, i.e. those $\cong (\mathbb{Z}/p)^i$

$\mathcal{E}' := \mathcal{E} \cap \mathcal{F}$ $\Rightarrow$ $E\mathcal{E}' \longrightarrow E\mathcal{F}$ is an $\mathcal{E}$-approximation.

$$\begin{tikzcd}
 & {|\mathcal{E}'|} \arrow[rr] \arrow[dd] & & {|\mathcal{E}|} \arrow[r] \arrow[dd] & C \arrow[dd, dashed] \\
E\mathcal{E}' \arrow[ur, "\sim"] \arrow[rr] \arrow[dd, "\mathcal{E}\text{-approx}"'] & & E\mathcal{E} \arrow[ur, "\sim"] \arrow[dd, "\mathcal{E}\text{-approx.}"] & & \\
 & {|\mathcal{F}|} \arrow[rr] & & * \arrow[r] & SK_n \\
E\mathcal{F} \arrow[ur, "\sim"] \arrow[rr] & & * \arrow[ur, "\sim"] & &
\end{tikzcd}$$

are isos on equiv. homology $H_*((E\Sigma_n X-)_{\Sigma_n}, \mathbb{F}_p)$

Suppose $n \neq p^i$.

Counting $\Rightarrow$ $\mathcal{E}' = \mathcal{E}$ $\Rightarrow$ $C \simeq *$.

Otherwise, the difference between $\mathcal{E}$ and $\mathcal{E}'$ is exactly the B_j.