Talbot 2012: The Calculus of Functors

Mentored by Gregory Arone and Michael Ching

Notes by Chris Kapulkin

Syllabus of Talks

(1) **Introduction and overview**, by Greg Arone (UVA).
(2) **Polynomial and analytic functors**, by Dan Lior (UIUC).
(3) **Constructing the Taylor tower**, by Geoffroy Horel (MIT).
(4) **Homogeneous functors**, by Matthew Pancia (UT Austin).
(5) **First examples**, by Joey Hirsh (CUNY).
(6) **The derivatives of the identity functor**, by Gijs Heuts (Harvard).
(7) **Operad and module structures on derivatives**, by Emily Riehl (Harvard).
(8) **Classification of polynomial functors**, by Michael Ching (Amherst).
(9) **Orthogonal Calculus I: theory**, by Kerstin Baer (Stanford).
(10) **Orthogonal Calculus II: examples**, by Sean Tilson (Wayne State).
(11) **Introduction to embedding calculus**, by Daniel Berwick-Evans (UC Berkeley).
(12) **Multiple disjunction lemmas**, by Greg Arone (UVA).
(13) **Embedding calculus, the little disks operad, and spaces of embeddings**, by Alexander Kupers (Stanford)
(14) **Factorization homology**, by Hiro Lee Tanaka (Northwestern).
(15) **Applications to algebraic K theory I**, by Pedro Brito (Aberdeen)
(16) **Applications to algebraic K theory II**, by Ernest E. Fontes (UT Austin).
(17) **Calculus of functors and chromatic homotopy theory**, by Tobias Barthel (Harvard).
(18) **Taylor tower of the identity functor, part 2**, by Vesna Stojanoska (MIT).
(19) **Where do we go from here?** by Greg Arone.

This PDF is a collection of hand-written notes taken by Chris Kapulkin at the 2012 Talbot Workshop. The workshop was mentored by Gregory Arone and Michael Ching, and the topic was the calculus of functors.

The aim of the Talbot Workshop is to encourage collaboration among young researchers, with an emphasis on graduate students. We make these notes available as a resource for the community at large, and more resources can be found on the Talbot website:

http://math.mit.edu/conferences/talbot/

Support for the 2012 Talbot Workshop came from the NSF grant DMS-1007096.

Greg Arone, Intro + Overview (1)

Let X, Y top. spaces. Consider $\mathrm{Map}(X,Y)$. What can we say about it knowing something about X and Y.

$$\pi_0 \mathrm{Map}(X,Y) = [X,Y]$$

For example, if X and Y are spheres we don't really know much about it

$Y \mapsto \mathrm{Map}(X,Y)$ is a very complicated

Having a decomposition $Y = Y_1 \cup_{Y_0} Y_2$ doesn't really help, example: $S^2 = D^2 \cup_{S^1} D^2$. But $[S^n, S^2]$ is rather hard

M, N smooth manifolds

$$\mathrm{Emb}(M,N)$$

Even worse: a complicated functor of both variables

5/14/12
GA

Some basic ideas:

1. Some functors deserve to be called <u>polynomial functors.</u>

2. General functors can be <u>approximated</u> with polynomial functors in two ways: interpolation polynomials and Taylor polynomials.

$$\begin{tikzcd} L_n F \arrow[r] & F \arrow[r] & P_n F \end{tikzcd}$$

interpolation polynomial

Taylor polynomial

nat. trans.

approximating from left and right

aka initial / terminal UMP

3. The n-th at a polynomial approximation is determined by the n-th cross-effect or the n-th derivative.

What are polynomial functors?

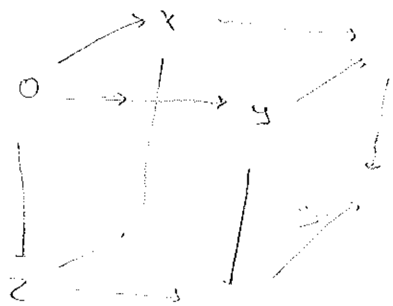
(1) What are linear functors?

$f(x+y-a) = f(x) + f(y) - f(a)$

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ Y \arrow[r] & X \cup_A Y \end{tikzcd} \quad \overset{F}{\longmapsto} \quad \begin{tikzcd} F(A) \arrow[r] \arrow[d] & F(X) \arrow[d] \arrow[dl, phantom, "\ulcorner"] \\ F(Y) \arrow[r] & F(X \cup_A Y) \end{tikzcd}$$

<u>First definition</u>: F takes homotopy pushout squares to homotopy pushout squares.

Polynomial of degree n: strongly cocartesian (every 2D face) (n+1)-cubes to cocartesian (n+1)-cubes.



2-nd cross effect

strongly cocartesian is stronger than cocartesian except deg 1: cocartesian means $x \to y$ equiv. cocartesian says that

Linear functors: $1_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$, $F(X) = K \times X$ for fixed K

Quadratic: $X \longmapsto X \times X$

⋮

$$F(X) = X \times X \times_{\Sigma_2} \mathbb{E}\Sigma_2$$

and more generally

$$F(X) = (k \times X \times X) \times_{\Sigma_2} \mathbb{E}\Sigma_2$$

are quadratic

Let F_n the category of finite sets of cardinality at most n.

$$G : \mathrm{Top} \longrightarrow \begin{cases} \mathrm{Top} \\ \mathrm{Spectra} \end{cases}$$

$$L_n G := L_{F_n}^{\mathrm{Top}} (G|_{F_n})$$

$$L_n G(X) = \operatorname*{hocolim}_{n \geq i \to X} G(i) = X^i \bigotimes_{i \in F} G(i)$$

This gives a functor $L_n G$ together with a natural transformation:

$$L_n(G) \longrightarrow G$$

Moreover, for $X \in F_n$ we have

$$L_n G(X) \longrightarrow X$$

is an equivalence, thus we can think of $L_n G$ as an interpolation of G at $0, 1, \dots, n$.

The functor $L_n G$ is polynomial of deg n.

5/14/12 GA

If G is contravariant, then there is a dual construction:

$$G: Top \longrightarrow Top^{op}$$

we can construct $R_n G: Top \to Top^{op}$ together with a natural transformation:

$$G \longrightarrow R_n G$$

where $R_n G(X) = \underset{i \in F_n}{Nat}(X^i, G(i))$

Let $\mathcal{M}^d$ denote the category of d-dimensional manifolds and embeddings

Unions of $\mathbb{R}^d - \mathcal{B}^d \subseteq \mathcal{M}^d$

$\mathcal{B}^d_n \subseteq \mathcal{B}$ union of at most n balls.

$$\mathcal{B}^d_n \xrightarrow{\pi_0} F_n$$

$$G: \mathcal{M}^d \longrightarrow Top$$

Top^{op}

Spectra

$\cdots \to L_n G \to L_{n+1} G \to \cdots \to G$ may not always converge

5/14/12 6.
GA

$$G : \mathcal{M}^d \longrightarrow \text{something}$$

$$L_n G(M) = \mathrm{Emb}(i \times \mathbb{R}^d, M) \otimes_{B_n^d} G(i \times \mathbb{R}^d)$$

If G is contravariant, then

$$R_n(M) = \underset{B_n^d}{\mathrm{Nat}}\left(\mathrm{Emb}(i \times \mathbb{R}^d, M), G(i \times M)\right)$$

(embedding calculus)

Back to homotopy case:

$$G : \mathrm{Top} \longrightarrow \begin{matrix}\mathrm{Top} \\ \text{Spectra}\end{matrix}$$

We have a sequence of approximations

$$L_0 G \longrightarrow L_1 G \longrightarrow L_2 G \longrightarrow \dots \longrightarrow L_n G \longrightarrow \dots \longrightarrow G$$

$$L_n G /_{L_n G}(X) = X^n/_{f.d} \wedge_{\Sigma_n} cr_n G$$

$$\begin{tikzcd}
G(n-2) \arrow[r] \arrow[d] & G(n-1) \arrow[d] \\
G(n-1) \arrow[r] & G(n) \arrow[dr] \\
& & cr_n G
\end{tikzcd}$$

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$

$$\left(L_n g - L_{n-1} g\right)(x) = \binom{x}{n} \cdot cr_n g$$

Taylor approximation.

$F: \mathcal{C} \longrightarrow \mathcal{D}$ is linear if it takes homotopy pushouts to homotopy pullbacks. Similarly, we can define polynomial of deg n.

$F: Top \longrightarrow Top$

$F(X \amalg Y) \xrightarrow{\simeq} FX \times FY$

$\Omega^\infty \Sigma^\infty X \rightsquigarrow e^{x-1}$

Note: 1_{Top} is not anymore a linear functor

$x = e^{x-1} \cdot e^{-\frac{x-1}{2}} \cdot \ldots \cdot e^{(-1)^{n+1}\frac{x-1}{n}} \ldots =$

$= e^{\ln(1+(x-1))} = x$

this gives a tower of fibrations

this has no counter-part in topology

Dan Lior, Polynomial and analytic functors (2)

Q: What is a polynomial functor of degree n?

$F : \mathcal{C} \longrightarrow \mathcal{D}$ for $\mathcal{C}, \mathcal{D} \in \{Top_+, Spectra\}$

homotopy functor (i.e. $X \xrightarrow{\simeq} Y \longmapsto F(X) \xrightarrow{\simeq} F(Y)$)

Cubes:

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & Z \end{tikzcd}$$

2-cube

$$\begin{tikzcd}[row sep=small, column sep=small] & X_{\{3\}} \arrow[rr] \arrow[dd] & & X_{\{2,3\}} \arrow[dd] \\ X_\emptyset \arrow[ur] \arrow[rr] \arrow[dd] & & X_{\{2\}} \arrow[ur] \arrow[dd] & \\ & X_{\{1,3\}} \arrow[rr] & & X_{\{1,2,3\}} \\ X_{\{1\}} \arrow[ur] \arrow[rr] & & X_{\{1,2\}} \arrow[ur] & \end{tikzcd}$$

3-cube

In general

$$\mathcal{X} : \underbrace{\mathcal{P}(n)}_{\substack{\text{poset of}\\ \text{subsets of } [n]\\ \text{regarded as a category}}} \longrightarrow \mathcal{C}$$

n-cube

There are natural maps

$$X \to \operatorname{holim}\left(\begin{array}{ccc} & & Y \\ & & \downarrow \\ Z & \to & W \end{array} \right) \quad \text{for a 2-cube}$$

in general: $X_\emptyset \xrightarrow{(*)} \operatorname{holim}\left((\mathcal{P}(n) - \{\emptyset\}) \hookrightarrow \mathcal{P}(n) \xrightarrow{\mathcal{X}} \mathcal{C} \right)$

<u>Def</u>: An n-cube $\mathcal{X}$ is
- <u>cartesian</u>, if $(*)$ is an equivalence, and
- k-cartesian, if $(*)$ is k-connected.

Note: we say cartesian for homotopy cartesian!

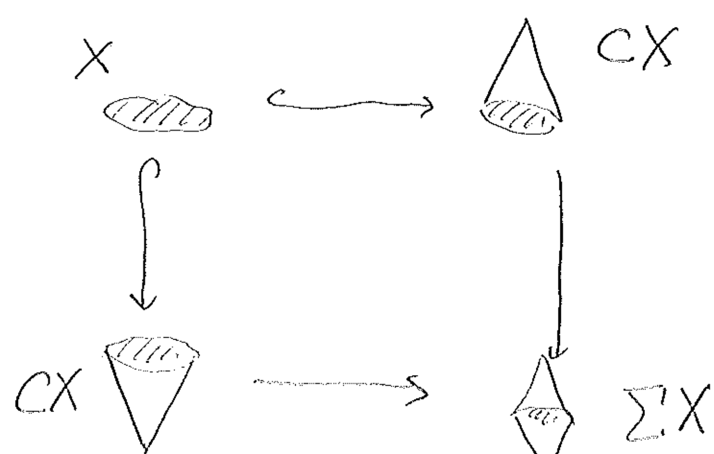
Recall: a map $f: X \to Y$ is k-connected, if the induced maps:

$\pi_i f : \pi_i X \to \pi_i Y$ are iso for $i < k$

$\pi_k f : \pi_k X \to \pi_k Y$ is surjective

<u>Examples:</u>

①
$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma X \end{tikzcd}$$



is cocartesian

(2)
$$\begin{tikzcd} \Omega X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} \Omega X \arrow[r] \arrow[d] & X^I \arrow[d] \\ X^I \arrow[r] & X \end{tikzcd}$$

is Cartesian

<u>Def:</u> F is <u>1-excisive</u>, if it takes coCartesian squares to Cartesian squares.

$f(x) = mx$ linear

$f(x) = mx + b$ 1-excisive

<u>Example:</u> $1_{\text{Spectra}} : \text{Spectra} \to \text{Spectra}$ is 1-excisive.

Recall:

<u>Thm (Blakers - Massey).</u> If the square $\mathcal{X}$:

$$\begin{tikzcd} X_\phi \arrow[r] \arrow[d] & X_{\{2\}} \arrow[d] \\ X_{\{1\}} \arrow[r] & X_{\{1,2\}} \end{tikzcd}$$

is coCartesian and

and $X_\phi \to X_{\{i\}}$ is k_i-connected. Then $\mathcal{X}$ is $(k_1 + k_2 - 1)$-Cartesian.

Note that by B-M Thm we know that in Spectra every diagram is Cartesian iff coCartesian. So 1_{Spectra} is a 1-excisive functor.

Non-example. $1_{Top_+} : Top_+ \to Top_+$ is not 1-excisive

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] & \Sigma S^0 \end{tikzcd}$$

is coCartesian but not cartesian because:

$$\begin{tikzcd} S^0 \arrow[r] \arrow[d, "\pi_0"] & \Omega \Sigma S^0 \\ 2 & \mathbb{Z} \end{tikzcd}$$

Example:

$$\begin{tikzcd} Top_+ \arrow[r, shift left, "\Sigma^\infty"] \arrow[r, phantom, "\perp"] & Spectra \arrow[l, shift left, "\Omega^\infty"] \end{tikzcd}$$

Both Σ^∞ and Ω^∞ are 1-excisive

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & W \end{tikzcd} \qquad \rightsquigarrow \qquad \begin{tikzcd} \Omega^\infty X \arrow[r] \arrow[d] & \Omega^\infty Y \arrow[d] \\ \Omega^\infty Z \arrow[r] & \Omega^\infty W \end{tikzcd}$$

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & W \end{array}$$ is coCartesian iff it is Cartesian

$$\Omega^\infty X \simeq \Omega^\infty\left(\mathrm{holim}\left(\begin{array}{ccc} & & Y \\ & & \downarrow \\ Z & \longrightarrow & W \end{array}\right)\right)$$

$$\simeq \mathrm{holim}\left(\begin{array}{ccc} & & \Omega^\infty Y \\ & & \downarrow \\ \Omega^\infty Z & \longrightarrow & \Omega^\infty W \end{array}\right)$$

is Cartesian

Example: $\Omega^\infty \Sigma^\infty : Top_+ \longrightarrow Top_+$ is 1-excisive, but $\Sigma^\infty \Omega^\infty$ is not 1-excisive. So the composite of 1-excisive functors is not necessarily 1-excisive.

Example: Let C be a fixed spectrum and define $G : Top_+ \longrightarrow Spectra$ by
$G(X) = C \wedge \Sigma^\infty X$

One can check that:

$$\left.\begin{array}{l} \text{(i) } G \text{ is 1-excisive} \\ \text{(ii) } G(*) \simeq * \end{array}\right\} \Rightarrow G \text{ linear}$$

(iii) G satisfies the "colimit axiom"

5/14/12 DL

<u>Colimit axiom.</u> If X is a filtered colimit of finite CW-complexes, then:

$$\operatorname{colim}_{\alpha}(FX_\alpha) \xrightarrow{\sim} F(\operatorname{colim}_{\alpha} X_\alpha)$$

$\boxed{\text{↳}F \to F \text{ is an equiv.}}$

Classification of functors $F: Top_* \to Spectra$ satisfying colim axiom

$$\frac{\dfrac{Hom(X,Y) \longrightarrow \Omega^\infty Hom(FX, FY) \quad \text{in Top}}{\Sigma^\infty Hom(X,Y) \longrightarrow Hom(FX, FY) \quad \text{in Spectra}}}{\Sigma^\infty Hom(X,Y) \wedge FX \longrightarrow FY}$$

Instantiating at $X = S^0$:

$$\Sigma^{+\infty} Y \wedge \underbrace{FS^0}_{=:C} \longrightarrow FY$$

Let $Y = S^{n+1}$. We have

$$\begin{tikzcd} FS^n \arrow[r] \arrow[d] & F* = * \arrow[d] \\ * = F* \arrow[r] & FS^{n+1} \end{tikzcd}$$

is pushout in Spectra hence it is determined by

$$\begin{tikzcd} FS^n \arrow[r] \arrow[d] & F* \\ * & \end{tikzcd}$$

and hence FS^1

5/14/12
DL

For CW-complexes:

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d, hook] & X \arrow[d] \\ D^n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & Y \\ * \arrow[u, phantom, "=" sloped] & \end{tikzcd}$$

is determined by

$$\begin{tikzcd} \partial D^n \arrow[r] \arrow[d] & X \\ D^n & \end{tikzcd}$$

and hence by $\partial D^n \to X$

and that extends to all spaces.

Example:

we start with those 3 maps and take pushouts

$$\begin{tikzcd}[row sep=small, column sep=small] & X_{\{3\}} \arrow[rr] \arrow[dd] & & X_{\{2,3\}} \arrow[dd] \\ X_\emptyset \arrow[ur, hook] \arrow[rr, hook, crossing over] \arrow[dd, hook] & & X_{\{2\}} \arrow[ur] & \\ & X_{\{1,3\}} \arrow[rr, dashed] & & X_{\{1,2,3\}} \\ X_{\{1\}} \arrow[ur] \arrow[rr, hook] & & X_{\{1,2\}} \arrow[ur] \arrow[from=uu, hook, crossing over] & \end{tikzcd}$$

<u>Def:</u> Any n-cube constructed this way is called <u>strongly coCartesian</u>.

<u>Def:</u> F is <u>n-excive</u>, if it takes strongly coCartesian $(n+1)$-cubes to $(n+1)$-Cartesian cubes.

Example: $X \longmapsto \Sigma^\infty(X \wedge X)$ is 2-excisive

<u>Thm. (Goodwillie).</u> If $G(X,Y)$ is bilinear, then $G(X,X)$ is 2-excisive.

Take $G(X,Y) = \Sigma^\infty(X \wedge Y) = \Sigma^\infty X \wedge \Sigma^\infty Y$

Thus the <u>Thm</u> gives that $\Sigma^\infty(X \wedge X)$ is 2-excisive.

$$\begin{tikzcd} \underline{2} = S^0 \vee S^0 \arrow[r] \arrow[d] & S^0 \vee * = \underline{1} \arrow[d] \\ \underline{1} = * \vee S^0 \arrow[r] & * \vee * = \underline{0} \end{tikzcd}$$

is taken to

$$\begin{tikzcd} \underline{4} \arrow[r] \arrow[d] & \underline{1} \arrow[d] \\ \underline{1} \arrow[r] & \emptyset = * \end{tikzcd}$$

not Cartesian.

Geoffroy Horel, Constructing the Taylor Tower (3)

We will concentrate on functors $Top_* \to Top_*$, but the theory works in other simplicial model categories.

Sometimes we will assume that $F(*) = *$.

We will construct a family of functors $\{P_n F\}_{n \in \mathbb{N}}$ together with a family of nat. trans

$$F \longrightarrow P_n F$$

s. th. (1) $P_n F$ is n-excisive

(2) for all G n-excisive and all $F \to G$ there exists a factorization

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & P_n F \\ & G \arrow[ur] & \end{tikzcd}$$

in the homotopy category of functors

(3) for any fiber sequence $F \to G \to H$ the sequence $P_n F \to P_n G \to P_n H$

n-excisive $\Leftarrow$ (n-1)-excisive

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_3 F \arrow[d] \\
 & P_2 F \arrow[d] \\
F \arrow[r] \arrow[ur] \arrow[uur] & P_1 F
\end{tikzcd}$$

<u>Definition.</u> X pointed space, S a set

<u>join</u> $$X * S := \text{hocofiber}\left(\bigvee_S X \to X\right)$$

Let n be an integer

$$\begin{aligned} \mathcal{P}([n+1]) &\longrightarrow Top_* \\ S &\longmapsto X * S \end{aligned}$$

This is strongly coCartesian

<u>Example:</u> $n=1$

$X * \phi = X$

$X * pt = CX$

$X * \{0,1\} = \Sigma X$

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & CX \arrow[d] \\
CX \arrow[r] & \Sigma X
\end{tikzcd}$$

Define

$$T_n F(X) = \underset{S \in \underbrace{\mathcal{P}_0}([n+1])}{\operatorname{holim}} F(X * S)$$

non-empty subsets!

$$F \xrightarrow{t_n F} T_n F$$

$$F X \xrightarrow{\simeq} F(X * \emptyset) \longrightarrow \underset{S \in \mathcal{P}_0([n+1])}{\operatorname{holim}} F(X * S)$$

If F is n-excisive, $F \xrightarrow{t_n F} T_n F$ is an equivalence.

$T_n F$ is a good approximation, but has no reason to be n-excisive.

<u>Def</u>: Define:

$$P_n F(X) = \operatorname{hocolim} \big(\underbrace{FX \xrightarrow{t_n F(X)} T_n F(X) \xrightarrow{t_n T_n F(X)} T_n^2 F(X) \xrightarrow{t_n T_n^2 F(X)} \cdots}_{} \big)$$

<u>why</u> does it terminate?

it is supposed to solve $F \cong T_n F$

We obtain:

$$F \xrightarrow{p_n F} P_n F$$

If F is n-excisive, $F \underset{p_n F}{\xrightarrow{\simeq}} P_n F$ is an equiv.

Remark. • $(X * S) * T = X * (S * T)$

• $\Sigma(X * S) = \Sigma X * S$

So: $T_n(F \circ \Sigma) = T_n F \circ \Sigma$

and hence: $P_n(F \circ \Sigma) = P_n F \circ \Sigma$. $\square$

Moral. $P_n F$ depends on the local behaviour of F around $*$. $\square$

Q: ???

Proposition. $P_n : \mathrm{Fun}(\mathrm{Top}_*, \mathrm{Top}_*) \to \mathrm{Fun}(\mathrm{Top}_*, \mathrm{Top}_*)$ commutes with
- filtered holimits
- finite holimits

Proof: In Top_* hocolims commute with hocolims

holims commute with holims

filtered hocolims commute with finite holims

i.e. finite nerve of a diag. $\square$

<u>Lemma.</u> Let $\mathcal{X}$ be a strongly coCartesian $(n+1)$-cube. Then $F(\mathcal{X}) \longrightarrow T_n F(\mathcal{X})$ factors through a Cartesian cube.

<u>Theorem.</u> $P_n F$ is n-excisive.

$$(*) \quad F\mathcal{X} \longrightarrow T_n F\mathcal{X} \longrightarrow T_n^2 F\mathcal{X} \longrightarrow \cdots$$

$$(**) \quad F\mathcal{X} \to C_1 \longrightarrow T_n F(\mathcal{X}) \longrightarrow C_2 \longrightarrow T_n^2 F(\mathcal{X}) \longrightarrow C_3 \to \cdots$$

total hfiber

hocolim $(*)$ = hocolim $(**)$ = hocolim$(C_1 \to C_2 \to C_3 \to \cdots)$

each C_i is cartesian, so hocolim$(C_1 \to C_2 \to \cdots)$ is cartesian

$P_n F(\mathcal{X})$ is cartesian, hence $P_n F$ is n-excisive.

Existence of factorization

Let G be n-excisive

$$\begin{array}{ccc} F & \longrightarrow & G \\ \downarrow & & \downarrow \\ P_n F & \longrightarrow & P_n G \end{array}$$

In the homotopy category the map $G \to P_n G$ is invertible, yielding

$$F \longrightarrow P_n F \longrightarrow G$$

<u>Lemma.</u> $P_n F \xrightarrow[P_n t_n F]{} P_n T_n F$ is a weak equivalence.

<u>Proof:</u> Let S be a finite set.

Define $J_S F(X) = F(X * S)$

$$\begin{array}{ccccc} P_n F & \xrightarrow{*} & P_n\left(\operatorname*{holim}_{U \in \mathcal{P}_0(n+1)} J_U F\right) & \xrightarrow{\simeq} & \operatorname*{holim}_{U} P_n J_U F \\ & & & & \downarrow \simeq \\ & & & & \operatorname{holim} J_U P_n F \end{array}$$

This composite is $t_n P_n F$ which is an equivalence because $P_n F$ is n-excisive. $\square$

<u>Corollary.</u> $P_n F \xrightarrow{P_n p_n F} P_n^2 F$ is an equivalence

Uniqueness of factorization:

Let $F \longrightarrow G$ with G n-excisive
with a factorization $F \longrightarrow P_n F \xrightarrow{v} G$

$$
\begin{tikzcd}
F \arrow[r, "p_n F"] \arrow[d, "p_n F"'] & P_n F \arrow[r, "v"] \arrow[d, "\simeq"', "p_n P_n F"] & G \arrow[d, "\simeq"] \\
P_n F \arrow[r] & P_n^2 F \arrow[r] & P_n G
\end{tikzcd}
$$

v is uniquely determined by $P_n v$.

$P_n v$ is uniquely determined by $P_n v \circ P_n p_n F$
$= P_n(v \circ p_n F)$

So v is determined by $v \circ p_n F$

As a final remark we will give a proof of

Lemma. $F\mathcal{X} \longrightarrow T_n F\mathcal{X}$ factors through a cartesian cube, if $\mathcal{X}$ is strongly cocartesian.

Proof (Rezk)

$\mathcal{X}$ cube; let $U \in \mathcal{P}(\underline{n+1})$ and define

$$\mathcal{X}_U(T) = \operatorname{hocolim}\left(\begin{array}{ccc} \coprod_{s\in U} \mathcal{X}(T) & \longrightarrow & \coprod_{s\in U} \mathcal{X}(T\cup\{s\}) \\ \downarrow & & \\ \mathcal{X}(T) & & \end{array}\right)$$

$$\mathcal{X}_U(T) \longrightarrow \mathcal{X}(T) * U$$

$$\begin{tikzcd} F(\mathcal{X}(T)) \arrow[r] \arrow[rr, bend right, "t_n F(\mathcal{X}(T))"'] & \operatorname{holim}_{U\in\mathcal{P}_0(\underline{n+1})} F(\mathcal{X}_U(T)) \arrow[r] & \operatorname{holim}_{U\in\mathcal{P}_0(\underline{n+1})} F(\mathcal{X}(T)*U) \end{tikzcd}$$

? If $\mathcal{X}$ is strongly coCartesian $\mathcal{X}_U(T) = \mathcal{X}(T\cup U)$

Matthew Pancia, Homogeneous functors (4)

$$P_n(f) = f(0) + f'(0)\,x + \frac{f''(0)}{2}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \dots$$

$$D_n(f) = P_n(f) - P_{n-1}(f) = \frac{f^{(n)}(0)}{n!}x^n$$

Properties of $D_n(f)$:

1. of deg n
2. exactly of deg n, homogeneous of deg n
3. $\frac{f^{(n)}(0)}{n!}$

For functors: $F : \mathrm{Top}_* \longrightarrow \begin{matrix}\mathrm{Top}_* \\ \mathrm{Spectra}\end{matrix}$

1. deg n $\longleftrightarrow$ n-excisive

"determined by values on $(n+1)$-points"

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[d] & & FX \arrow[rr] \arrow[dd] \arrow[dr] & & FY \arrow[dd] \\
Z \arrow[r] & Z \cup_X Y \arrow[r, shorten=1em, ""] & {} & & \lim \arrow[ur] \arrow[dl] \arrow[dr] & \\
& & & FZ \arrow[rr] & & F(Z \cup_X Y)
\end{tikzcd}$$

shortly what it means to be n-excisive (recover X)

② $P_{n-1} F = *$

<u>Def:</u> A functor F is homogeneous of degree n, if:

① F is n-excisive.

② $P_{n-1} F = *$ (n-reduced)

<u>Example:</u> Layers of the Taylor Tower

$$D_n F := \text{hofiber}\,(P_n F \longrightarrow P_{n-1} F)$$

<u>Proposition.</u> $D_n F$ is homogeneous of degree n.

<u>Proof:</u> Follows from

① P_n preserves homotopy fiber sequences

② $P_{n-1} P_n \longrightarrow P_{n-1} P_{n-1}$ is an equivalence

③ $F \to G \to H$ fiber sequence of functors, If G & H are n-excisive, so is F

$$D_n F \to P_n F \to P_{n-1} F$$

$$P_{n-1} D_n F \to P_{n-1} P_n F \longrightarrow P_{n-1} P_{n-1} F$$

✓

<u>Example.</u> $F :$ Spectra $\to$ Spectra $\quad X \mapsto X^{\wedge n}$

$G :$ $Top_* \longrightarrow$ Spectra $\quad X \mapsto \Sigma^{+\infty}(X^{\wedge n})$

These are both homogeneous of degree n.

<u>Lemma.</u> $L : C^n \to D$ is k_i-excisive in each slot, then the composite functor

$$C \xrightarrow{\Delta} C^n \xrightarrow{L} D \text{ is } \left(\sum k_i\right)\text{-excisive}$$

<u>Lemma.</u> If $L : C^n \longrightarrow D$ is reduced in each slot, then $C \xrightarrow{\Delta} C^n \xrightarrow{L} D$ is n-reduced (i.e. $P_{n-1}(L \circ \Delta)(X) = *$)

Let $\tilde{F}(X_1, \ldots, X_n) = X_1 \wedge \ldots \wedge X_n$

Example. C a fixed spectrum

$$F(X) = C \wedge X^{\wedge n}, \qquad G(X) = C \wedge \Sigma^{\infty} X^{\wedge n}$$

are n-homogeneous.

Moreover, if C has a Σ_n-section, then so does $F(X) = (C \wedge X^{\wedge n})_{h\Sigma_n}$.

A nice property is:

Thm. $F: \mathrm{Top}_* \to \mathrm{Top}_*$ homogeneous of deg n, then FX is an infinite loop space for any $X \in \mathrm{Top}_*$. $\square$

Example: (or proof for a linear functor)

$$\begin{tikzcd} X \arrow[r] \arrow[d] & CX \arrow[d] \\ CX \arrow[r] & \Sigma X \end{tikzcd} \longrightarrow \begin{tikzcd} FX \arrow[r] \arrow[d] & F(CX)^{\simeq *} \arrow[d] \\ F(CX) \arrow[r] & F(\Sigma X) \end{tikzcd}$$

$$\Omega F(\Sigma' X) \simeq FX \longrightarrow * \longrightarrow F(\Sigma' X)$$

$F(\Sigma X)$

F excisive $\Rightarrow F \circ \Sigma'$ excisive

repeatedly

For homogeneous of deg $n \neq 1$, harder, but doable. We need:

<u>Lemma.</u> F is reduced, then there is a homogeneous functor of deg n $R_n F$, which fits into a fiber sequence:

$$P_n F \to P_{n-1} F \to R_n F$$

<u>Example:</u> $f(x)$ linear iff $\hat{f}(x_1, x_2) \equiv 0$

$$\hat{f}(x_1, x_2) := f(x_1+x_2) - f(x_1) - f(x_2) + f(0)$$

Suppose $f(x) = ax^2 + bx + c$

$$\hat{f}(x_1, x_2) = (ax_1 a_2) \cdot 2$$

If f is a deg n polynomial, then

$$\hat{f}(x_1, \ldots, x_n) = n! \cdot a \cdot x_1 x_2 x_3 \cdots x_n$$

Def: The n^{th} cross effect, $cr_n F$ is the functor of n variables given by applying F to the cube:

$$\mathcal{X}(\underline{n} - T) = \bigvee_{s \in T} X_s$$

and taking the total homotopy fiber

$$\begin{tikzcd} X_1 \vee X_2 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ X_1 \arrow[r] & * \end{tikzcd} \longrightarrow \begin{tikzcd} F(X_1 \vee X_2) \arrow[r] \arrow[d] & FX_2 \arrow[d] \\ FX_1 \arrow[r] & F* \end{tikzcd}$$

$$F(X_1 \vee X_2) - FX_1 - FX_2 + F*$$

<u>Proposition.</u> If F is n-excisive, then $cr_{m+1} F$ is $(n-m)$-excisive in each variable. In particular, if F n-excisive, then $cr_n F$ is symmetric multilinear and if F is $(n-1)$-excisive, $cr_n F$ is trivial

$$cr_n : \text{homogeneous, deg } n \text{ functors} \longrightarrow \text{symmetric multilinear functors}$$

L is symmetric multilinear $L : C^n \to D$, then $C \xrightarrow{\Delta} C^n \xrightarrow{L} D$

$(L(X_1, \ldots, X_n))_{h\Sigma_n}$ is homogeneous of deg n

We have an equivalence:

$$\left\{\begin{matrix}\text{symmetric} \\ \text{multilinear} \\ \text{functors } Top_* \to Spectra\end{matrix}\right\} \underset{cr_n}{\overset{\Delta_n}{\rightleftarrows}} \left\{\begin{matrix}\text{homogeneous deg } n \\ \text{functors } Top_* \to Spectra\end{matrix}\right\}$$

where $\Delta_n : L \mapsto (L \circ \Delta)_{h\Sigma_n}$

Suppose we have a fixed spectrum C

$$L(X_1, \ldots, X_n) = C \wedge X_1 \wedge \ldots \wedge X_n$$

is symmetric multilinear

on the other hand:

if L is symmetric multilinear, then

$$L(S^0, S^0, \ldots, S^0) \wedge X_1 \wedge X_2 \wedge \ldots \wedge X_n \to L(X_1, \ldots, X_n)$$

The layers of the Taylor Tower are

homogeneous of deg n, hence:

$$(cr_n D_n)(X_1, \ldots, X_n) = C \wedge X_1 \wedge X_2 \wedge \ldots \wedge X_n$$

$$(D_n F)(X) = (\underbrace{C}_{\substack{\| \\ (cr_n D_n F)(S^0, \ldots, S^0) := \partial^{(n)}(*)}} \wedge X^{\wedge n})_{h\Sigma_n}$$

$cr_n D_n F := D^{(n)} F$ the n^{th} differential of F

$$(D_n F)(X) = (\partial^{(n)}(*) \wedge X^{\wedge n})_{h\Sigma_n}$$

Thm. The n^{th} differential $D^{(n)}F$ is equivalent to the multilinearization of $cr_n F$.

(both loosely are $\operatorname{hocolim}_{(k_1,\dots,k_n)} \Omega^{k_1+k_2+\dots+k_n} cr_n F(\Sigma^{k_1}X_1, \dots, \Sigma^{k_n}X_n)$)

So far, we've been evaluating the derivative at $*$ (in $\mathbb{R}\to\mathbb{R}$ corresponds to Maclaurin) but we may want to try doing that on other objects.

Q & A session Monday:

Let $F : Top_* \longrightarrow Top_*$ be a homotopy functor

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & P_3 F \arrow[d] \\
 & P_2 F \arrow[d] \\
F \arrow[r] \arrow[ur] \arrow[uur] & P_1 F
\end{tikzcd}
\qquad F \longrightarrow \underset{\overleftarrow{n}}{\mathrm{holim}}\, P_n F$$

and ... ?

$X \xrightarrow{r} Y$ $\qquad \ell(r) = e^{-\mathrm{conn}(r)}$ length

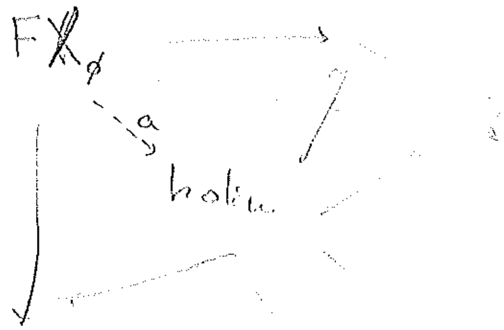
map between spaces, spectra,

F is $E_n(c, \kappa)$, if

$\mathrm{conn}(X_\phi \to X_i) \geq \kappa \Rightarrow \mathrm{conn}(a) \geq -c + \sum$

$F : Top_* \longrightarrow Top_*$

$\{X_\phi \longrightarrow X_i\}$ $\qquad i = 1, \ldots, n+1$



F has $\underline{E_n(c,k)}$, if whenever $\mathrm{conn}(X_\emptyset \to X_i) \geq k$ for $i=1,\dots,n+1$, we have $\mathrm{conn}(a) \geq -c + \sum_{i=1}^{n+1} \mathrm{conn}(X_\emptyset \to X_i)$

Given $\rho \in \mathbb{N}$, we say that F is $\underline{\rho\text{-analytic}}$, if there is $q \in \mathbb{Z}$ such that F has $E_n(n\rho - q, \rho+1)$ for all $n \geq 1$

F $\underline{\text{analytic}}$, if it is ρ-analytic for some ρ.

<u>Theorem.</u> If F is ρ-analytic and $\mathrm{conn}(X) > \rho$, then $FX \xrightarrow{\simeq} \mathrm{holim}\, P_n F(X)$. $\square$

Let $F : \mathrm{Top}/X \longrightarrow \mathrm{Top} \;/\; \mathrm{Spectra} \;/ \ldots$

$$\begin{tikzcd} & P_n^X F(Y) \arrow[d, dashed] \\ FY \arrow[r] & P_0^X(Y) = FX \end{tikzcd}$$

$$D_n F := \mathrm{hofib}(P_n F \to P_{n-1} F)$$

For X a CW-complex

$$D_n F(X) \simeq \Omega^\infty \left(\partial_n F \wedge (\Sigma^\infty X)^{\wedge n} \right)_{h\Sigma_n}$$

Every homogeneous functor $\mathrm{Top}_* \to \mathrm{Top}_*$ factors thru:

$$\begin{tikzcd} \mathrm{Top}_* \arrow[r, "D_n F"] \arrow[d, "\Sigma^\infty"'] & \mathrm{Top}_* \\ \mathrm{Spectra} \arrow[r, "D_n F"'] & \mathrm{Spectra} \arrow[u, "\Omega^\infty"'] \end{tikzcd}$$

Joey Hirsch, First examples (5)

Outline:

1. $1_{S\text{-}Alg}$
2. $Map_X(K, -)$

---

(1) S-Mod : model category of Spectra

$\wedge$ symmetric monoidal product

S is the unit

$$S\text{-}Alg = ComMon(S\text{-}Mod, \wedge)$$

Let's compute $P_1 F$ where $F : \mathcal{C} \to \mathcal{C}$ & $F* = *$

$$P_1 F = \underset{n\to\infty}{hocolim}\ \Omega^n F \Sigma^n$$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & * \arrow[d] \\ * \arrow[r] \arrow[ur, phantom, "\ulcorner"] & \Sigma X \end{tikzcd} \qquad \begin{tikzcd} T_1F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F* \arrow[d] \\ F* \arrow[r] & F(\Sigma X) \end{tikzcd}$$

$$P_1(1_{\mathcal{C}}) = \Omega^\infty \Sigma^\infty \qquad \Sigma X = hocolim\begin{pmatrix} X \to * \\ \downarrow \\ * \end{pmatrix}$$

$$\Omega X = holim\begin{pmatrix} * \\ \downarrow \\ * \to X \end{pmatrix}$$

$$\mathcal{C} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\underset{\leftarrow}{\overset{\rightarrow}{\perp}}}} Stab(\mathcal{C})$$

$P_1(1_{\mathcal{C}}) =$ "stable $\mathcal{C}$-homotopy"

When $\mathcal{C} = R\text{-}Alg$ Bestera - Mendell

$$Stab(R\text{-}Alg) \simeq R\text{-}Mod$$

via this equivalence $\Sigma^\infty \simeq \underline{TAQ}(\cdot) =$ derived indecomp

Topological Andre Quillen

$\underbrace{I(A)}_{R\text{-module}} = hofib(A \to R)$ algebra over R

$$TAQ(A) = I(A)/I(A)^2$$

$\Omega^\infty(M) \simeq R \vee M$

when $R = S$, $P_1(\mathbb{1}_{S\text{-Alg}})(A) = S \vee TAQ(A)$

$D_1(\mathbb{1}_{S\text{-Alg}})(A) = TAQ(A)$

<u>Goal:</u> compute $D_n \mathbb{1}_{S\text{-Alg}}(A)$

We know that: $\left(\mathrm{MultiLin}(cr_n(\mathbb{1})) \circ \Delta(A)\right)_{h\Sigma_n} = D_n \mathbb{1}_{S\text{-Alg}}(A)$

<u>Claim.</u> $cr_n(\mathbb{1}_{S\text{-Alg}})(A_1, -, A_n) = I(A_1) \wedge \ldots \wedge I(A_n)$

<u>Claim.</u> $\mathrm{MultiLin}(cr_n(\mathbb{1}_{S\text{-Alg}}))(A_1, \ldots, A_n) = TAQ(A_1) \wedge \ldots \wedge TAQ(A_n)$

So $D_n(\mathbb{1}_{S\text{-Alg}})(A) = \left(TAQ(A)^{\wedge n}\right)_{h\Sigma_n} = \text{"}\Omega^\infty(\partial_n(\mathbb{1}) \wedge (\Sigma^\infty A)^{\wedge n})_{h\Sigma_n}$

<u>Fact.</u> If $I(A)$ is 0-connected, then the map $A \longrightarrow \operatorname{holim}_{n \to \infty} P_n(\mathbb{1}_{S\text{-Alg}})(A) = P_\infty(A)$ is an equivalence

<u>Corollary.</u> Let $f: A \longrightarrow B$ & $I(A), I(B)$ 0-connected. If $TAQ(A) \xrightarrow{\simeq} TAQ(B)$, then $f: A \longrightarrow B$ is an equivalence.

<u>Proof:</u>

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\simeq"'] & B \arrow[d] \\
P_\infty A \arrow[r, "\simeq", "P_\infty f"'] & P_\infty B
\end{tikzcd}$$

$$\begin{tikzcd}
& P_{n+1} A \arrow[rr] \arrow[dd] & & P_{n+1} B \arrow[dd] \\
D_{n+1} A \arrow[ur] \arrow[rr] & & D_{n+1} B \arrow[ur] & \\
& P_n A \arrow[rr, "\simeq", "P_n f"'] & & P_n B
\end{tikzcd}$$

$\Longrightarrow$ by the Five Lemma $\quad P_{n+1} A \xrightarrow{\sim} P_{n+1} B$

(2) $\mathrm{Map}_*(K,-)$.

Notation. K based finite CW-complex

X based space

Top_X the category of spaces containing X as a retract i.e.

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend right, "1_X"'] & Y \arrow[r, "\alpha"] & X \end{tikzcd}$$

K_1^n will denote an equivariant subquotient of K^n (i.e. $\exists\, K_a^n \subseteq K_b^n \subseteq K^n$ with $K_1^n = K_b^n / K_a^n$)

Def: $\overline{\mathrm{Map}_*}\left(K_1^n, (Y/X)^{\wedge n} \wedge \mathrm{Map}(K,X)_*\right) =$

$$= \left\{ \begin{array}{l} f \in \mathrm{Map}_*\left(K_1^n, (Y/X)^{\wedge n} \wedge \mathrm{Map}(K,X)_*\right) \\ \forall k_i^n \in K_1^n \setminus f^{-1}(*) \quad P_{\mathrm{Map}(K,X)}(f(k_1^n))(P_i(k_i^n)) \\ \qquad\qquad\qquad\qquad \parallel \\ \qquad\qquad \alpha\left(P_i \cdot P_{(Y/X)}(f(k_1^n))\right) \quad \forall_{1 \leq i \leq n} \end{array} \right.$$

$$\begin{tikzcd} K_1^n \arrow[r, "F"] \arrow[d, "P_i"'] & (Y/X)^{\wedge n} \arrow[d, "\alpha \cdot P_i"] \\ K \arrow[r, "f"'] & X \end{tikzcd}$$

Spoiler alert. $X = *$

Let $\mathcal{M}$ = the category of finite sets and surjections

$\mathcal{M} =$ —— / / —— $\leq n$ - / / ——

$$Y^{\wedge -} : \mathcal{M}_n^{op} \longrightarrow Top_*$$

$$\begin{tikzcd} m \arrow[r, mapsto] \arrow[d, "\Theta"'] & Y^{\wedge m} \\ n \arrow[r, mapsto] & Y^{\wedge n} \arrow[u, "\Theta^*"'] \end{tikzcd} \qquad \begin{tikzcd} y_{\Theta(1)} \wedge \cdots \wedge y_{\Theta(m)} \\ y_1 \wedge \cdots \wedge y_n \arrow[u, mapsto] \end{tikzcd}$$

<u>Def:</u> Fix K and Y. $F_n, G_n : \mathcal{M}_n^{op} \longrightarrow Top_*$

$$F_n^K(m) = \Sigma^\infty K^{\wedge m} \; ; \; G_n^Y(m) = \Sigma^\infty Y^{\wedge m}$$

<u>Def:</u> We give $\underset{M_n}{Nat}(\Sigma^\infty K^{\wedge \cdot}, \Sigma^\infty Y^{\wedge \cdot})$ the subspace topology by:

$$\underset{M_n}{Nat}(\Sigma^\infty K^{\wedge \cdot}, \Sigma^\infty Y^{\wedge \cdot}) \subseteq \prod_{m \in M_n} Map(\Sigma^\infty K^{\wedge m}, \Sigma^\infty Y^{\wedge m})$$

<u>Observe:</u> ① $Nat(F_n, G_n) \xrightarrow{rest} Nat(F_{n-1}, G_{n-1})$ is a fibration

② $\Omega^\infty \Sigma^\infty Map(K, Y) \xrightarrow{\sigma_n} Nat(F_n, G_n)$

$$Map(K, Y) \longrightarrow Map(K^{\wedge n}, Y^{\wedge n})$$

$$f \longmapsto f^{\wedge n}$$

③
$$\begin{tikzcd} & Nat(F_{n-1}, G_{n-1}) \arrow[d] \\ \cdot \arrow[ur, "\sigma_n"] \arrow[r, "\sigma_{n-1}"'] \arrow[ur, phantom, "="', near end] & Nat(F_{n-1}, G_{n-1}) \end{tikzcd}$$

Thm.

$$\begin{tikzcd} & \vdots \arrow[d] \\ QMap(K_i) \arrow[r] & Nat(F_n, G_n) \arrow[d] \\ & \vdots \end{tikzcd}$$

is the Taylor Tower for $QMap(K,-)$, where $Q = \Omega^\infty \Sigma^\infty$. $\square$

$$D_n(QMap(K,-)) = Map_*(\Sigma^\infty K^{\wedge n} / \underbrace{\Delta^n K}_{\text{fat diagonal}}, \Sigma^\infty Y^n)^{\Sigma_n}$$

Surprise. $P_\infty(QMap(K,-)) = \underbrace{Nat(F,G)}_{\substack{\text{maps of right} \\ \text{modules over the} \\ \text{commutative operad}}}$

$$\left\{ \begin{array}{c} \text{finite sets} \\ \text{with surjections} \end{array} \right\} \simeq \left\{ \begin{array}{c} \text{Commutative} \\ \text{Operads} \end{array} \right\}$$

Claim. $\mathcal{M} \xrightarrow{\mathcal{A}} Top_*$ product preserving are just commutative algebras in spaces.

Proof: $\mathcal{A}(1) =: A$

$$\begin{tikzcd} \mathcal{A}(2) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathcal{A}(1) \arrow[d, equal] \\ A^{\times 2} \arrow[r, "\mu"'] & A \end{tikzcd}$$

Gijs Heuts, The derivatives of the Identity Functor (G)

We consider $\mathbb{1}_{Top_*} : Top_* \longrightarrow Top_*$

Analiticy of $\mathbb{1}_{Top_*}$

<u>Thm (BM, ES, G).</u> Let $\mathcal{X}$ be a strongly coCartesian n-cube. If for $1 \le i \le n$ the maps $X_\emptyset \longrightarrow X_i$ are k_i-connected, then $\mathcal{X}$ is $(1-n+\sum k_i)$-Cartesian.

<u>Corollary.</u> $\mathbb{1}_{Top_*}$ satisfies $E_n(n,k)$ for all $k \in \mathbb{Z}$ and all $n \geq 1$. Hence $\mathbb{1}_{Top_*}$ is 1-analytic. In particular, Taylor tower converges on simply-connected spaces. □

<u>Rmk.</u> Convergence for suitably nilpotent spaces.

Derivatives

$$\begin{array}{c} \operatorname{colim} \Omega^{k_1+\dots+k_n} cr_n(\mathbb{1}_{Top_*})(\Sigma^{k_1}X_1, \dots, \Sigma^{k_n}X_n) \\ \big\downarrow \\ \Omega^\infty(\partial_n id \wedge X_1 \wedge \dots \wedge X_n) \end{array}$$

<u>Construction:</u> Let $\mathcal{X}$ be an n-cube of spaces.

For $U \subseteq \{1, \dots, n\}$, let

$$I^U = \{(t_1, \dots, t_n) \in I^n \mid t_i = 0, \text{ if } i \notin U\}$$

A point in $\mathrm{hfib}(\mathcal{X})$ is a collection

$\{\phi_U\}_{U \subseteq \underline{n}}$, where $\phi_U : I^U \longrightarrow \mathcal{X}_U$

satisfying:

(a)
$$\begin{tikzcd} I^V \arrow[r] \arrow[d] \arrow[dr, phantom, "="] & \mathcal{X}_V \arrow[d] \\ I^U \arrow[r] & \mathcal{X}_U \end{tikzcd} \qquad V \subseteq U$$

(b) if $t_i = 1$ for some i, then

$$\phi_U(t_1, \dots, t_n) = *$$

<u>Construction of T_n</u>

A point in $cr_n(1_{Top_*})(X_1, \dots, X_n)$ consists of maps

$$\phi_U : I^U \longrightarrow \bigvee_{i \notin U} X_i$$

In particular, get maps:

$$\underline{I}^{n-1} \cong \underline{I}^{\underline{n}\setminus\{i\}} \longrightarrow X_i$$

Get $T_n^1 : cr_n(1_{Top_*})(X_1, \dots, X_n) \to Map_*(\underline{I}^{n(n-1)}, \prod_{i=1}^n X_i)$

Compose with $\prod_{i=1}^n X_i \longrightarrow \bigwedge_{i=1}^n X_i$ to get

$$T_n^n : cr_n(1_{Top_*})(X_1, \dots, X_n) \to Map_*(\underline{I}^{n(n-1)}, \bigwedge_{i=1}^n X_i)$$

Make the identifications via action:

$$\underline{I}^{n(n-1)} = \{ (t_{ij})_{1\le i,j\le n} \mid t_{ii} = 0 \ \forall i \}$$

$$= \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ * & & 0 \end{pmatrix} \begin{matrix} \leftarrow \underline{I}^{n-1} \\ \leftarrow \underline{I}^{n-1} \end{matrix} \right\}$$

Def: ① $Z := \{ t \in \underline{I}^{n(n-1)} \mid t_{ij} = 1$ for some $i,j \}$

② $W_{ij} := \{ t \in \underline{I}^{n(n-1)} \mid t_{ik} = t_{jk} \ \forall_{1\le k\le n} \}$

③ $K_n := \underline{I}^{n(n-1)} \big/ (Z \cup \bigcup_{i<j} W_{ij})$

Get a map $T_n : cr_n(\mathbb{1}_{Top_*})(X_1, \dots, X_n) \to Map(K_n, X_1 \wedge \dots \wedge X_n)$

<u>Claim.</u> This map is Σ_n-equivariant.

<u>Claim.</u> This map becomes an equivalence after multilinearizing.

<u>Prop.</u> Non-equivalently, $K_n = \bigvee_{(n-1)!} S^{n-1}$ Exercise!

$$Map_*(K_n, X_1 \wedge \dots \wedge X_n) \simeq \prod_{i=1}^{(n-1)!} \Omega^{n-1}(X_1 \wedge \dots \wedge X_n)$$

Let's consider the "first step":

$$L_n : \Omega cr_n(\mathbb{1}_{Top_*})(\Sigma X_1, \dots, \Sigma X_n) \to \prod_{i=1}^{(n-1)!} \Omega^n \Sigma^n (X_1 \wedge \dots \wedge X_n)$$

<u>Thm. (Hilton - Milnor).</u>

$$\Omega \Sigma (X_1 \vee \dots \vee X_n) \xrightarrow{\simeq} \prod_{\substack{\text{monomials} \\ \text{in a standard} \\ \text{basis of Lie}(n)}} \Omega \Sigma (X_1^{\wedge a_1} \wedge \dots \wedge X^{\wedge a_n})$$

where Lie(n) free Lie algebra on generators $x_1, \dots, x_n$. a_i = number of x_i's in a given monomial.

<u>Cor.</u> $cr_n(\Omega\Sigma)(X_1,\dots,X_n) = \prod_{\substack{\text{monomials} \\ \text{s.th. } a_i \geq 1 \\ \text{for all } i}} \Omega\Sigma(X_1^{\wedge a_1} \wedge \dots \wedge X_n^{\wedge a_n})$

<u>Cor.</u> If all the X_i are k-connected,

$\pi_m(cr_n(\Omega\Sigma)(X_1,\dots,X_n)) \to \pi_m\left(\prod_{i=1}^{(n-1)!} \Omega\Sigma(X_1\wedge\dots\wedge X_n)\right)$

is an iso, if $0 \leq m \leq (n+1)(k+1)-1$

Since $\pi_m(\Omega\Sigma(X_1\wedge\dots\wedge X_n)) = \pi_m(X_1\wedge\dots\wedge X_n)$ in this range (BM, Freudenthal), we get

<u>Prop.</u> For $0 \leq m \leq (n+1)(k+1)-1$

$\pi_m(cr_n(\Omega\Sigma)(X_1,\dots,X_n)) \simeq \pi_m\left(\prod_{i=1}^{(n-1)!} X_1\wedge\dots\wedge X_n\right)$

$\pi_m\left(\prod_{i=1}^{(n-1)!} \Omega^n\Sigma^n(X_1\wedge\dots\wedge X_n)\right) \simeq \pi_m\left(\prod_{i=1}^{(n-1)!} X_1\wedge\dots\wedge X_n\right)$

and L_n induces iso's on π_m in those degrees

This gives

$\Omega^{ln} cr_n(1_{Top_*})(\Sigma^l X_1,\dots,\Sigma^l X_n) \to \Omega^{ln} Map_*(K_n, \Sigma^l X_1\wedge\dots\wedge\Sigma^l X_n)$

$\|$

$$\wr\wr$$

$$\Omega^{\ell n-1} cr_n(\Omega, \Sigma^1)(\Sigma^{\ell-1}X_1, \ldots, \Sigma^{\ell-1}X_n)$$

$$\longrightarrow \Omega^{\ell n+1} \prod_{i=1}^{(n-1)!} \Omega^n \Sigma^n(\Sigma^{\ell-1}X_1 \wedge \ldots \wedge \Sigma^{\ell-1}X_n)$$

induces isos on π_m for

$$0 \leq m \leq \underbrace{(n+1)(k+1) - 1 - (\ell n - 1)}_{= 1 + \text{junk}}$$

$\Rightarrow$ T_n becomes an equivalence after multilinearizing

$$\operatorname*{colim}_{\ell\to\infty} \Omega^{\ell n} Map_*(K_n, \Sigma^{\ell n} X_1 \wedge \ldots \wedge X_n)$$

$$\wr\wr$$

$$Map_*(K_n, Q(X_1 \wedge \ldots \wedge X_n))$$

<u>Thm.</u> $\partial_n(1_{Top_*}) = \mathbb{D}K_n$

Spanier-Whitehead

Non-equivariantly, $\partial_n(1_{Top_*}) = \bigvee_{i=1}^{(n-1)!} S^{1-n}$

Def: $\mathrm{Part}(n)$ is a poset of partitions of $\{1, \dots, n\}$

$\mathrm{Part}_{<1}(n)$ is the subset of non-trivial partitions

i.e. $\mathrm{Part}(n) \setminus \{\underline{1}\}$

$\mathrm{Part}_{>0}(n)$ is the subset of partitions

i.e. $\mathrm{Part}(n) \setminus \{\underline{0}\}$

Exercise. $K_n \simeq \left| N(\mathrm{Part}(n)) \big/ N(\mathrm{Part}_{<1}(n)) \cup N(\mathrm{Part}_{>0}(n)) \right|$

$\simeq \sum S \left| N(\mathrm{Part}(n) - \{\underline{0}, \underline{1}\}) \right|$ if $n > 1$

Emily Riehl, Operads and chain rule for the calculus of functors (7)

Context: $\mathcal{C}, \mathcal{D}, \mathcal{E} = Top_*$, Spectra ($=$ EKMM S-modules)

$F: \mathcal{C} \to \mathcal{D}$ homotopy functor (homotopical)

$\partial_n F \circlearrowright \Sigma_n$
spectrum

$\partial_* F$ forms a symmetric sequence in Spectra

Let Σ = the category of finite sets and isos

$\partial_* F : \Sigma \longrightarrow$ Spectra

Q: What extra structure is present on $\partial_* F$?

Example. $\partial_* I_{Spectra} = \begin{cases} S & \text{if } n=1 \\ * & \text{oth.} \end{cases}$

since $P_0 I = *$ $\quad P_1 I = I$

$D_1 I = \text{hfib}(I \longrightarrow *) = I$

$D_n F = (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$

$\partial_1 F = S$ and $P_n I = I$ $\quad$ hfib $= *$

We'll denote $\mathbb{1} := \partial_* I_{Spec}$.

<u>Example.</u> $\partial_* I_{Top_*}$ is an operad i.e. a monoid in $(Spectra^{\Sigma}, \circ, \mathbb{1})$

composition product

<u>NB.</u> Not a symmetric monoidal category. Still have L- and R-modules.

<u>Main Thm 1.</u> Let $F: \mathcal{C} \to \mathcal{D}$ homotopical. Then $\partial_* F$ form a $(\partial_* I_{\mathcal{D}}, \partial_* I_{\mathcal{C}})$-bimodule.

<u>Main Thm 2.</u> Let $F: \mathcal{D} \to \mathcal{E}$ and $G: \mathcal{C} \to \mathcal{D}$ both homotopical and reduced, and moreover F finitary. Then $\partial_*(FG) = \partial_* F \circ_{\partial_* I_{\mathcal{D}}} \partial_* G$

In Spectra composition product is just usual $\circ$.

<u>Context.</u> $(\mathcal{C}, \wedge, S)$ symmetric monoidal category, finitely bicomplete.

$\mathcal{C}^{\Sigma}$ symmetric sequences in $\mathcal{C}$

$$(A \circ B)(n) = \bigvee_{\substack{\text{partitions}\\ \text{of } n}} A(k) \wedge B(n_1) \wedge \ldots \wedge B(n_k) =$$

$$= \bigvee_{k=1}^{n} \Big(\bigvee_{\underline{n} \twoheadrightarrow \underline{k}} A(k) \wedge \big(B(n_1) \wedge \ldots \wedge B(n_k)\big)_{\Sigma_k} \Big)$$

<u>Lemma.</u> $\mathcal{C}$ is closed $\Rightarrow$ $(\mathcal{C}^{\Sigma}, \circ, \mathbb{1})$ is a monoidal category.

<u>Proof:</u> Exercise for undergrads. $\square$

<u>Def:</u> An <u>operad</u> P is a monad

$$P \circ P \xrightarrow{\mu} P \qquad \mathbb{1} \xrightarrow{\eta} P$$

associative & unital.

$$\mu : P(k) \wedge P(n_1) \wedge \ldots \wedge P(n_k) \longrightarrow P(n_1 + \ldots + n_k)$$

<u>Def:</u> A right P-module R is a symmetric sequence together with associative and unital action $R \circ P \longrightarrow R$.

<u>Rmk.</u> Analogously, one defines a left P-module : $P \circ L \to L$.

Exercise. $\mathbb{P}$ is reduced iff $\mathbb{1} \to \mathbb{P}$ is an iso in $\mathcal{C}$

$\mathbb{1}$ is L and R module over $\mathbb{P}$.

F: Spectra $\longrightarrow$ Spectra : how to get structure on $\partial_* F$?

Dual derivatives (also in Spectra$^{\Sigma}$)

$$\partial^* F = \mathrm{Nat}(FX, X^{\wedge n})$$

Rules: • tactically restricting F to Spec$^{\Sigma}$

• EKMM S-modules $\Rightarrow$ all objects are fibrant

To get the desired homotopy type, need cofibrancy for F:

$Q: F \mapsto \overline{F}$ cofibrant replacement construction in $[\text{Spectra}^{fin}, \text{Spectra}]_{proj}$ by Small Object Argument.

This is a presented cell spectrum. In particular, get $\mathrm{Sub}(QF)$ a filtered category of finite subcomplexes.

So $\partial^n F = \{\mathrm{Nat}(CX, X^{\wedge n})\}_{C \in \mathrm{Sub}(QF)}$

is a pro-object in Spectra.

Thm. There exists a Quillen equivalence

$$\mathbb{D} : \mathrm{Pro}(\mathrm{Spectra})^{op} \underset{\mathrm{Map}(-,S)}{\overset{\mathrm{Map}(-,S)}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathrm{Ind}(\mathrm{Spectra}) \underset{\text{trivial}}{\overset{\mathrm{colim}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathrm{Spectra}$$

$\mathbb{D}$ defines the Spanier-Whitehead dual.

$$\partial_* F = \mathbb{D}\{\underbrace{\mathrm{Nat}(CX, X^{\wedge n})}_{\text{pointwise cofibrant replacement}}\}_{C \in \mathrm{Sub}(QF)} =$$

$$= \operatorname*{hocolim}_{C \in \mathrm{Sub}(QF)} \mathrm{Map}(\mathrm{Nat}(CX, X^{\wedge n}), S)$$

Claim. This is a model for the Goodwillie derivatives. Furthermore, structures on $\partial^* F$ correspond to dual structures on $\partial_* F$.

Example. (Good situation). F homotopical and a comonad presented cell functor. Then $\partial^* F$ is an operad

$$\Sigma^\infty = S_c \wedge - : sSets_* \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} Spectra : \underline{Spectra}(S_c, -)$$

$\underline{Spectra}(S_c,-) = \Omega^\infty$

sSets-valued hom

Let S_c = cofibrant replacement of S

$$S_c \xrightarrow{\sim} S$$

$\Sigma^\infty \Omega^\infty X = S_c \wedge \underline{Spectra}(S_c \wedge X)$ and $* \to S_c$ is a generating cofibration. So this is a cell object as desired. $\square$

$$\partial^n(\Sigma^\infty \Omega^\infty) = Nat\left(S_c \wedge Hom(S_c, X), X^{\wedge n}\right)$$
$$\cong Map(S_c, S_c^{\wedge n})$$
$$\simeq Map(S, S^{\wedge n}) \quad \text{(internal hom in Spectra)}$$

Upshot: dual derivatives of $\Sigma^\infty \Omega^\infty$ are equivalent to S.

Operad structure coincides with coendomorphism operad structure.

Turns out $\partial_* I_{Top_*} \simeq \mathbb{D}B(1, S, 1)$
$= \mathbb{D}BS$ the bar constr.

<u>Slogan.</u> $\partial_* I_{Top_*}$ is Koszul dual to commutative operad and

$$\mathbb{D}BS \simeq \mathbb{D}(1, \partial^*(\Sigma^{\infty}\Omega^{\infty}), \mathbb{1})$$

<u>Proof:</u> $\partial_* I_{Top_*}$ is $\mathbb{D}\left(\begin{smallmatrix}\text{partition}\\ \text{poset}\end{smallmatrix}\right) \cong B(\mathbb{1}, S, \mathbb{1})$
homeomorphism

□

<u>Bar construction:</u>

<u>Thm. (Ching).</u> P a reduced operad, R a right P-module, L a left P-module:

- $B(\mathbb{1}, P, \mathbb{1})$ is a cooperad BP.
- $B(\mathbb{1}, P, L)$ is a left BP-comodule.
- $B(R, P, \mathbb{1})$ is a right BP-comodule.

In general, $(\mathcal{C}, \wedge, S)$ symmetric monoidal
$\Rightarrow (\mathcal{C}^{op}, \wedge, S)$ symmetric monoidal
But closedness does not transport this way

Composition product applied to $\mathcal{C}^{op}$ gives rise to dual composition product $(\mathcal{C}^{\Sigma}, \hat{\circ}, \mathbb{1})$.
But this dual composition product is not associative in general.

$$A \hat{\circ} B(n) = \prod_{k=1}^{n} \Big(\prod_{\underline{n} \twoheadrightarrow \underline{k}} A(k) \wedge B(n_1) \wedge \ldots \wedge B(n_k) \Big)_{\Sigma_k}$$

Def.: Q is a cooperad iff it is a comonoid w.r.t. $\hat{\circ}$ in $\mathcal{C}^{\Sigma}$

$$\begin{tikzcd} & Q \hat{\circ} Q \arrow[r, "\delta \hat{\circ} 1"] & (Q \hat{\circ} Q) \hat{\circ} Q \arrow[dr] & \\ Q \arrow[ur, "\delta"] \arrow[dr, "\delta"'] & & & Q \hat{\circ} Q \hat{\circ} Q \\ & Q \hat{\circ} Q \arrow[r, "1 \hat{\circ} \delta"'] & Q \hat{\circ} (Q \hat{\circ} Q) \arrow[ur] & \end{tikzcd}$$

Def.: P operad, L a left P-module, R a right P-module

$$B_\bullet(R, P, L) : \Delta^{op} \longrightarrow \mathcal{C}^{\Sigma}$$

$$B_k(R, P, L) = R \circ \underbrace{P \circ P \circ \ldots \circ P}_{k-1 \text{ times}} \circ L$$

<u>Rmk.</u> $P = \mathbb{1}$ gives $B(R, \mathbb{1}, L) = R \circ L$

so $\partial_*(FG) = B(\partial_* F, \partial_* I_{\otimes}, \partial_* G)$

<u>Lemma.</u> If $R \xrightarrow{\simeq} R'$, $L \xrightarrow{\simeq} L'$, and $P \xrightarrow{\simeq} P'$, then $B(R, P, L) \xrightarrow{\simeq} B(R, P, L)$.

<u>Main ingredient.</u> $B(R, P, L) \to B(R, P, \mathbb{1}) \hat{\circ} B(\mathbb{1}, P, L)$

Connect back to sSets & chain rules

$$\mathcal{C} \xrightarrow{G} sSets_* \xrightarrow{F} \mathcal{D}$$

$F\Omega^\infty$ right $\Sigma^\infty \Omega^\infty$-comodule

$\Omega^\infty G$ left $\Sigma^\infty \Omega^\infty$-comodule

$\Rightarrow \partial^*(\Sigma^\infty G)$ and $\partial^*(F\Omega^\infty)$ are left and right modules (respectively) over $\partial^*(\Sigma^\infty \Omega^\infty)$ operad.

<u>Thm.</u> F, G pointed, simplicial, homotopical and F finitary. Then $P_n(FG) \xrightarrow{\simeq} \widetilde{\mathrm{Tot}}(P_n(F\Omega(\Sigma^\infty\Omega^\infty)\Sigma^\infty G))$

take Reedy cofibrant replacement, then totalization

<u>Unifying fact:</u> $\partial^*(FG) \simeq \partial^*(F\Omega^\infty) \circ_{\partial^*(\Sigma^\infty\Omega^\infty)} \partial^*(\Sigma^\infty G)$

if $F = G = I_{Top_*}$, then:

$$\partial^*(I_{Top_*}) \simeq B(\mathbb{1}, \partial^*(\Sigma^\infty\Omega^\infty), \mathbb{1})$$

$\Rightarrow I_{Top_*}$ Koszul dual to $\partial^*(\Sigma^\infty\Omega^\infty)$

Michael Ching, Classification of polynomial functors (8)

Story so far : $F : \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & \arrow[d] & \arrow[l] \\
F \arrow[ur] \arrow[r] \arrow[dr] & P_n F \arrow[d] & D_n F \arrow[l] \\
 & P_{n-1} F \arrow[d] & D_{n-1} F \arrow[l] \\
 & \vdots &
\end{tikzcd}$$

$$D_n F(X) \simeq \Omega^\infty (\partial_n F \wedge X^{\wedge n})_{h\Sigma_n}$$

$\partial_n F$: spectrum with Σ_n- action

$\partial_* F$: symmetric sequence of spectra

$\partial_* \mathrm{Id}_{\mathrm{Top}_*}$: operad

$\partial_* F$: $\partial_* \mathrm{Id}_{\mathrm{Top}_*}$- bimodule

Question : How can we describe the information needed to reconstruct the tower from the derivatives $\partial_* F$?

General framework for answering questions of the form: given a functor

$$L : \mathcal{A} \longrightarrow \mathcal{B}$$

can we recover $A \in \mathcal{A}$ from LA together with extra information?

This is <u>descent theory</u>.

Suppose that L has a right adjoint

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "R"] \end{tikzcd}$$

[We will apply this framework to $[\mathrm{Top}_*^{fin}, \mathrm{Top}_*]$ with $[\mathrm{Top}_*^{fin}, \mathrm{Top}_*] \underset{\perp}{\xrightarrow{\partial_*}} \partial_* \mathrm{Id}_{\mathrm{Top}_*}\text{-BiMod}$

key observation: existence of adjoint]

We have maps:

$$1_{\mathcal{A}} \xrightarrow{\eta} RL$$

$$LR \xrightarrow{\varepsilon} 1_{\mathcal{B}}$$

and hence

$$LR \xrightarrow{L\eta R} LRLR$$

$$LR \xrightarrow{\varepsilon} 1_{\mathcal{B}}$$

$K := LR$ is a comonad on $\mathcal{B}$.

$$\begin{tikzcd} K \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & KK \arrow[d, "\Delta K"] \\ KK \arrow[r, "K\Delta"'] & KKK \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r] \arrow[dr, equal] & KK \arrow[d, shift right, "K\varepsilon"'] \arrow[d, shift left, "\varepsilon K"] \\ & K \end{tikzcd}$$

and for any $A \in \mathcal{A}$

$$LA \xrightarrow{L\eta_A} LRLA$$

which makes $B := LA$ into a K-coalgebra

$$\begin{tikzcd} B \arrow[r, "\theta"] \arrow[d, "\theta"'] & KB \arrow[d, "K\theta"] \\ KB \arrow[r, "\Delta"'] & KKB \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "\theta"] \arrow[dr, equal] & KB \arrow[d, "\varepsilon"] \\ & B \end{tikzcd}$$

This is our "extra information" on LA

We try to recover A from LA using a <u>cobar construction</u>:

for any K-coalgebra B we have

$$\begin{tikzcd} RB \arrow[r, shift left=2, "R\theta"] \arrow[r, shift right=2, "\eta_{RB}"'] & RLRB \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & RLRLRB \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

The cobar construction on B is

$$\mathrm{cobar}(R, LR, B) = \mathrm{Tot}(\text{simplicial object})$$

If $B = LA$

$$A \longrightarrow \mathrm{cobar}(R, LR, LA)$$

Question: what are general conditions that make this map an equivalence?

Example:

$$\mathrm{Top}_* \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathrm{Spectra}$$

$X \longrightarrow \mathrm{cobar}(\Omega^\infty, \Sigma^\infty\Omega^\infty, \Sigma^\infty X)$ is an equivalence, if X is nilpotent (i.e. $\pi_1(X)$ nilpotent + π_1 acts nilpotently on π_n)

Theorem. The following functors have right adjoints.

pointed simplicially enriched functors $[Sp^{fin}, Sp] \xrightarrow{\partial_*} Sp^{\Sigma}$

$[\mathrm{Top}_*^{fin}, Sp] \xrightarrow{\partial_*} \left\{\begin{matrix}\text{Right } \partial_* \mathrm{Id}_{\mathrm{Top}_*} \\ \text{modules}\end{matrix}\right\} \xrightarrow{\text{forget}} Sp^{\Sigma}$

$[Sp^{fin}, \mathrm{Top}_*] \xrightarrow{\partial_*} \left\{\begin{matrix}\text{Left } \partial_* \mathrm{Id}_{\mathrm{Top}} \\ \text{modules}\end{matrix}\right\}$

$[\mathrm{Top}_*^{fin}, \mathrm{Top}_*] \xrightarrow{\partial_*} \{\partial_* \mathrm{Id}_{\mathrm{Top}_*}\text{-BiMod}\} \xrightarrow{\text{forget}} \left\{\begin{matrix}\text{Left } \partial_* \mathrm{Id}_{\mathrm{Top}_*} \\ \text{modules}\end{matrix}\right\}$

<u>Corollary.</u> Let $[C^{fin}, D] \underset{\phi}{\overset{\partial_*}{\rightleftarrows}} \mathcal{M}$ be one of the above with right adjoint ϕ. Then $K = \partial_* \phi : \mathcal{M} \to \mathcal{M}$ is a comonad and for any $F \in [C^{fin}, D]$ $\partial_* F$ has a K-coalgebra structure.

<u>Theorem.</u> ① If F is N-excisive for some N (or if $F \xrightarrow{\simeq} \operatorname{holim} P_n F$), then $F \xrightarrow{\simeq} \operatorname{cobar}(\phi, \partial_* \phi, \partial_* F)$ i.e. F can be recovered from $\partial_* F$ with its K-coalgebra structure.

② More generally, for any $F \in [C^{fin}, D]$

$$\begin{tikzcd} P_n F \arrow[r, "\simeq"] \arrow[d] & \operatorname{cobar}(\phi, \partial_* \phi, \partial_{\leq n} F) \arrow[d] \\ P_{n-1} F \arrow[r, "\simeq"'] & \operatorname{cobar}(\phi, \partial_* \phi, \partial_{\leq n-1} F) \end{tikzcd}$$

③ $$\operatorname{Nat}(F, G) \longrightarrow \operatorname{Map}_K(\partial_* F, \partial_* G)$$

space/spectrum of nat. trans. $F \to G$

space/spectrum of derived K-coalgebra maps $\partial_* F \longrightarrow \partial_* G$

is an equivalence, if $G \xrightarrow{\simeq} \operatorname{holim} P_n G$

(4) There is an equivalence of homotopy theories:

$$\left\{ \begin{matrix} N\text{-excisive} \\ F : C^{fin} \to D \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} N\text{-truncated} \\ K\text{-coalgebras} \end{matrix} \right\}$$

<u>Proof of (1).</u> Induction on Taylor Tower

$$\begin{array}{ccc} P_n F & \longrightarrow & \mathrm{Tot}\, P_n(\phi\, (\partial_* \phi)^\bullet\, \partial_* F) \\ \downarrow & & \\ P_{n-1} F & \longrightarrow & \mathrm{Tot}\, P_{n-1}(\phi\, (\partial_* \phi)^\bullet\, \partial_* F) \\ \downarrow & & \\ \Omega^{-1} D_n F & \xrightarrow[(*)]{} & \mathrm{Tot}\, \Omega^{-1} D_n(\phi(\partial_* \phi)^\bullet \partial_* F) \end{array}$$

<u>Claim:</u> for any n, $(*)$ is an equivalence.

<u>Proof of claim:</u> $D_n = \Psi_n \partial_*$, where $\Psi_n(A) = \Omega^\infty (A_n \wedge X^{\wedge n})_{h\Sigma_n}$

So $(*)$ is:

$$\Omega^{-1} \Psi_n \partial_* F \xrightarrow{\simeq} \mathrm{Tot}\,(\Omega^{-1} \Psi_n(\partial_* \phi)\, (\partial_* \phi)^\bullet\, \partial_* F)$$

using an extra codegeneracy (given by unit)

$$\begin{tikzcd}
F \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathrm{Tot}\left(\phi(\partial_* \phi)^\bullet \partial_* F\right) \arrow[d, "\simeq \text{ follows from}"] \\
P_N F \arrow[r] & \mathrm{Tot}\, P_N\left(\phi(\partial_* \phi)^\bullet \partial_* F\right)
\end{tikzcd}$$

<u>Lemma.</u> If A is n-truncated sym. seq., ϕA is N-excisive.

$\square$

Can we be more explicit about what K-coalgebra structure really is?

We need an explicit description of ϕ (the right adjoint to ∂_*)

$$[Sp^{fin}, Sp] \xrightarrow{\partial_*} Sp^{\Sigma}$$

Idea: P_n preserves hocolim.. of spectra valued functor

$$[Top_*^{fin}, Top_*] \xrightarrow{\partial_*} \partial_* Id\text{-}BiMod$$

preserves hocolim

$$
\begin{tikzcd}
F \arrow[d] \\
\Sigma^\infty F \Omega^\infty \arrow[d] \\
\partial_*(\Sigma^\infty F \Omega^\infty) \arrow[d, "\text{Koszul dual}"'] \\
\partial_* F
\end{tikzcd}
$$

Bicomodule over $\partial_*(\Sigma^\infty \Omega^\infty)$

$$\partial_* F = \mathrm{cobar}\left(\mathbb{1}, \partial_*(\Sigma^\infty \Omega^\infty), \partial_*(\Sigma^\infty F \Omega^\infty), \partial_*(\Sigma^\infty \Omega^\infty), \mathbb{1}\right)$$

$$[Sp^{fin}, Sp] \xrightarrow{\partial_*} Sp^{\Sigma}$$

Define ∂_* by left Kan extension from representable functors

<u>Dfn</u>: $X \in Sp^{fin}$ $\quad R_X : Sp^{fin} \to \mathcal{S}$

$$R_X(-) = \Sigma^\infty \mathrm{Hom}(X, -)$$

coYoneda Lemma. $R_X(-) \underset{X \in Sp^{fin}}{\wedge} FX \xrightarrow{\cong} F(-)$

$$\Sigma^\infty \mathrm{Hom}(X,Y) \underset{X}{\wedge} FX \longrightarrow F(Y)$$

So we define $\partial_* : [Sp^{fin}, Sp] \to Sp^\Sigma$

by $\partial_* F := (\partial_* R_X) \underset{X \in Sp^{fin}}{\wedge} FX$

which has a right adjoint

$$\phi : Sp^\Sigma \longrightarrow [Sp^{fin}, Sp]$$

$$\phi(A) : X \longrightarrow \mathrm{Map}_{Sp^\Sigma}(\partial_* R_X, A)$$

$$\| $$

$$\prod_{n \geq 1} \mathrm{Map}(\partial_n R_X, A_n)^{h\Sigma_n}$$

$$\|$$

$$\prod_{n \geq 1} (A_n \wedge X^{\wedge n})^{h\Sigma_n}$$

$$\partial_n R_X \simeq \mathbb{D} X^{\wedge n}$$

Classification of polynomial functors ADDENDA

by Michael Ching (8')

Case:

$$\mathrm{Top}_* \longrightarrow \mathrm{Spectra}$$

$$[\mathrm{Top}_*^{fin}, \mathrm{Spectra}] \underset{\Phi}{\overset{\partial_*}{\rightleftarrows}} \mathrm{Spectra}^{\Sigma}$$

$$K = \partial_* \Phi \ : \ \mathrm{Spectra}^{\Sigma} \longrightarrow \mathrm{Spectra}^{\Sigma}$$

$$\left\{ \begin{array}{c} n\text{-excisive} \\ F : \mathrm{Top}_*^{fin} \to \mathrm{Spectra} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} n\text{-truncated} \\ K\text{-coalgebras} \end{array} \right\}$$

$$F \longmapsto \partial_* F$$

$$R_X : \mathrm{Top}_*^{fin} \longrightarrow \mathrm{Spectra} \qquad X \in \mathrm{Top}_*^{fin}$$

$$R_X(-) := \Sigma^{\infty} \mathrm{Hom}(X, -)$$

$$\partial_n R_X := \mathbb{D}(X^{\wedge n} / \Delta^n X)$$

Define $\partial_*(F) := \partial_*(R_X) \underset{X \in T_{\varphi_*}^{\mathrm{fin}}}{\wedge} FX$

$$\Phi(A) : X \longmapsto \mathrm{Map}_{\mathrm{Spectra}^{\Sigma}}(\partial_* R_X, A)$$

$$\rotatebox{90}{$\cong$}$$

$$\prod_{n \geq 1} \mathrm{Map}(\partial_n R_X, A_n)^{h\Sigma_n}$$

we could take cofib. replacement of $\partial_* R_X$

$$\prod_{n \geq 1} (A_n \wedge X^{\wedge n}/\Delta^n X)^{h\Sigma_n}$$

$$\nwarrow N \text{ the norm map}$$

<u>Claim.</u> N is a weak equivalence. $\prod_{n \geq 1} (A_n \wedge X^{\wedge n}/\Delta^n X)_{h\Sigma_n}$

$$K = \partial_* \Phi : \mathrm{Spectra}^{\Sigma} \longrightarrow \mathrm{Spectra}^{\Sigma}$$

If A is N-truncated,

$$K(A) = \partial_* \prod_{n=1}^{N} (A_n \wedge X^{\wedge n}/\Delta^n X)_{h\Sigma_n}$$

$$\simeq \prod_{n=1}^{N} \Big(\partial_*(A_n \wedge X^{\wedge n}/\Delta^n X)\Big)_{h\Sigma_n}$$

$$X^{\wedge n}/\Delta^n X \simeq B(X^{\wedge *}, \mathrm{Com}, 1)(n)$$

"$B(R,P,1) \simeq R \circ^{\mathbb{L}}_{P} 1$"

$M \otimes_A k$

5/17/12 3.
MC
Adalonda

Com : commutative operad in $Top_{\otimes}$

$Com(n) = S^0$ for all $n \in \mathbb{N}$

$X^{\wedge *}$: right Com-module (equivalently: $Epi^{op} \longrightarrow Top_{*}$)

fin. sets with surjections

$$X^{\wedge k} \wedge S^0 \wedge S^0 \wedge \ldots \wedge S^0 \longrightarrow X^{\wedge n}$$

where $n \twoheadrightarrow k$

$$X \wedge X \to X \wedge X \wedge X$$

$$(x, y) \longmapsto (x, x, y)$$

$B(X^{\wedge *}, Com, 1)$: right comodule over cooperad $B(1, Com$

$DB(X^{\wedge *}, Com, 1)$: right module over $\partial_* I$

$$\cong \mathbb{D}(X^{\wedge n}/\Delta^n X) \simeq \partial_* R_X$$

$$K(A) \simeq \prod_{n=1}^{N} \left(\partial_*(A_n \wedge \Sigma^\infty X^{\wedge n}/\Delta^n X) \right)_{h\Sigma_n}$$

$$\simeq \prod_{n=1}^{N} \left(A_n \wedge B(\partial_r(\Sigma^\infty X^{\wedge \bullet}), Com, 1)(n) \right)_{h\Sigma_n}$$

$$\simeq \prod_{n=1}^{N} \left(A_n \wedge B\left(\begin{bmatrix} \vdots \\ * \\ \Sigma_r \\ * \\ \vdots \end{bmatrix}, Com, 1 \right)(n) \right)_{h\Sigma_n}$$

$$B\left(\begin{bmatrix} \vdots \\ * \\ \Sigma_r \\ * \\ \vdots \end{bmatrix}, Com, 1 \right)(n) \simeq \coprod_{n = n_1 + \dots + n_r} B(1, Com, 1)(n_1) \wedge \dots \wedge B(1, Com, 1)(n_r)$$

coproduct

product in Spectra

really unordered partitions of $\underline{n}$ into r pieces

$$K_r(A) \simeq \prod_{n=1}^{N} \left(\coprod_{n_1 + \dots + n_r = n} A_n \wedge B(1, Com, 1)(n_1) \wedge \dots \wedge B(1, Com, 1)(n_r) \right)_{h\Sigma_n}$$

$$\simeq \prod_{n=1}^{N} \left(\prod_{n = n_1 + \dots + n_r} Map(\partial_{n_1} I \wedge \dots \wedge \partial_{n_r} I, A_n) \right)_{h\Sigma_n}$$

A K-coalgebra structure on A consists of

$$A \longrightarrow KA$$

$$\begin{array}{c} A_r \longrightarrow \left(\prod_{\substack{n=n_1+\dots+n_r \\ \text{i.e. } \underline{n} \to \underline{r}}} \mathrm{Map}(\partial_{n_1} I \wedge \dots \wedge \partial_{n_r} I, A_n) \right)_{h\Sigma_n} \\ \Big\downarrow N \\ \left[\prod_{n=n_1+\dots+n_r} \mathrm{Map}(\partial_{n_1} I \wedge \dots \wedge \partial_{n_r} I, A_n \right]^{h\Sigma_n} \end{array}$$

This composite gives Σ_n-equivalent maps

$$A_r \longrightarrow \prod_{n=n_1+\dots+n_r} \mathrm{Map}(\partial_{n_1} I \wedge \dots \wedge \partial_{n_r} I, A_n)$$

i.e.

$$A_r \wedge \partial_{n_1} I \wedge \dots \wedge \partial_{n_r} I \longrightarrow A_n$$

i.e. a right $\partial_* I$-module structure

We refer to a K-coalgebra or a <u>divided power</u> right $\partial_* I$-module.

Kerstin Baer, Orthogonal calculus : theory (9)

We will consider $E : \mathcal{J} \to Top$

$\mathcal{J} = \left\{ \begin{array}{l} \text{finite dimensional inner product} \\ \text{subspaces of } \mathbb{R}^\infty \end{array} \right\}$

$mor(V, W) = O(V, W)$ embeddings

We require E to be cts : $mor(V, W) \times E(V) \to E(W)$

Examples:

$O(V)$

$BO(V)$

$Conf(n, V)$

$Emb(M, N)$

$\Omega^\infty(V^c \wedge \Theta)$

$\Omega^\infty(((\mathbb{R}^n \otimes V)^c \wedge \Theta)_{hO(n)})$

Def: Let $\mathcal{E} = Cat(\mathcal{J}, Top)$. $E \in \mathcal{E}$ is a polynomial of degree n, if $E(V) \longrightarrow \underbrace{\operatorname{holim}_{0 \neq U \subseteq \mathbb{R}^{n+1}} E(U \oplus V)}_{\tau_n E(V)}$ is a htpy equivalence for all $V \in \mathcal{J}$.

Taylor Polynomials : $T_n E = \operatorname{hocolim}(E \to \tau_n E \to \tau_n^2 E \to \dots)$

$T_n E : \mathcal{E} \to \mathcal{E}$

$\eta_n : T \to T_n$

Remarks. (a) Every polynomial of degree $n-1$, then it is a polynomial of degree n.

(b) $T_n E$ is a polynomial of degree n

(c) If E is a polynomial of deg n, then

$$\eta_n : E \longrightarrow T_n E$$

is an equivalence.

(d) $T_n(\eta_n) : T_n E \longrightarrow T_n^2 E$ is an equivalence

Universality of T_n

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & P_n \\
T_n E \arrow[ur, dashed] &
\end{tikzcd}$$

$\exists!$ up to htpy ?

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & P_n \arrow[d, "\simeq"] \\
T_n E \arrow[r] \arrow[ur, dashed] & T_n P_n
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\eta_n"] \arrow[d, "\eta_n"'] \arrow[rr, bend left] & T_n E \arrow[r] \arrow[d, "\simeq"', "\eta_n(\eta_n)"] & P_n \arrow[d, "\simeq"] \\
T_n E \arrow[r, "\simeq", "T_n\eta_n"'] \arrow[rr, bend right] & T_n^2 E \arrow[r, "g"] & T_n P_n
\end{tikzcd}$$

Corollary.

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
 & T_2E \arrow[d] & \\
E \arrow[ur] \arrow[r] \arrow[dr] & T_1E \arrow[d] & D_1E \arrow[l] \\
 & T_0E &
\end{tikzcd}$$

Def: E is homogeneous of degree n, if $E \xrightarrow{\simeq} \tau_n E$ and $T_{n-1}E \simeq *$.

Thm. If $E \in \mathcal{E}_0$, then

$$D_nE \simeq \Omega^\infty\left(\left((\mathbb{R}^n \otimes V)^c \wedge \Theta\right)_{hO(n)}\right)$$

Def: $$\operatorname{hofib}\left(E(V) \longrightarrow \tau_n E(V)\right) =: E^{(n+1)}(V)$$

$$E^{(1)}(V) = \operatorname{hofib}\left(E(V) \longrightarrow E(\mathbb{R} \oplus V)\right)$$

$$E(V) = BO(V)$$

$$E^{(1)}(V) = \operatorname{hofib}\Big(\underset{\displaystyle \frac{EO(\mathbb{R}\oplus V)}{O(n)}}{\underset{\|}{BO(V)}} \longrightarrow \underset{\displaystyle \frac{EO(\mathbb{R}\oplus V)}{O(n+1)}}{\underset{\|}{BO(\mathbb{R}\oplus V)}}\Big) = \frac{O(n+1)}{O(n)} \simeq V^c$$

Define: $\mathrm{mor}_n(V,W) = \mathrm{Thom}\left\{ (f,x) \;\middle|\; \begin{matrix} f \in \mathrm{mor}(V,W) \\ x \in n\cdot \mathrm{coker} f \end{matrix} \right\}$

$J_{(n)} := J$ with mor_n

$J_m \hookrightarrow J_n$ for all $m \leqslant m$

$\mathcal{E}_{(n)} = \{ J_n \to \mathrm{Top}_* \mid \text{cts, pointed} \}$

$$\begin{aligned} \mathrm{mor}_n(V,W) &\longrightarrow \mathrm{Map}_*(E(V_1), E(V_2)) \\ * &\longmapsto \mathrm{const}_* \end{aligned}$$

The inclusion $J_m \hookrightarrow J_n$ induces a restriction

$$\mathrm{res}^n_m : \mathcal{E}_n \longrightarrow \mathcal{E}_m$$

The map res^n_m has a right adjoint ind^n_m

$E^{(n)} = \mathrm{ind}^n_0 E = \mathrm{nat}_0(\mathrm{mor}_n(V,-), E(-))$

If $E \in \mathcal{E}_0$, then $E^{(n)} \in \mathcal{E}_n$.

$$\mathrm{mor}_n(V,W) \wedge E^{(n)}(V) \longrightarrow E^{(m)}(V)$$

Setting $W = \mathbb{R} \oplus V$ we get $\mathrm{mor}_n(V, \mathbb{R}\oplus V) \wedge E^{(n)}(V) \to E^{(n)}(\mathbb{R}\oplus V)$

$$S^n \wedge E^{(n)}(V) \longrightarrow E^{(n)}(\mathbb{R}\oplus V)$$

$$\left(\Theta E^{(n)}\right)_{k\cdot n} := E^{(n)}(\mathbb{R}^k)$$

$$\begin{tikzcd}
E^{(1)}(V) \arrow[r] \arrow[d] & E(V) \arrow[r] \arrow[d] & E(\mathbb{R}\oplus V) \arrow[d] \\
E^{(1)}(\mathbb{R}\oplus V) \arrow[r] & E(\mathbb{R}\oplus V) \arrow[r] & E(\mathbb{R}^2\oplus V)
\end{tikzcd}$$

$$E(V) = \Omega^\infty\left(\left((\mathbb{R}^n\otimes V)^c \wedge \Theta\right)_{hO(n)}\right)$$

$$\Theta E^{(k)}(V) = \begin{cases} * & \text{if } k\neq n \\ \Theta & \text{if } k=n \end{cases}$$

Sean Tilson, Orthogonal calculus: Examples (10)

① Derivatives of $BO(V)$ & $BU(V)$

② Derivatives of $\Sigma^\infty C(t_y V)$, $\Sigma^\infty \bar{C}(M, N)$

③ $Q\, Mor(V_0, V)_+$

Thm. (Arone) The n-th derivative of $Aut(V)$ is $Map_*(L_n^F, \Sigma^\infty S^{Ad_n})$, where $S^{Ad_n} = (Ad_{Aut(F^n)})^c$

Def: $\mathcal{L}_n^F$ is the poset of direct sum decompositions of F^n. $\Lambda \subseteq \Lambda'$, if every summand of Λ is a subspace of a summand of Λ'

Def: $\mathcal{O}_{k,n}$ decreasing chains in $\mathcal{L}_n$

$(\lambda_0, \lambda_1, \dots, \lambda_k)$, $\lambda_F = F^n = 1$ with a basepoint

s_i = repeats i^{th} guy

d_i = omits i^{th} guy if $i \neq k$

$i = k$, then $d_k(\tau) = \begin{cases} * & \text{if } \lambda_{k-1} \neq 1 \\ (\lambda_0, \dots, \lambda_{k-1}) & \text{if } \lambda_{k-1} = 1 \end{cases}$

$|\mathcal{O}_{\bullet, n}| = L_n^F$, where $L_n^F = \Sigma |\mathcal{L}_n - \{\text{final objects}\}|$

Thm. ① Exists a $O(n-1)$-equivariant weak equivalence

$$\mathrm{Map}_*(L_n^{\mathbb{R}}, \Sigma^\infty S^{Ad_n^{\mathbb{R}}}) \simeq \mathrm{Map}(S^1 \wedge K_n, \Sigma^\infty S^0) \wedge_{\Sigma_n} O(n-1)$$

② Exists a $U(n-1)$-equivariant weak equivalence

$$\mathrm{Map}_*(L_n^{\mathbb{C}}, \Sigma^\infty S^{Ad_n^{\mathbb{C}}}) \simeq \mathrm{Map}_*(S^1 \wedge K_n, \Sigma^\infty S^{-n}) \wedge_{\Sigma_n} U(n-1)$$

where $K_n = |\mathrm{Part}(n) \setminus \{\text{initial \& final}\}|$

$$\Sigma^\infty C(k,V) \quad \text{or} \quad H\mathbb{Z} \wedge C(k,V)$$

$$C(k,V) = \mathrm{Emb}(\underline{k}, V) = V^k \setminus \Delta^k V$$

$$\Delta^k V = \{\bar{x} \in V^k \mid x_i = x_j \text{ for some } i \neq j\}$$

$$\Delta^k V = \bigcup_{\Lambda \in P_k^\circ} V^{c(\Lambda)} \quad \text{where} \quad c(\Lambda) = k/\Lambda$$

$$C(k,V) = \bigcap_{\Lambda \in P_k^\circ} V^k \setminus V^{c(\Lambda)} = \lim_{\Lambda \in P_k^\circ} V^k \setminus V^{c(\Lambda)}$$

Proposition: $\Sigma^\infty C(k,V) \longrightarrow \operatorname*{holim}_{\Lambda \in P_k^\circ} \Sigma^\infty V^k \setminus V^{c(\Lambda)}$

Proof: Just sketch ... Change the indexing category

$S = 2^{\binom{k}{2}}$ graphs with k vertices

5/16/12
ST

$$S \longrightarrow \mathcal{P}_k$$

$$U \longmapsto \Lambda(U) = \text{path components of } U$$

$$S' = S \setminus \{\emptyset\} \longrightarrow \mathcal{P}_k^{\,0}$$

$$F : \mathcal{P}_k^{\,0} \longrightarrow \text{Spectra}$$

$$\tilde{F} : S' \longrightarrow \mathcal{P}_k^{\,0} \xrightarrow{F} \text{Spectra}$$

$\operatorname{holim}_{\mathcal{P}_k^0} F \longrightarrow \operatorname{holim}_{S'} \tilde{F}$ is a weak equivalence

$\mathcal{X} : S' \longrightarrow$ Spectra is an n-cube

$$U \longmapsto \tilde{F}(U) = F(\Lambda(U))$$

$$\Sigma^{\infty} V^k \setminus V^{c(\Lambda(U))}$$

$$\mathcal{X} : S \longrightarrow \text{Top}$$

$$U \longmapsto \begin{cases} \mathcal{E}(k, V) & \text{if } U = \emptyset \\ V^k \setminus V^{c(\Lambda(U))} & \text{oth.} \end{cases}$$

Fact. ① $\forall U \subseteq \binom{k}{2}$ $\quad \mathcal{X}(U) \overset{\text{open}}{\subseteq} \mathcal{X}\left(\binom{k}{2}\right) = V^k \setminus V$

② $\forall U \subseteq \binom{k}{2}$ non-empty, $\quad \mathcal{X}(U) = \bigcup_{x \in U} \mathcal{X}(\{x\})$

③ $\mathcal{X}(\emptyset) = \bigcap_{x \in \binom{k}{2}} \mathcal{X}(\{x\})$

<u>Lemma</u>. Let X is a space with $X_1, \dots, X_k$ open in X and such that $X = \bigcup X_i$, $X_0 = \bigcap X_i$.
If $\mathcal{X}(U) = \bigcup_{i \in U} X_i$, then $\mathcal{X}$ is a homotopy pushout

$\mathcal{X}$ is htpy p.o.

$\Rightarrow \Sigma^\infty \mathcal{X}$ is htpy p.o. as well

$\Rightarrow \Sigma^\infty \mathcal{X}$ is htpy p.b.

$$\Sigma^\infty \mathcal{X}(\emptyset) \xrightarrow{\simeq} \operatorname{holim}_{S'} \Sigma^\infty \mathcal{X}(U) \xrightarrow{\simeq} \operatorname{holim}_{P_k^0} \Sigma^\infty V^k \setminus V^{c(\Lambda)}$$

$$T_n \Sigma^\infty C(k, V) = T_n \operatorname{holim}_{\Lambda \in P_k^0} \Sigma^\infty V^k \setminus V^{c(\Lambda)}$$

$$= \operatorname{holim}_{\Lambda \in P_k^0} T_n \Sigma^\infty V^k \setminus V^{c(\Lambda)}$$

$$V^k \cong V^{e(\Lambda)} \oplus V^{c(\Lambda)} \qquad V^k \setminus V^{c(\Lambda)} = V^{e(\Lambda)} \setminus 0$$

$$e(\Lambda) = k - c(\Lambda) \qquad \simeq S^{e(\Lambda)}$$

$$\Sigma^{\infty} V^k \backslash V^{c(\Lambda)} \simeq \Sigma^{-1}\Sigma^{\infty} S^{e(\Lambda)\cdot V}$$ homogeneous

$$T_n \Sigma^{\infty} V^k \backslash V^{c(\Lambda)} = \begin{cases} * & \text{if } e(\Lambda) > n \\ \Sigma^{-1}\Sigma^{\infty} S^{e(\Lambda)\cdot V} & \text{if } e(\Lambda) \leq n \end{cases}$$

$$T_n \Sigma^{\infty} C(k,V) = \operatorname*{holim}_{\substack{\Lambda \in P^{\circ} \\ e(\Lambda)\leq n}} \Sigma^{-1}\Sigma^{\infty} S^{e(\Lambda)\cdot V}$$

$$\mathbb{D}_n \Sigma^{\infty} C(k,V) = \operatorname*{holim}_{\substack{\Lambda \in P^{\circ}_k \\ e(\Lambda)\leq n}} F(\Lambda) = \begin{cases} * & \text{if } e(\Lambda) < n \\ \Sigma^{-1}\Sigma^{\infty} S^{e(\Lambda)\cdot V} & \text{if } e(\Lambda) = n \end{cases}$$

with a
little bit
of thought

Dan Berwick-Evans, Intro to Embedding Calculus (II)

Goal: $F : \mathcal{O}(M)^{op} \longrightarrow Top$ understand good of these

Examples. (1.) $Emb(-, N)$ $m \leq n$ ("m" = dim M, "n" = dim N)

(2) $Imm(-, N)$

Let $\mathcal{F}$ be the category of "good" sub functors

Thm. $T_1 Emb(-, N) \simeq Imm(-, N)$ (it means)

Thm. The following are analytic:

(1) $Emb(-, N)$, if $n - m \geq 3$ (if $n > m$ or M has no compact component)

(2) $Imm(-, N)$ (· if $n = m$, M has no compact components
· if $n > m$, no conditions etc.)

We can understand $T_k Emb(-, N)$ as sheafifications.

Sheaves.

Def: A <u>presheaf</u> is a functor $F : \mathcal{O}(M)^{op} \to \mathcal{C}$

A presheaf is a <u>sheaf</u>, when

$$FU \xrightarrow{\sim} \operatorname{holim}\left(\prod_i FU_i \rightrightarrows \prod_{i,j} FU_{ij} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \right)$$

for $\{U_i \to U\}_i$ a covering.

Def: $\{U_i \to U\}_{i \in I}$ is a $\mathcal{J}_k$-covering, if $\forall S \subset M$ $|S| < k$, $\exists_i$ $S \subseteq U_i$.

We can sheafify w.r.t. $\mathcal{J}_k$-coverings

$$\begin{tikzcd} F \arrow[rr] \arrow[dr] & & G \\ & (F)_k \arrow[ur, dashed, "\exists!"'] & \end{tikzcd}$$

$$Sh_{\mathcal{J}_k}((F)_k, G) \simeq PrSh(F, U(G))$$

$$(F)_k(U) = \operatorname*{hocolim}_{\substack{\mathcal{U} \in \mathcal{J}_k \\ \| \\ \{U_i \to U\}}} \operatorname{holim}\left(\prod_i FU_i \rightrightarrows \prod_{i,j} FU_{ij} \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \cdots \right)$$

Polynomial Functors.

Let $A_0, \dots, A_k \underset{\text{closed}}{\subseteq} U$ $\quad A_i \cap A_j \overset{i \neq j}{=} \emptyset$

$$\begin{aligned} \chi : P_{k+1} &\longrightarrow \mathrm{Top} \\ \rotatebox{90}{$\in$} \quad & \\ S &\longmapsto F\Big(U \setminus \bigcup_{i \in S} A_i\Big) \end{aligned}$$

Rmk. This is strongly coCartesian.

Polynomial of degree $\leq k$ means that those cubes go to cartesian ones under F.

Thm. F is a sheaf w.r.t. $\mathcal{J}_k$-coverings iff F is polynomial of deg $\leq k$.

Cor. $T_k F \simeq (F)_k$

k^{th} polynomial approximation

sheafification w.r.t. $\mathcal{J}_k$

Idea of proof:

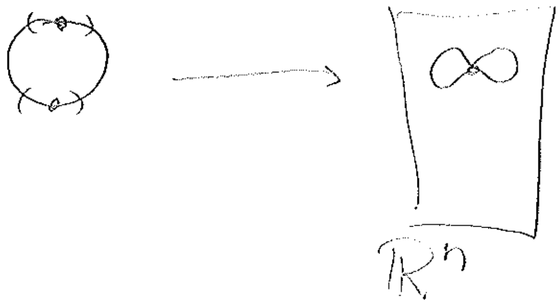
$$T_k F(M) \simeq \operatorname*{holim}_{U \in \mathcal{O}_k(M)} F(M) \simeq (F)_k (M)$$

$\mathcal{J}_k$-sheaves are determined by values on $\mathcal{O}_k(M)$

$$\mathcal{O}_k(M) := \{ \coprod_j \mathbb{R}^m \longrightarrow M \mid j \leq k \} \subseteq \mathcal{O}(M)$$

Some pictures: looking at $Emb(-, N)$

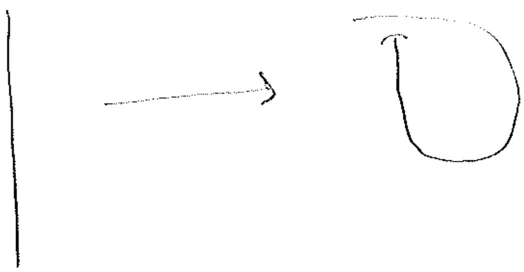
① $M = S^1$, $N = \mathbb{R}^n$



$\in T_1 \, Emb(M, N)$

$\notin T_2 \, Emb(M, N)$

? ② $M = \mathbb{R}$, $N = \mathbb{R}^n$



$\in T_k \, Emb(M, N)$ k finite

$\notin \underbrace{T_\infty \, Emb(M, N)}_{\text{limit of the tower}} \simeq Emb(M, N)$

Layers.

Start off by choosing a basepoint in $F(M)$

| for Emb choose $M \subseteq N$

$$L_k F U := \mathrm{hfib}\left(T_k F(U) \longrightarrow T_{k-1} F(U)\right)$$

$$L_k Emb(M, N) = \Gamma_\circ \left(\begin{array}{c} E_k \\ \downarrow \\ \binom{M}{k} \end{array} \right)$$

sections vanishing near fat diagonal

configuration space i.e. k points in M

$$\pi_k^{-1}(s) = \text{total hfib}(\mathcal{X})$$

$$\begin{array}{ccc} \mathcal{X} : P_{|S|} & \longrightarrow & \mathrm{Top}_* \\ \rotatebox{90}{$\in$} & & \\ S \supseteq R & \longmapsto & \mathrm{Emb}(R, N) \end{array}$$

<u>Convergence:</u>

$$\mathrm{Emb}(M,N) \longrightarrow T_k \mathrm{Emb}(M,N)$$

is $(3-n+k(n-m+2))$-connected

So for $n-m \geq 3$ $\mathrm{Emb}(M,N) \xrightarrow{\simeq} T_\infty \mathrm{Emb}(M,N)$

Greg Arone, Multiple disjunctions lemmas (12)

Thm. (Blacker-Massey) Let $X_\bullet$ a k-dimensional cubical diagram. If $X_\bullet$ is strongly cocartesian and the maps $X_0 \longrightarrow X_i$ are k_i-connected for $i=1,\dots,n$, then $X_\bullet$ is $1-k+\sum_{i=1}^{k} k_i = 1+\sum (k_i-1)$ - cartesian

Corollary. If X is d-connected, then the map $X \longrightarrow P_n(\mathrm{Id})(X)$ is $((n+1)d+1)$-connected.

Let $L_1, \dots, L_k, N$ be manifolds

Let $L_\bullet$ be the k-dimensional cube

$$S \longmapsto \coprod_{i\in S} L_i$$

Consider the cubical diagram $\mathrm{Emb}(L_\bullet, N)$

Let l_i be the ~~dimension~~ handle index of L_i and n be the dimension of N

Thm. The cube $\mathrm{Emb}(L_\bullet, N)$ is $3-n+\sum_{i=1}^{k}(n-l_i-2)$-cartesian.

Cor. The map $\mathrm{Emb}(M,N) \to T_k \mathrm{Emb}(M,N)$ is $(3-n+(k+1)(n-m-2))$-connected.

Easy multiple disjunction lemma. The ~~embe~~ $\widetilde{Emb}(L_\cdot, N)$

is $\left(3-n+\sum(n-2l_i-2)\right)$-cartesian

I think that easy disjunctions holds with
GA

the word "cartesian" replaced by "cocartesian"

$$\Sigma^\infty Emb(M,N)$$

$$\overline{Emb} = hfib(Emb \longrightarrow Imm)$$ is of some dim. so what?

Proof of Blakers-Massey:

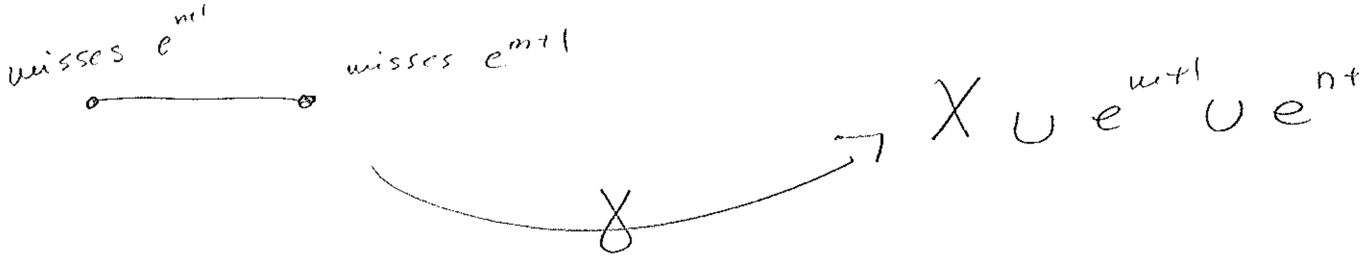
$$\begin{tikzcd}
X \arrow[rr, hook, "m\text{-connected}"] \arrow[dr, dashed] \arrow[dd, hook, "n\text{-connected}"'] & & X \cup e^{m+1} \arrow[dd] \\
& P \arrow[ur] \arrow[dl] & \\
X \cup e^{n+1} \arrow[rr] & & X \cup e^{m+1} \cup e^{n+1}
\end{tikzcd}$$

Want: the square $(m+n-1)$-cartesian.

(P, X) is $(m+n-1)$-connected.

$\pi_i(P, X) = 0$ for $i \leq m+n-1$.

$P = \{\gamma : [0,1] \to X \cup e^{n+1} \cup e^{m+1} \mid \gamma(0) \in e^{m+1}, \gamma(1) = e^{n+1}\}$



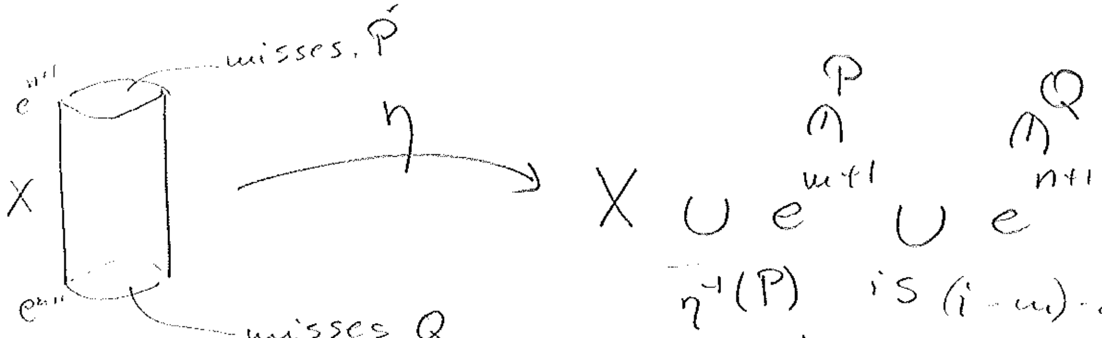
What represents an element of $\pi_i(P, X)$

A map $\eta : D^i \times [0,1] \to X \cup e^{m+1} \cup e^{n+1}$

$$\partial D^i \times I \longrightarrow X$$

$$D^i \times \{0\} \longrightarrow X \cup e^{m+1}$$

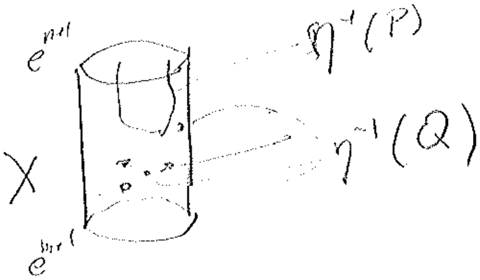
$$D^i \times \{1\} \longrightarrow X \cup e^{n+1}$$



$\eta^{-1}(P)$ is $(i-m)$-dimensional
$\eta^{-1}(Q)$ is $(i-n)$-dimensional

Without loss of generality we may assume that $P \in e^{m+1}$, $Q \in e^{n+1}$ are such that η is smooth on η^{-1}(small neighborhoods of P and Q)

$$\dim(\eta^{-1}(P)) + \dim(\eta^{-1}(Q)) = 2i - m - n \le i - 1.$$



$\underline{k=3:}$

$$\begin{tikzcd}[row sep=small, column sep=small]
 & X\cup e^{m+1} \arrow[rr] \arrow[dd] & & X\cup e^{m+1}\cup e^{n+1} \arrow[dd] \\
X \arrow[ur, "m\text{-conn}"] \arrow[rr, "n\text{-conn}"] \arrow[dd, "r\text{-conn}"'] & & X\cup e^{n+1} \arrow[ur] \arrow[dd] & \\
 & X\cup e^{m+1}\cup e^{r+1} \arrow[rr] & & X\cup e^{m+1}\cup e^{n+1}\cup e^{r+1} \\
X\cup e^{r+1} \arrow[ur] \arrow[rr] & & X\cup e^{n+1}\cup e^{r+1} \arrow[ur] &
\end{tikzcd}$$

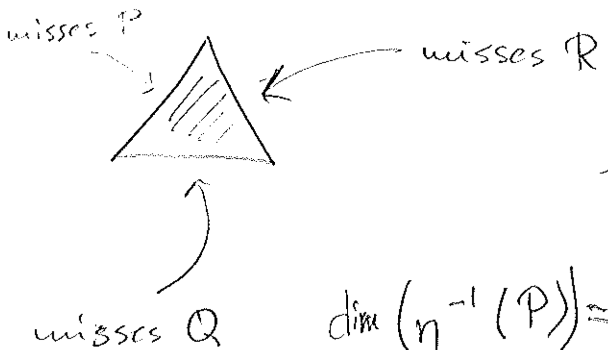
The cube is:

$(m+n+r-2)$ - cartesian

In particular, it is 0-cartesian, if

$m+n+r \geq 2$

A point in the homotopy pullback



$$\xrightarrow{\eta} X\cup e^{m+1}\cup e^{n+1}\cup e^{r+1}$$

$$\begin{aligned}
&\operatorname{dim}(\eta^{-1}(P)) = 1-m \\
&\operatorname{dim}(\eta^{-1}(Q)) = 1-n \\
&\operatorname{dim}(\eta^{-1}(R)) = 1-r \\
\hline
&3-m-n-r \leq 1
\end{aligned}$$

Ψ
P

$\boxed{\text{Picture}}$ by assumption

Multiple disjunctions:

$$\begin{tikzcd}
\mathrm{Emb}(L_1, N-L_2) \arrow[r] \arrow[d] & \mathrm{Emb}(L_1 \amalg L_2, N) \arrow[r] \arrow[d] \arrow[dl, phantom, "\mathrm{LHS}"] & \mathrm{Emb}(L_2, N) \arrow[d] \arrow[dl, phantom, "\mathrm{RHS}"] \\
\mathrm{Emb}(L_1, N) \arrow[r] & \mathrm{Emb}(L_1, N) \arrow[r] & \mathrm{Emb}(\emptyset, N) = *
\end{tikzcd}$$

RHS

<u>Equivalent statement:</u> Let $M, N, L_1, \ldots, L_n$ be manifolds, $L_i \subseteq N$ disjoint ($L_i \cap L_j \overset{i \neq j}{=} \emptyset$)

The cube $\mathrm{Emb}(M, N - L_\bullet)$ is

$$\left(1 + \sum (n - m - l_i - 2)\right)\text{- cartesian}$$

<u>Claim 1.</u> The cube $\mathrm{Emb}(M, N \setminus L_\bullet)$ is ~~[illegible]~~ cocartesian

<u>Claim 2.</u>

$$\begin{tikzcd}
\mathrm{Emb}(M, N \setminus (L_1 \cup L_2)) \arrow[r, "(n-m-l_2-1)\text{-connected}"] \arrow[d] & \mathrm{Emb}(M, N \setminus L_1) \arrow[d] \\
\mathrm{Emb}(M, N \setminus L_1) \arrow[r] & \mathrm{Emb}(M, N)
\end{tikzcd}$$

5/17/12 6.
G/A

Strong disjunctions:

Tom Goodwillie: true except on π_0

Goodwillie-Klein: proved for π_0, (Poincaré embeddings)

Compare:

$\mathrm{Emb}(M,N)$ $\qquad\qquad$ $\mathrm{Map}(K,X)$

vs.

$\Sigma^\infty \mathrm{Emb}(M,N)$ $\qquad\qquad$ $\Sigma^\infty \mathrm{Map}(K,X)$

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
\mathrm{Emb}(M,N) \arrow[r] \arrow[dr] & T_{k+1}\mathrm{Emb}(M,N) \arrow[d] \\
 & T_k \mathrm{Emb}(M,N) \arrow[d] \\
 & \vdots
\end{tikzcd}$$

this map is as connected as this one

Alexander Kupers, Embedding calculus, little disks operad, spaces of embeddings (13)

$F: \mathcal{O}(M)^{op} \longrightarrow Top$ good isotopy

$T_k F(M) = (F)_k(M) = \underset{U \in \mathcal{O}_k(M)}{\mathrm{holim}} F(U)$

sheaf w.r.t. $\mathcal{J}_k$

poset of open subsets of M homeomorphic to a disjoint union of $\leq k$ balls

Little n-disks operad $\quad \mathbb{B}_n(k) = sEmb(\coprod_k D^n, D^n)$

Goals of the lecture:

① If M^m open submanifold of $\mathbb{R}^n$, F "context-free"

we seek general expression for $T_k F(M)$ in terms of module

② $H\mathbb{Q} \wedge Emb = hfib(Emb \to Imm)$

Let M open submanifold of $\mathbb{R}^n$.
A standard ball in M is a ball in $\mathbb{R}^n$ that is also contained in M

$$\mathcal{O}_k^s(M) \hookrightarrow \mathcal{O}_k(M)$$

<u>Thm.</u> This inclusion induces a weak equivalence

$$\operatorname*{holim}_{U \in \mathcal{O}_k(M)} FU \xrightarrow{\simeq} \operatorname*{holim}_{V \in \mathcal{O}_k^s(M)} FV$$

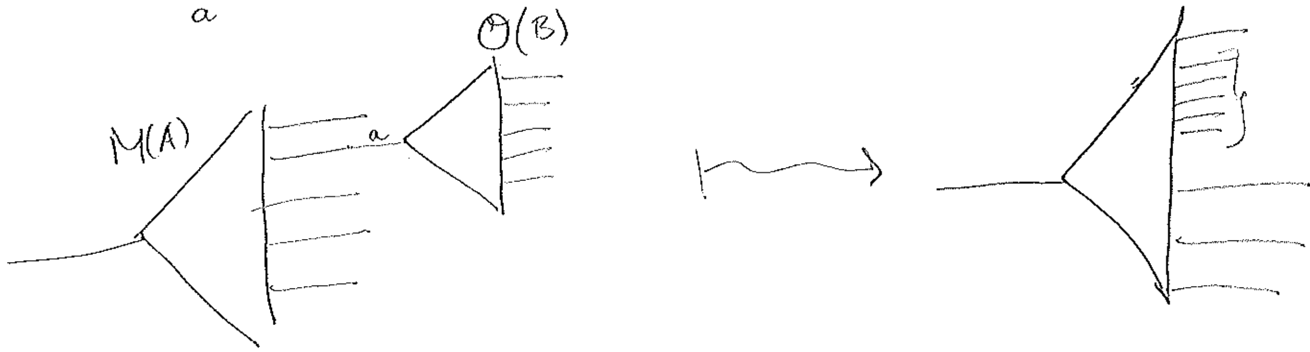
Some operad theory

$\mathcal{O}$ operad $\rightsquigarrow$ $F\mathcal{O}$ "$\mathcal{O}$-labelled forest"

+ some bullshit

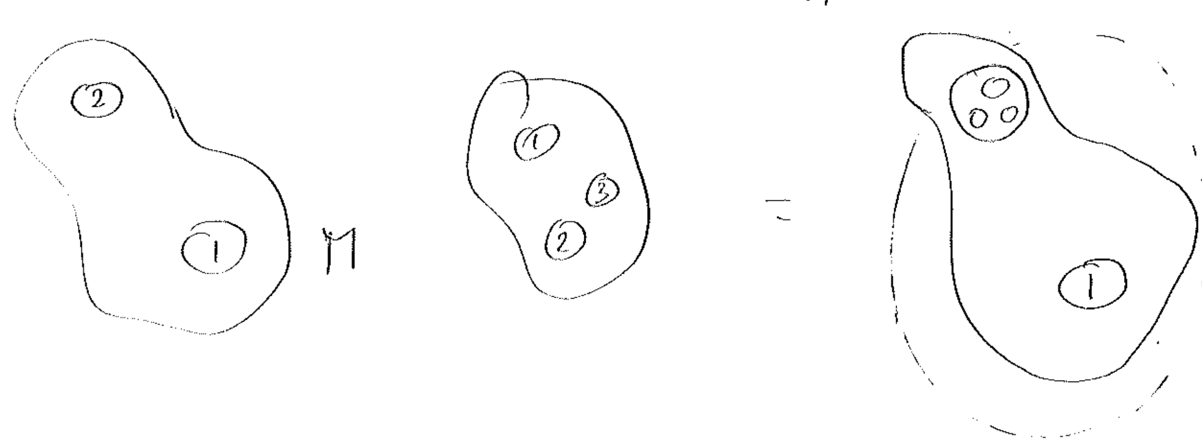
Def: A (weak) right module over $\mathcal{O}$ is a symmetric sequence M with composition maps

$$- \circ_a - : M(A) \otimes \mathcal{O}(B) \to M(A \cup_a B)$$



Examples: Every operad is a right module over itself

Example: $M(A) = s\mathrm{Emb}(A \times D^m, M)$ is a right module over $\mathbb{B}_m$



Lemma. There is an equivalence of categories:

$$\left\{\begin{matrix}\text{right modules} \\ \text{over } \mathcal{O}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{functors} \\ \mathrm{Fun}\left(F(\mathcal{O})^{op}, \mathcal{D}\right)\end{matrix}\right\}$$

Proof:

$$M \longmapsto M(A) =: \mathcal{M}(A)$$

$$\mathcal{M}(B) \otimes \coprod_{f:A\to B} \bigotimes_{b\in B} \mathcal{O}(f^{-1}(b)) \to \mathcal{M}(A)$$

given by using $-\circ_b-$ repeatedly

$$M(A) := \mathcal{M}(A) \longleftarrow\!\!| \ \mathcal{M}$$

$\square$

Lemma. $\int_{s\mathrm{Emb}^{\delta}(-,M)} F(\mathbb{B}_n^{\delta}) \xrightarrow{ev^{\delta}} \mathcal{O}_{\infty}^{\delta}(M)$ is an equivalence of categories

5/17/12 6.
AK

$$T_n F(M) = \operatorname*{holim}_{U \in \mathcal{O}_k^{\delta}(M)} F U$$

$$= \operatorname*{holim}_{\substack{(c,x)\in \int sEmb^{\delta}(-,M) \\ F(\mathbb{B}_m^{\delta})_{\leq n}}} F(\mathrm{im}(x))$$

~~Def: If F is~~

$$\mathcal{C}^{op} \underset{G}{\overset{F}{\rightrightarrows}} \mathcal{M}$$

model category

$$\operatorname*{holim}_{\int_{\mathcal{C}} F} G \circ \pi_{\mathcal{C}} \simeq \mathrm{hNat}_{\mathcal{C}}(F, G)$$

$$\int_{\mathcal{C}} F \xrightarrow{\pi_{\mathcal{C}}} \mathcal{C}$$

$$\overline{Emb}(M,V) \simeq \mathrm{hfib}\big(Emb(M,V) \longrightarrow Imm(M,V)\big)$$

euclidean

$$M = \coprod_k D^n, \quad V = \mathbb{R}^n$$

$$Imm(M,V) \simeq \prod_k \big(GL_n(\mathbb{R}) \times V\big)$$

$$Emd(M,V) \simeq \prod_k \big(GL_n(\mathbb{R}) \times C(k,V)\big)$$

$$\overline{Emb}(M,V) \simeq C(k,V)$$

Hiro Lee Tanaka, Factorization homology (14)

(and Manifold Calculus)

Def: Mfld is a Top-enriched category whose objects are n-manifolds and whose morphisms are defined via $\mathrm{Hom}(X, Y) = \mathrm{Emb}(X, Y)$.

Everything here is smooth.

We did this:

$$F : \mathcal{O}(M)^{op} \longrightarrow \mathcal{E}$$

But we would be better off with:

$$F : (\mathrm{Mfld}/M)^{op} \longrightarrow \mathcal{E}$$

But really we want to study that:

$$F : \mathrm{Mfld}^{op} \longrightarrow \mathcal{E}$$

Def: Let Disk be the full subcategory of Mfld, whose objects are of the form $\coprod_j \mathbb{R}^n$ for $0 \leq j < \infty$.

Def:

$$\begin{tikzcd}
A : \mathrm{Mfld} \arrow[r] & \mathcal{C} \\
\mathrm{Disk} \arrow[u, hook] & \\
\vdots \arrow[u, hook] & \\
\mathrm{Disk}_{\leq j} \arrow[u, hook] & \\
\vdots \arrow[u, hook] & \\
\emptyset \simeq \mathrm{Disk}_{\leq 0} \arrow[u, dotted] &
\end{tikzcd}$$

The left Kan extension of $A|_{\mathrm{Disk}_{\leq j}}$ along the inclusion $\mathrm{Disk}_{\leq j} \hookrightarrow \mathrm{Mfld}$ will be denoted by $T_j A$ and will be called the j^{th} polynomial approximation to A.

$$\begin{tikzcd}
\mathrm{hocolim}\big(T_0 A \arrow[r] \arrow[d] & T_1 A \arrow[r] \arrow[dl] & T_2 A \arrow[r] \arrow[dll] & \dots\big) =: T_\infty A \arrow[dlll, dashed] \\
A & & &
\end{tikzcd}$$

We say that A is analytic, if the map $T_\infty A \to A$ is a weak equivalence.

We say that $T_\infty A(M)$ is the factorization homology of M with coefficients in A denoted by $\int_M A$.

<u>Example.</u> $Mfld^{fr} \hookrightarrow Top$ is analytic.

<u>Proof:</u> This is corepresented by $Hom_{Mfld^{fr}}(\mathbb{R}^n, -)$.

$$T_\infty A(M) = Emb^{fr}(-, M) \underset{Disk}{\otimes} Emb^{fr}(\mathbb{R}^n, -) \quad \text{coYoneda}$$

$$\simeq Emb^{fr}(\mathbb{R}^n, M)$$

$$\simeq M \qquad \square$$

<u>Example.</u> Let $U = \mathbb{R}^n - \{0\}$. Then $Emb^{fr}(U, -)$ is <u>not</u> analytic.

~~<u>Proof:</u> This functor agrees with the previous one because one can always fill in the point. So~~

Mfld is symmetric monoidal with $\amalg$.
Suppose $\mathcal{C}$ is also symmetric monoidal, and restrict our attention to symmetric monoidal functors $(Mfld, \amalg, \emptyset) \to (\mathcal{C}, \otimes, 1)$

Typical examples:

$$A : (\mathrm{Mfld}, \amalg) \longrightarrow \underbrace{(\mathcal{C}, \otimes)}$$

$\longrightarrow$ $(\mathrm{Spaces}, \times)$

$(\mathrm{Chain}_k, \oplus)$

$(\mathrm{Chain}_k, \otimes_k)$

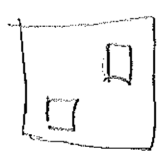
$(\mathrm{Spectra}, \wedge)$

Obs. $A|_{\mathrm{Disk}}$ defines an E_n^{fr}-algebra

$A|_{\mathrm{Disk}^{fr}}$ defines an E_n-algebra

Example. $n=2$ | $\mathbb{R}^n \amalg \mathbb{R}^n \xrightarrow{f} \mathbb{R}^n$

$$A(\mathbb{R}^n) \otimes A(\mathbb{R}^n) \longrightarrow A(\mathbb{R}^n)$$



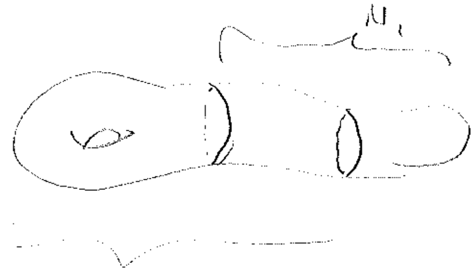
$n=1$

$A|_{\mathrm{Disk}^{fr}}$ defines an A_∞-algebra.

Thm (excision). Given $M = N_0 \amalg_{V \times \mathbb{R}} N_1$, we have:

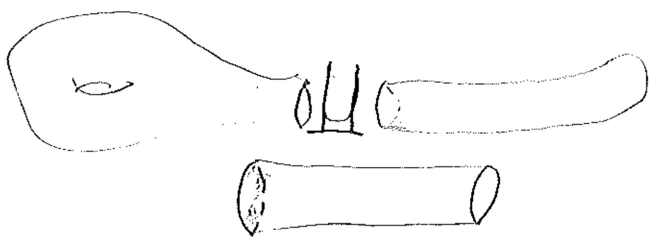
$$\int_M A \simeq \int_{N_0} A \bigotimes_{\int_{V\times\mathbb{R}} A} \int_{N_1} A$$

the homotopy tensor product



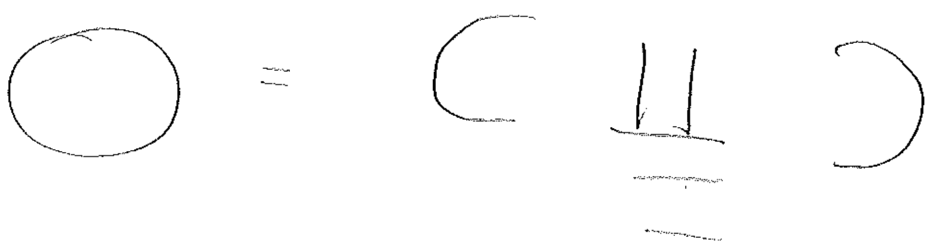
N_1

N_0



$V \times \mathbb{R}$

Example. $M = S^1$:



$$\int_{S^1} A = \int_{\mathbb{R}} A \bigotimes_{\int_{\mathbb{R} \sqcup \mathbb{R}} A} \int_{\mathbb{R}} A \quad \text{but} \quad \int_{\mathbb{R} \sqcup \mathbb{R}} A \cong \int_{\mathbb{R}} A \otimes \left(\int_{\mathbb{R}} A\right)^{op}$$

Def: A homology theory for n-manifolds is a symmetric monoidal functor

$$H : (\mathrm{Mfld}, \sqcup) \longrightarrow (\mathcal{C}, \otimes)$$

s.th. (1) H is cts as a functor between topological categories i.e. $\mathrm{Hom}(X,Y) \longrightarrow \mathrm{Hom}(HX, HY)$ cts.

(2) H is excisive.

5/17/12 HT 6.

Thm. $\mathbb{E}_n^{fr}\text{-alg}(\mathcal{C}) \simeq \left\{ \begin{array}{l} \text{Homology theories} \\ \text{for } n\text{-mflds} \end{array} \right\}$

$\mathbb{E}_n\text{-alg}(\mathcal{C}) \simeq \left\{ \begin{array}{l} \text{Homology theories for} \\ \text{framed } n\text{-mflds} \end{array} \right\}$

Manifold calculus.

$$F : \mathrm{Mfld}^{op} \longrightarrow \mathcal{C}$$

$$\begin{array}{c} \mathrm{Mfld}^{op} \\ \uparrow \\ \mathrm{Disk}^{op} \\ \uparrow \\ \vdots \\ \uparrow \\ \mathrm{Disk}^{op}_{\leq j} \\ \vdots \end{array}$$

the j^{th} polynomial approximation to F

Thm/Def: $T_j F :=$ Right Kan extension of $F|\mathrm{Disk}^{op}_{\leq j}$ along $\mathrm{Disk}_{\leq j} \hookrightarrow \mathrm{Mfld}$

Thm. $T_\infty F(M) \simeq \mathrm{Hom}_{R\text{-}\mathrm{Mod}_{\mathrm{Disk}}}(\mathrm{Emb}(-, M), F)$

Factorization Homology	Manifold Calculus
$A: \mathrm{Mfld} \longrightarrow \mathcal{C}$	$F: \mathrm{Mfld}^{op} \longrightarrow \mathcal{C}$
Left Kan	Right Kan
$T_\infty A(M) = \mathrm{Tor}_{\mathrm{Disk}}(\mathrm{Emb}(-,M), A)$	$T_\infty F(M) = \mathrm{Ext}_{\mathrm{Disk\text{-}Mod}}(\mathrm{Emb}(-,M), F)$

Conj/Thm. Let F symmetric monoidal. Given $M = N_0 \coprod_{V \times \mathbb{R}} N_1$, we have:

$$T_\infty F(M) \simeq \mathrm{cobar}\left(\mathrm{Tot}(N_0), \mathrm{Tot}(V \times \mathbb{R}), \mathrm{Tot}(N_1)\right)$$

Def: A homology cotheory for n-manifolds is a symmetric monoidal functor

$$H: (\mathrm{Mfld}^{op}, \amalg) \longrightarrow (\mathcal{C}, \otimes)$$

s.th. 1. H is ds_g

2. H satisfies coexcision.

+ analogous theorem as before

Pedro Brito, Applications to K-theory I (15)

$A(X)$ Waldhausen A-theory

$A: \mathrm{Top} \longrightarrow \mathrm{Spectra}$

$$A(X) = K(\underbrace{\mathbb{S}[\Omega X]}_{\mathbb{S} \wedge (\Omega X)_+})$$

Thm (Waldhausen). (i) $A(X) \simeq \Sigma^\infty_+ X \times Wh^{Diff}$

(ii) $\Omega^2 \Omega^\infty Wh^{Diff}(X) \simeq \mathcal{C}(X)$ stable isotopy theory

Main idea.

$$A(X) \xrightarrow{\text{trace map}} \mathcal{L}(X) := \Sigma^\infty_+ \mathrm{Map}(S^1, X)$$

(1) Compute derivatives of the free loop spaces.

$$F(K) = \Sigma^\infty_+ \mathrm{Map}(K, X) \qquad F: \mathrm{Top}^{op} \longrightarrow \mathrm{Spectra}$$

$T_k F(K)$ = homotopy sheafification of F wrt. J_k

$= \mathbb{R}\mathrm{Hom}_{\Gamma_k}(\mathrm{Map}(-, K), F)$

where $\Gamma_k' = \mathrm{FinSets}_{\leq n}$

If K is a finite complex, then we may take

$= \mathrm{Hom}_{\Gamma_n}(\mathrm{Map}(-, K), F)$

$\left\{\begin{array}{l}\text{finite sets}\\ \text{of } \# \leq n\\ \text{and surjection}\end{array}\right\} = \Omega_n \hookrightarrow \Gamma_n$

$F = \Sigma^\infty \mathrm{Map}(-, X)$

We can also take

$$\simeq \mathrm{Hom}_{\Omega_n}(K^{\wedge -}, \Sigma^\infty X^{\wedge -})$$

This turns out to be $P_n F$.

in the sense Goodwillie (i.e. Taylor, not interpolation)

Relative case: $\partial_x \mathcal{L}(X) :=$ coefficient (i.e. $X \vee_x S^0$) of the linearization
$x \in X$

$$\begin{array}{c} Z \\ \downarrow \\ X \end{array} \longmapsto \mathrm{hfib}(F(Z) \to F(X))$$

$$\partial_x \mathcal{L}(X) \simeq \underbrace{\Gamma(\Sigma^\infty_+ W \longrightarrow S^1)}_{\text{Spectrum fibered over } S^1}$$

where $W := \{ (f, k) \in X^{S^1} \times S^1 \mid f(k) = x \}$

Goodwillie's notation for $\Gamma(\Sigma^\infty_+ W \to S^1)$ is

$$\int^{k \in K} \Sigma^\infty_+ \mathrm{Map}_*((S^1, k), (X, x))\, dk$$

$$\Gamma(\Sigma^\infty W \longrightarrow S^1) \simeq \mathrm{Map}_*\left(S^1_+, \Sigma^\infty \Omega_x X\right),$$

the derivative of $\underline{f}$

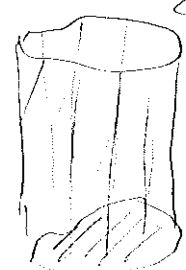
② A-theory.

$$\mathcal{E}(X) = \operatorname*{hocolim}_k C(X \times I^k)$$

$$C(Z) := \mathrm{Diff}(Z \times I, \text{ rel } Z \times \{0\} \cup \partial Z \times I)$$

concordence space of Z

Z



The map $C(Z) \longrightarrow C(Z \times I)$ is $\frac{\dim Z}{3}$-conn.

Prop. $\mathcal{E}$ is a homotopy functor (at least on compact manifolds).

Want: calculate $\partial_x \mathcal{E}(X)$

coefficient of the linearization of

$$Z \longmapsto \mathrm{hfib}\left(\mathcal{E}(Z) \longrightarrow \mathcal{E}(X)\right)$$

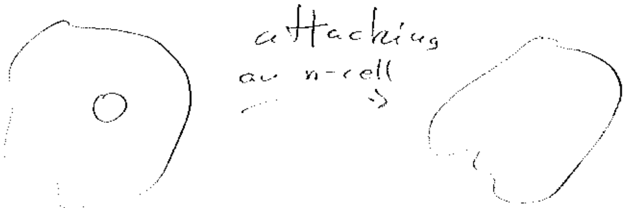
$$\operatorname{hocolim}_n \Omega^n \,\mathrm{hfib}\big(\mathcal{E}(X \vee_x S^n) \longrightarrow \mathcal{E}(X)\big) = \,?$$

It suffices to look at

$$\mathrm{hfib}\big(C(X \vee_x S^n) \longrightarrow C(X)\big)$$

unstable concordance space

$X' \simeq X \vee_x S^n$



$$C(\underbrace{X \vee_x S^n}_{X'}) \longrightarrow C(X) \longrightarrow CE(\mathbb{X})$$

where $CE_X(\mathbb{X}) := \mathrm{Emb}\big(\{*\}\times I,\ X \times I,\ \mathrm{rel} \sqcup\big)$

$(*,0) \longmapsto (x,0)$

concordance embeddings

Using manifold calculus we obtain the approximation: $CE_X(\mathbb{X}) \longrightarrow T_2 CE(\mathbb{X})$

that, by B-M, is $\sim 2n$-connected.

$$\begin{tikzcd}
 & T_2\, CE_X(I) \arrow[d] & L_2\, CE_X(I) \arrow[l] \arrow[dr] & \\
CE_X(I) \arrow[ur] \arrow[r] & T_1\, CE_X(I) & & \Gamma(E \to I^{(2)})
\end{tikzcd}$$

$$\text{total fiber}\left(\begin{tikzcd}
CE(2) \arrow[r] \arrow[d] & CE(1) \arrow[d] \\
CE(1) \arrow[r] & CE(*)
\end{tikzcd}\right)$$

Similarity with Disk

$E—∃ = 1$

$E— —∃ = 2$

$$CE(2) \simeq X \times \underbrace{\left((X \times I)^2 \setminus \Delta\right)}_{\text{conf. space of 2 points}}$$

$$E \simeq \text{hfib}\left(\begin{tikzcd}
(X \times I)^2 \setminus \Delta \arrow[r] \arrow[d] & X \times I \arrow[d] \\
X \times I \arrow[r] & *
\end{tikzcd}\right)$$

$$\simeq \text{hfib}\left(\underbrace{(X \times I)\setminus\{pt\}}_{\simeq X \vee S^n} \longrightarrow \underbrace{X \times I}_{\simeq X}\right)$$

$\simeq$

$\simeq \Sigma^n \Omega_x X$

$$T_2\, CE_x(I) \simeq \Omega^2 \Sigma^n \Omega_x X$$

$$CE_x(I) \longrightarrow \Omega^2 \Sigma^n \Omega_x X \quad 2n\text{-connected}$$

$$\partial_x \mathcal{E}(X) \simeq \Omega^2 \Sigma^\infty_* \Omega_x X$$

$$\Rightarrow \quad \partial_x A(X) \simeq \Sigma^\infty_+ \Omega_x X$$

<u>Recap.</u> $\quad A(X) \longrightarrow \mathcal{L}(X)$

$$\partial_x \mathcal{L}(X) = Map_*(S^1, \Sigma^\infty_+ \Omega_x X)$$

$$\left(\partial_x \mathcal{L}(X)\right)^{hS^1} \simeq \Sigma^\infty_+ \Omega_x X$$

$$\Rightarrow \quad \partial_x A(X) \simeq \left(\partial_x \mathcal{L}(X)\right)^{hS^1}$$

<u>Thm. (Goodwillie)</u> $\quad F \longrightarrow G$ homotopy fun. + ρ-analytic, and $\partial_x F(X) \xrightarrow{\simeq} \partial_x G(X)$ $\forall (X, x \in X)$

$\Rightarrow \quad \tilde{F}(X) \simeq \tilde{G}(X)$ for all ρ-connected X.

Ernest Fontes, Applications to K-theory 2. (16)

Goal: $D_1 K \xrightarrow{\simeq} THH(B; M)$

K-theory:

Def: A Waldhausen category $\mathcal{C}$ is a category $\mathcal{C}$, pointed, equipped with $cof(\mathcal{C})$ and $w(\mathcal{C})$ subcategories of $\mathcal{C}$ s. th.

1. $\text{iso}\, \mathcal{C} \subseteq cof(\mathcal{C}), w(\mathcal{C})$

2. $$\begin{tikzcd} \cdot \arrow[r, tail] \arrow[d] & \cdot \arrow[d] \\ \cdot \arrow[r, tail] \arrow[ur, phantom, "\ulcorner"] & \cdot \end{tikzcd}$$

3. $cof(\mathcal{C}), w(\mathcal{C}) \cap cof(\mathcal{C})$ are pushout stable

4. $0 \rightarrowtail C$ for all $C \in \mathcal{C}$

Example: Let R ring, $\mathcal{P}_R$ category of fin. gen. R-modules, proj., $w\,\mathcal{C} =$ isos

$a \rightarrowtail\!\!\!\twoheadrightarrow b$ iff injection with quotient in $\mathcal{P}_R$

Rmk.
$$\begin{tikzcd} a \arrow[r, tail] \arrow[d] & b \arrow[d, two heads] \\ x \arrow[r, tail] & c \end{tikzcd}$$

Def: $S_n\mathcal{C}$ is the category defined via

$$* = a_{00} \rightarrowtail a_{01} \rightarrowtail a_{02} \rightarrowtail \cdots \rightarrowtail a_{0n}$$

$$\forall\, 0 \le i \le j \le n \qquad a_{0i} \rightarrowtail a_{0j} \twoheadrightarrow a_{ij} \qquad * = a_{ii}$$

$$\begin{tikzcd}
* = a_{00} \arrow[r, tail] & a_{01} \arrow[r, tail] \arrow[d, two heads] & a_{02} \arrow[r, tail] \arrow[d, two heads] & a_{03} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & a_{0n} \arrow[d, two heads] \\
 & * = a_{11} \arrow[r, tail] & a_{12} \arrow[r, tail] \arrow[d, two heads] & a_{13} \arrow[r, tail] \arrow[d, two heads] & \cdots \arrow[r, tail] & a_{1n} \arrow[d, two heads] \\
 & & * = a_{22} \arrow[r, tail] & a_{23} \arrow[r, tail] & \cdots \arrow[r, tail] & a_{2n}
\end{tikzcd}$$

simplicial Waldhausen category

Def: $K(\mathcal{C}) := \Omega |N_\bullet w(S_\bullet \mathcal{C})|$.

Rmk. ① $|N_\bullet w(S_\bullet^{n-1}\mathcal{C})| \xrightarrow{\simeq} \Omega |N_\bullet w(S_\bullet^{n}\mathcal{C})|$

$\Longrightarrow K(\mathcal{C})$ is a spectrum

② $|N_\bullet w(S_\bullet \mathcal{C})| \xrightarrow{\simeq} |N_\bullet w S_\bullet \mathcal{C}|$

$K(\mathcal{C}) \simeq \Omega |\mathrm{ob}(w S_\bullet \mathcal{C})|$

$\Big\}$ $\mathcal{C}$ additive

③ $|N_\bullet w S_\bullet^p \mathcal{C}|$ is $(p-1)$-connected.

Idea of Hochschild homology:

$$[n] \longmapsto \underbrace{R \otimes R \otimes \dots \otimes R}_{n+1}$$

$$\pi_* |R \otimes R^{\otimes n}|$$

Def: For $\mathcal{C}$ spectral category

$$N_k^{cyc}(\mathcal{C}) = \bigvee_{c_0, -, c_n} \mathcal{C}(c_0, c_1) \wedge \mathcal{C}(c_1, c_2) \wedge \dots \wedge \mathcal{C}(c_n, c_0)$$

$$THH(\mathcal{P}_R) = \Omega |N_\bullet^{cyc} \, wS_\bullet \mathcal{P}_R|$$

$$\|$$

$THH(R)$ (topological Hochschild homology)

$$K(R) = K(\mathcal{P}_R) \simeq \Omega | ob(S_\bullet \mathcal{P}_R)| \xrightarrow{tr} \Omega |N_\bullet^{cyc} \, wS_\bullet \mathcal{P}_R|$$

$$\| \\ THH(R)$$

Interlude about TC (topological cyclic homology)

$THH(\mathcal{C})$ as an S^1-spectrum

Fix p.

$$F, R: THH(\mathcal{C})^{C_{p^n}} \longrightarrow THH(\mathcal{C})^{C_{p^{n-1}}}$$

$$TC(\mathcal{C}, p) = \operatorname{holim} THH(\mathcal{C})^{C_{p^n}}$$

Theorem. (Dundas-McCarthy). Suppose $R \xrightarrow{f} S$ is a map of simplicial rings s. th. $\pi_0(f)$ has nilpotent kernel. Then

$$\begin{tikzcd} K(R)^\wedge \arrow[r] \arrow[d] & TC(R) \arrow[d] \\ K(S)^\wedge \arrow[r] & TC(S) \end{tikzcd}$$

is homotopy cartesian. $\square$

Rmk. $K(\Sigma^\infty_+ \Omega X) = A(X)$ compare.

Let R a ring, M an R-bimodule, simplicial.

$$K(R;M) = K(R \oplus M)$$

$M \wedge X = M \otimes X / M \otimes *$ for X finite ssets

$K(R;M;X)$ and analogously $THH(R;M;X)$

$$\Vert$$

$$K(R \oplus (M \wedge X))$$

$$K(R;M;X) \xrightarrow{f} K(R) = K(R;0;*)$$

Define: $\tilde{K}(R;M;X)$ as $\mathrm{hfib}(f)$

$R \oplus M$ - modules

R-module equipped with $P \to P \oplus M$

$$D_1 K(R;M;X) = \mathrm{hocolim}\, \Omega^n \tilde{K}(R;M;\Sigma^n X)$$

$$K(R;M) = \Omega \Big| \coprod_{c \in S_\bullet \mathcal{P}_R} \underbrace{\mathrm{Hom}(c, c \otimes M)}_{\text{simplicial abelian groups}} \Big|$$

$$THH(R;M) = \bigoplus_{c_0, \dots, c_n \in \mathcal{P}_R} \mathrm{Hom}(c_n, c_\varphi) \oplus \dots \oplus \mathrm{Hom}(c_{n-1}, c_n) \oplus \mathrm{Hom}(c_n, c_0 \otimes M)$$

$$\simeq \Omega \Big| \bigoplus_{c \in S_\bullet \mathcal{P}_R} \mathrm{Hom}(c, c \otimes M) \Big|$$

$$K(R;M) = K(R) \times \tilde{K}(R;M)$$

Thm. If M is conn(M), —

$$\begin{array}{ccc} \tilde{K}(R,M) & \xrightarrow{tr} & THH(R;M) \\ \downarrow & & \downarrow \\ D_1 \tilde{K}(R,M) & \xrightarrow{\simeq} & \end{array}$$

<u>Proof</u>: $A_p := \left| \coprod_{c \in S^p_\cdot \mathcal{P}_R} \mathrm{Hom}(c, c \otimes M) \right|$

$B_p := \left| \bigoplus_{c \in S^p_\cdot \mathcal{P}_R} \mathrm{Hom}(c, c \otimes M) \right|$

$C_p := \left| S^{(p)}_\cdot \mathcal{P}_R \right|$

$$\begin{tikzcd}
\widetilde{K}(R;M) \arrow[rr, "tr"] \arrow[d, equal] & & THH(R;M) \arrow[d, "\simeq"] \\
\Omega^p fib_p \arrow[r, "\Omega^p \alpha"] & \Omega^p cof_p \arrow[r] & \Omega^p B_p
\end{tikzcd}$$

A_p, C_p, cof_p all $(p-1)$-connected, so by B-M:

$$\text{total hfib}\left(\begin{tikzcd} A_p \arrow[r] \arrow[d] & B_p \arrow[d] \\ cof_p \arrow[r] & * \end{tikzcd}\right)$$

is $(p-1+p-1-1)$-connected

$\mathrm{hfib}(fib_p \to cofib_p)$ is $(p-1+p-1-1)$-connected

$\therefore$ α_p is $(2p-3)$-connected

$\mathcal{B} : \mathrm{cof}_p \longrightarrow \mathcal{B}_p$

$\bigvee_{S^m}$ (m - connected spaces)

$\mathbb{Z}_{n'} = \mathrm{hfib}(\bigvee(\;) \longrightarrow \oplus$

blah blah blah

Toby Bartels, Calculus and chromatic homotopy theory (17)

Intro to chromotopy.

$S^0 \xrightarrow{\cdot 2} S^0 \to S/2 \to S^1$

Adams. $\Sigma^\infty S/2 \xrightarrow{\alpha} S/2$ s.th.

$K(\alpha)$ iso (and hence $\alpha^k \neq 0$)

$$S^{8k} \longrightarrow \Sigma^{8k} S/2 \xrightarrow{\alpha^k} S/2 \longrightarrow S^1$$

$\Rightarrow \alpha_k \in \pi_{8k-1} \mathbb{S} \quad \forall k$

From now on: everything is p-local

"We need to define Morava K-theory. And by defining I mean: mention their existence"

Morava K-theory. $K(n)$ $\forall n > 0$

- $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$ $\quad |v_n| = 2(p^n - 1)$
- htpy comm ring spectra, complex or in char $p \neq 2$
- Künneth formula (this formula essentially characterizes them)

5/18/12 TB 2.

Def: Let X finite spectrum. We say that X has <u>type n</u>, if $K(n)_* X \neq 0$, but $K(n-1)_* X = 0$

$f : \Sigma^d X \longrightarrow X$ is v_n-self-map, if

$$K(m)_*(f) = \begin{cases} \text{iso} & m = n \\ \text{nilpotent} & \text{oth.} \end{cases}$$

<u>Thm. (Periodicity theorem).</u> X finite

- X has type $n \Longrightarrow$ it admits a v_n-self-map $f : \Sigma^d X \to X$
- if (X, f, d) and (Y, g, e) are such and there is $\varphi : X \to Y$, then $\exists r, s \in \mathbb{N}$ with $dr = es$

$$\begin{tikzcd}
\Sigma^{dr} X \arrow[r, "\Sigma^{dr}\varphi"] \arrow[d, "f^r"'] \arrow[dr, phantom, "="] & \Sigma^{es} Y \arrow[d, "g^s"] \\
X \arrow[r, "\varphi"'] & Y
\end{tikzcd}$$

Telescopes: X finite of type n

$\Rightarrow$ exists v_n-self-map of X

$$\underset{\substack{\uparrow \\ \text{telescopes} \\ \text{are Bousfield} \\ \text{equivalent}}}{T(n)} = T(X) = T(X, f) := \operatorname{hocolim}(X \xrightarrow{f} \Sigma^{-d} X \to \ldots)$$

Corollary. (Resolutions). Exists a finite spectrum $X(i)$ of type n,

$$\begin{tikzcd} X(0) \arrow[r] \arrow[d] & X(1) \arrow[r] \arrow[dl] & \ldots \\ S^0 & & \end{tikzcd}$$

such that $\operatorname{hocolim} X(i) \to S^0$ is an $T(m)$-equivalence for all $m \geq n$.

Localizations (Bousfield-Kuhn functor).

Def: Let E and X be spectra. We say that X is E-acyclic, if $X \wedge E \simeq *$. Let $\mathcal{C}_E$ be the category E-acyclic spectra.

• Y a spectrum is called E-local, if

$$[X, Y] = 0 \quad \text{for all } X \in \mathcal{C}_E$$

<u>Thm. (Bousfield).</u> Let E be a spectrum. Then $\exists\, L_E : \text{Spectra} \longrightarrow \text{Spectra}$ an idempotent functor with a natural transformation $\eta_E : \mathrm{Id} \longrightarrow L_E$ s.th.

(1) $L_E X$ is E-local for all X.

(2) $X \longrightarrow L_E X$ is an E-equivalence. $\square$

<u>Example.</u> $L_{T(n)}$ telescopic localization

$L_{K(n)}$ localization at Morava K-theory

$$\mathcal{C}_{T(n)} \subseteq \mathcal{C}_{K(n)} \Rightarrow L_{K(n)} = L_{K(n)} L_{T(n)}$$

<u>Telescope conjecture</u> ?

Example. $L_{K(1)} X = L_{T(1)} X =$

$$= \operatorname{holim}_n \left(\operatorname{hocolim}\left(X/p^n \xrightarrow{\alpha} \Sigma^{-\infty} X/p^n \xrightarrow{\alpha} \cdots \right)\right)$$

Thm. (Bousfield, Kuhn). For all $n \in \mathbb{N}$, there exists a functor $\Phi_n : \mathrm{Top}_* \longrightarrow \mathrm{Spectra}$ s.th.

$$\begin{tikzcd} \mathrm{Spectra} \arrow[r, "L_{T(n)}"] \arrow[d, "\Omega^\infty"'] \arrow[dr, phantom, "="] & \mathrm{Spectra} \\ \mathrm{Top}_* \arrow[ur, "\Phi_n"'] & \end{tikzcd}$$

and for all $Z \in \mathrm{Top}_*$ $\Phi_n(Z)$ is $T(n)$-local. $\square$

Corollary. $\Sigma^\infty \Omega^\infty X \xrightarrow{\varepsilon} X$ admits a section after $T(n)$-localization.

Proof:

$$\begin{tikzcd} & \Omega^\infty \Sigma^\infty \Omega^\infty X \arrow[dr, "\Omega^\infty \varepsilon_X"] & \\ \Omega^\infty X \arrow[ur, "\eta_{\Omega^\infty X}"] \arrow[rr, "1_{\Omega^\infty X}"] & & \Omega^\infty X \end{tikzcd}$$

Applying Φ_n:

$$\begin{tikzcd}
 & L_{T(n)}\Sigma^\infty\Omega^\infty X \arrow[dr, "L_{T(n)}\varepsilon_X"] & \\
L_{T(n)}X \arrow[ur, "\eta_n"] \arrow[rr, equal] & & L_{T(n)}X
\end{tikzcd}$$

<u>Localized Goodwillie calculus.</u> (All theorems from now on are due to Kuhn).

<u>Thm.</u> Let F : Spectra $\to$ Spectra. The fiber sequence:

$$D_d F(X) \longrightarrow P_d F(X) \longrightarrow P_{d-1} F(X)$$

splits $T(n)$-locally.

<u>Corollary.</u> $\operatorname{holim}_n L_{T(n)} P_d T(X) = \prod_{i=0}^{\infty} L_{T(n)} D_i F(X)$

<u>Short recap of Tate spectra:</u>

$\boxed{x \mapsto \sum_{g\in G} g\cdot x}$

Y spectrum with G-action

$$Y_{hG} \xrightarrow{N_G} Y^{hG}$$

<u>Klein:</u> Norm map uniquely chosen by being an equivalence if Y is finite free

$$\mathrm{cofib}\left(Y_{hG} \xrightarrow{N_G} Y^{hG} \right) = \underbrace{Y^{tG}}_{\text{Tate spectrum}}$$

<u>Prop. (McCarthy).</u> There exists a homotopy (dual calculus) pullback:

$$\begin{tikzcd}
D_d F(X) \arrow[r, "\simeq"] \arrow[d] & (\Delta_d F(X))_{h\Sigma_d} \arrow[d] \\
P_d F(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\Delta_d F(X))^{h\Sigma_d} \arrow[d] \\
P_{d-1} F(X) \arrow[r] & (\Delta_d F(X))^{t\Sigma_d}
\end{tikzcd}$$

<u>Proof:</u> $\alpha : FX \longrightarrow \left(\Delta_d F(X)\right)^{h\Sigma_d}$. $\square$

<u>Rmk.</u> TFAE:

1. $\{F$ d-excisive functor$\}$
2. - G $(d-1)$-excisive functor aka $P_{d-1}F$
   - H d-homogeneous functor aka $D_d F$
   - $G \longrightarrow (\underset{\text{diagonal}}{\Delta_d}\, cr H)^{t\Sigma_d}$

Proof:

$$
\begin{tikzcd}
D_d F(X) \arrow[r, "\simeq\ L_{T(n)}"] \arrow[d] & D_d L_{T(n)} F(X) \arrow[r, "\simeq"] \arrow[d] & (\Delta_d L_{T(n)} F(X))_{h\Sigma_d} \arrow[d] \\
P_d F(X) \arrow[r] \arrow[d] \arrow[u, dashed, bend right, "L_{T(n)}"'] & P_d L_{T(n)} F(X) \arrow[r] \arrow[d] & (\Delta_d L_{T(n)} F(X))^{h\Sigma_d} \arrow[d] \arrow[u, dashed, bend right] \\
P_{d-1} F(X) \arrow[r] & P_{d-1} L_{T(n)} F(X) \arrow[r] & (\Delta_d L_{T(n)} F(X))^{t\Sigma_d} \\
 & & *
 \arrow[from=3-3, to=4-3, phantom, "\wr\!\wr"]
\end{tikzcd}
$$

Thm. $L_{T(m)}(L_{T(n)} S^0)^{t\mathbb{Z}/p} \simeq *$ $\Longrightarrow$ Thm. $T(n)$-locally

$\square$

Vesna Sojanoska, The Taylor Tower of the idendity 2. (18)

$\partial_* \mathrm{Id}$

$\mathrm{Id}: \mathrm{Top}_* \longrightarrow \mathrm{Top}_*$

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
X \arrow[r] \arrow[dr] & P_n X \arrow[d] & D_n X \arrow[l] \\
 & P_{n-1} X \arrow[d] & \\
 & \vdots &
\end{tikzcd}$$

stable!

$D_n X \simeq \Omega^\infty \underbrace{D_n X}_{\simeq (\partial_n \wedge X^{\wedge n})_{h\Sigma_n}}$

In good cases $X \simeq \operatorname{holim}_n P_n X$

GSS:

$E_1(X) = \pi_*^S D_n X \Rightarrow \underbrace{\pi_* X}_{\text{unstable}}$ (more complicated than stable)

How can we compute $E_1(X)$, $E_1(S^k)$?

$H_*(D_n S^k, \mathbb{F}_p) = H_*(\partial_n) \otimes H_k(S^{kn})$

$H_*(\Sigma_n, H_*(D_n S^k, \mathbb{F}_p)) = H_*(\Sigma_n, H_*(\partial_n) \otimes H_k(S^{kn}))$

$\Downarrow$ s.s.

$E_1(S^k) \Leftarrow$ input in a s.s. which $H_*((\partial_n \wedge S^{kn})_{h\Sigma_n})$ is a module over Steenrod algebra

Chromatic Approach

↳ decomposing into frequencies
$p, v_1, v_2, \ldots$ colors

Type n complexes know about v_n-periodicity
Decompose $\pi_* S^k$ into v_m-periodic parts.

<u>Thm.</u> Let k be odd. Then

$$D_n S^k \simeq \begin{cases} * & n = p^i \\ D_{p^i}(S^k) \text{ has type } i \leftarrow \text{knows about } v_i\text{-periodic htpy in } S^k \end{cases}$$

Let's look at:

$$\mathrm{Id} \xrightarrow{E} \Omega \Sigma \xrightarrow{H} \Omega \Sigma\, Sq$$

$$Sq : X \longmapsto X \wedge X$$

$$\tilde{H} : \Sigma \Omega \Sigma X \xrightarrow{\simeq} \Sigma \bigvee_i X^{\wedge i} \longrightarrow \Sigma (X \wedge X)$$

H is adjoint to $\tilde{H}$

2-locally: $\exists$ fiber sequences

in Top $P_n S^k \xrightarrow{E} P_n(\Omega_1 \Sigma_1)(S^k) \longrightarrow P_n(\Omega_2 \Sigma^1 Sq)(S^k)$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\Omega_1 P_n(S^{k+1})}$

in Spectra $D_n S^k \longrightarrow D_n(\Omega \Sigma^1)(S^k) \longrightarrow D_n(\Omega_2 \Sigma^1 Sq)(S^k)$

$\qquad\qquad\qquad\qquad \overset{\shortparallel}{\Omega D_n S^{k+1}}$

<u>Lemma.</u> Let $F : Top_* \to Top_*$, (stably i-excisive for all i). Then

(i) $P_n(F \circ Sq) \simeq P_{\lfloor \frac{n}{2} \rfloor}(F)(Sq)$

(ii) $D_n(F \cdot Sq) \simeq \begin{cases} D_{n/2}(F)(Sq) & n \text{ even} \\ * & n \text{ odd} \end{cases}$

<u>Proof:</u> Chain rule. $\square$

So by the Lemma:

$$P_n(\Omega_1 \circ \Sigma_1^{-1} \cdot Sq)(S^k) \simeq \Omega P_{\lfloor \frac{n}{2} \rfloor} S^{2k+1} \quad \text{in Top}$$

and

$$D_n(\Omega_1 \circ \Sigma_1 \cdot Sq)(S^k) = \begin{cases} D_{n'}(S^{2k+1}) & n = 2n' \\ * & n \text{ odd} \end{cases}$$

Want to show $D_n(S^k) \simeq *$, if $n \neq 2^j$ $j \geq 0$

n odd

$$D_n S^k \xrightarrow{\simeq} \Omega D_n(S^{k+1}) \xrightarrow{\simeq} \Omega^2 D_n(S^{k+2}) \to \dots$$

$$\dots \longrightarrow \underbrace{\Omega^\infty D_n \Sigma^\infty S^k}_{= \operatorname{hocolim}_p \Omega^p D_n \Sigma^p S^k} = D_1 D_n S^k \simeq *$$

Induction hypothesis $n = s \cdot 2^j$ s odd $s > 1$

and all spheres

Want: $s \cdot 2^{j+1} = 2n$

$$D_{2n} S^k \to \Omega D_{2n} S^{k+1} \longrightarrow \Omega D_n S^{2k+1} \simeq *$$

$\wr\wr$

$$D_1(\Omega D_{2n})(S^{k+1}) \simeq *$$

$\partial_n = (\Sigma_\cdot SK_n)^\vee$

partition complex

$\mathcal{K}$ = poset of non-trivial partitions (>1 & $<n$ sets in a partition)

$K_n = |\mathcal{K}|$

$\partial_n \simeq (\bigvee S^{n-1})^\vee$, $K_n \simeq \bigvee S^{n-3}$

$n=2 \Rightarrow K_2 = \phi \Rightarrow \partial_2 = S^{-1}$ trivial Σ_2-action

Find a smaller complex B_k s.th.

Tits building for $GL_k(\mathbb{Z}/p)$

$K_{p^k} \simeq B_k$

$\simeq \bigvee S^{k-1}$

and, along the way, show $K_n \simeq *$ for $n \neq p^k$

B_k : simplicial set of flags in $(\mathbb{F}_p)^k$

$0 \subset V_1 \subset V_2 \subset \ldots \subset V_s \subset \mathbb{F}_p^k$

subspaces

$\mathbb{F}_p^k \cong \underline{n}$ and think of flags as giving a partition

$$B_k \longrightarrow K_{p^k}$$

For general n:

$$\begin{array}{ccl} \mathcal{H} & \xleftrightarrow[\text{and order-preserving}]{\text{bijective}} & \mathcal{S} = \text{poset of stabilizers} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \lambda & \longmapsto & H_\lambda \subset \Sigma_n \end{array}$$

stabilizes each subset of λ up to conjugacy

$H_\lambda = \Sigma_{n_1} \times \ldots \times \Sigma_{n_d}$

$K_n \simeq |\mathcal{S}|$

$\mathcal{C}$: collection of subgroups $\subseteq G$ (closed under conjugacy)

$X \circlearrowleft G \qquad X \longmapsto \mathrm{Iso}(X) \ni H$ which stabilizes simplicices of X

X has <u>$\mathcal{C}$-isotropy</u>, if $\mathrm{Iso}(X) \subseteq \mathcal{C}$

$X \to Y$ is <u>$\mathcal{C}$-equivalence</u>, if $\forall_H \; X^H \xrightarrow{\simeq} Y^H$

Prop. There exists a unique functorial $\mathcal{E}$-approximation such that for all X, exists $X_{\mathcal{E}} \longrightarrow X$ s. th. it is a $\mathcal{E}$-equivalence and $X_{\mathcal{E}}$ has $\mathcal{E}$-isotropy

Example. if $\mathcal{E}$ = all subgps $\quad X_{\mathcal{E}} = X$

$\mathcal{E} = \{G\} \quad X_{\mathcal{E}} = X^G$

$\mathcal{E} = \{\{e\}\} \quad X_{\mathcal{E}} = EG \times X$

Define: $F\mathcal{E} = (*)_{\mathcal{E}}$

$\mathcal{E}$ is a poset $\Rightarrow$ $E\mathcal{E} \longrightarrow |\mathcal{E}|$ not an equivalence

$$(E\mathcal{E})^H = |H \downarrow \mathcal{E}| = \begin{cases} * & \text{if } H \in \mathcal{E} \\ & \end{cases}$$

$$|H \downarrow \mathcal{E}| = \{H' \in \mathcal{E} \mid H \subset H'\}$$

$\mathcal{F}$ = collection of non-transitive, non-trivial subgroups of Σ_n

Let $\mathcal{S} \subseteq \mathcal{F}$ & it turns out $E\mathcal{S} \xrightarrow{\simeq} E\mathcal{F}$

5/18/12 VS 8.

This means $K_n \simeq |\mathcal{F}|$

$\mathcal{E}$: non-trivial elementary abelian subgroups of Σ_n $\quad (\simeq (\mathbb{Z}/p)^j)$

$\mathcal{E}' = \mathcal{E} \cap \mathcal{F}$

$E\mathcal{E}' \longrightarrow E\mathcal{F}$ is $\mathcal{E}$-approximation

$$\begin{tikzcd}
& {|\mathcal{E}'|} \arrow[rr] \arrow[dd] & & {|\mathcal{E}|} \arrow[dd] \\
E\mathcal{E}' \arrow[ur, "w.e."] \arrow[rr] \arrow[dd, "\mathcal{E}\text{-approx.}"'] & & E\mathcal{E} \arrow[ur, "w.e."] \arrow[dd] & \\
& {|\mathcal{F}|} & & * \\
E\mathcal{F} \arrow[ur, "w.e."] \arrow[rr] & & * \arrow[ur, "w.e."] &
\end{tikzcd}$$

are iso on equiv. homology

and something, something, something