CALCULUS OF FUNCTORS AND CHROMATIC HOMOTOPY THEORY

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Contents

1. Motivation

Chromatic homotopy theory approaches the computations of stable homotopy groups (of spheres, say) by decomposing them into periodic families. Such families can then be detected by designer cohomology theories, and it thus becomes an important issue to better understand these. The calculus of functors opens up completely new methods for approaching this problem.

Another way in which the calculus of functors interacts with chromatic homotopy theory is by means of interpolations between the stable and unstable theory. Work of Arone, Dwyer and Mahowald as well as the EHP spectral sequence computations of Mark Behrens, to be discussed in the next talk, are among the many instances of this general principle.

2. Chromatic stable homotopy theory

Arguably, stable homotopy theory is all about studying the stable homotopy groups of spheres, which are related to topology, analysis, number theory and so on. So what's better than constructing a non-trivial element in the stable stem? Constructing infinitely many!

The roots of chromatic homotopy theory reach down all the way to Adams' work on the image of J. In [Ada66], he showed that at $p = 2$, the mod 2 Moore spectrum $M(\mathbb{F}_2) = S^0 \cup_2 e^1$ admits a self map

$$\alpha : \Sigma^8 M(\mathbb{F}_2) \to M(\mathbb{F}_2)$$

which is an isomorphism in K-theory. Therefore, iterating this map will never be zero, and we obtain a family of non-trivial elements in the stable homotopy groups of spheres:

$$S^{8k} \to \Sigma^{8k} M(\mathbb{F}_2) \xrightarrow{\alpha^k} M(\mathbb{F}_2) \to S^1$$

where the first map is the inclusion of the bottom cell and the last map pinches to the top cell. In fact, this is a geometric incarnation of Adams periodicity in the E_2 term of the Adams spectral sequence.

After this seminal work, people tried to construct other families like this, but they couldn't really go beyond 'the third layer'. In the early 70s, there was even some controversy about the non-triviality of the first element in the gamma family, which gained public attention when people interpreted it as a first sign of the decline of mathematics. Incipit Ravenel! As a vast generalization of Adams' work, providing completely new insights into the global structure of the stable homotopy category, Ravenel proposed his conjectures which where proven in the late 80s by Devinatz, Hopkins and Smith. We will restrict ourselves here to the part of the story that is most relevant to the interactions with the calculus of functors to be discussed in the second half of the talk.

To this end, we have to introduce a remarkable family of spectra known as Morava K-theories. Everybody loves them. For the rest of the talk, let us work localized at a fixed prime p. Then, for n any non-negative integer, the n-th Morava K-theory $K(n)$ (for p) is a homotopy commutative[1] complex oriented ring spectrum with $K(n)_* = \mathbb{F}_p[v_n, v_n^{-1}]$, a graded field, where $|v_n| = 2(p^n - 1)$. Furthermore,

[1] that's not quite true for $p = 2$

the associated cohomology theories satisfy a Künneth formula, and this property essentially characterizes the Morava K-theories. One way to think about these spectra is as the 'fields' of stable homotopy theory.

Before we can state one version of the main theorem of Devinatz-Hopkins-Smith, we need to introduce a little bit of notation. Let's say a finite p-local spectrum is of type n if $K(n-1)_*X = 0$ but $K(n)_*X \neq 0$. A map $f : \Sigma^d X \to X$ is said to be a v_n-self map if $K(m)_*f$ is an isomorphism for $m = n$ and nilpotent otherwise. These are the higher analogues of Adams' map α. Now we are ready:

Theorem 2.1 (Periodicity theorem)**.** *A finite spectrum X is of type n if and only if it admits a v_n-self map. Furthermore, these are compatible in the sense that if $f : \Sigma^d X \to X$ and $g : \Sigma^e Y \to Y$ are two v_n-self maps and $\phi : X \to Y$ is a map of spectra, then there exist integers k, l with $dk = el$ and such that the following square commutes:*

$$\begin{array}{ccc} \Sigma^{dk}X & \xrightarrow{\Sigma^{dk}\phi} & \Sigma^{el}Y \\ {\scriptstyle f^k}\downarrow & & \downarrow{\scriptstyle g^l} \\ X & \xrightarrow[\phi]{} & Y. \end{array}$$

As an immediate consequence, we see that for $\Sigma^d X \xrightarrow{f} X$ as in the theorem, the spectrum

$$T(n) = T(X, f) = \mathrm{hocolim}(X \xrightarrow{f} \Sigma^{-d}X \xrightarrow{f} \Sigma^{-2d}X \xrightarrow{f} ...)$$

is independent of the choice of the self-map; moreover, its Bousfield class is independent of the type n spectrum X. Another, non-trivial consequence of this theorem is the following result of Kuhn:

Corollary 2.2. *For any n, there exists a diagram of type n spectra*

$$F(1) \to F(2) \to ...$$

over S^0 such that $\mathrm{hocolim}_k F(k) \to S^0$ *is an isomorphism in $T(m)$-cohomology for all $m \geq n$.*

Sketch of proof. Induction. □

Morally, the perspective on stable homotopy theory provided by the solution of the Ravenel conjecture is like this: The stable homotopy groups of a finite spectrum, say the sphere spectrum, decompose into periodic families, very much like light decomposes into periodic waves of different lengths; hence the term 'chromatic'.

3. Bousfield localization and the Bousfield-Kuhn functors

In order to study the local pieces of the stable homotopy category, as exhibited by the periodicity theorem, one by one, we need to introduce localization functors. The idea goes back to Adams, again, but non-changing-the-universe-does-the-trick set-theoretical problems prevented him from solidly laying down the foundations of the theory. Bousfield later resolved these issues and proved the next result.

Definition 3.1. Let E be any spectrum. The category of E-acyclic spectra $\mathcal{C}_E$ is the full subcategory of the category of spectra on objects X such that $E \wedge X \sim *$. A spectrum Y is called E-local if, for any $X \in \mathcal{C}_E$, $[X, Y] = 0$. Equivalently, any E-equivalence $A \to B$, i.e., a morphism that becomes an isomorphism in E-homology, induces an isomorphism $[B, Y] = [A, Y]$.

Theorem 3.2 (Bousfield)**.** *Let E be any spectrum, then there exists an idempotent functor $L_E : Sp \to Sp$ together with a natural transformation $\eta : Id \to L_E$ satisfying the following:*

(1) *$L_E X$ is E-local*
(2) *$\eta_X : X \to L_E X$ is an E-equivalence*

In fact η_X is terminal with the above property (2), or initial among maps from an X to an E-local object. (Exercise!)

Remark 3.3. One can construct a nice (Bousfield) model structure on spectra in which the fibrant objects are precisely the E-local objects, with weak equivalences the stable E-equivalences, and the cofibrations the usual ones. Alternatively, we could work in the context of ∞-categories.

In general, it is difficult to describe the localized spectrum or its homotopy groups explicitly. However, if both E and X are connective, Bousfield showed that the E-localization of X is essentially determined by the arithmetic of $\pi_0 E$.

Example 3.4. If $E = M(\mathbb{F}_p)$, then $L_p X = L_E X = \mathrm{holim}(\cdots \to X \wedge M(\mathbb{Z}/p^2) \to X \wedge M(\mathbb{Z}/p))$, the p-completion of X. Similarly, $E = M(\mathbb{Z}_{(p)})$ gives p-localization.

As should be clear from the last section, the functors $L_{K(n)}$ and $L_{T(n)} = L_{T(n)}$ are of central importance to stable homotopy theory. Since $\mathcal{C}_{T(n)} \subset \mathcal{C}_{K(n)}$, formally we get $L_{K(n)} = L_{K(n)}L_{T(n)}$, hence it is enough to consider the $L_{T(n)}$.

Example 3.5. $K(1)$-resolutions: $L_{K(1)}X = \varprojlim_n \varinjlim\{X/p^n \xrightarrow{\alpha} \Sigma^{-dk}X/p^n \xrightarrow{\alpha} \cdots\}$.

Remark 3.6. The telescope conjecture, the only one of Ravenel's conjecture that remains open and is widely believed to be false in general, says that $L_{K(n)} = L_{T(n)}$. If time permits, we will come back to it at the end of the talk.

A remarkable result, due to Bousfield for $n = 1$ and generalized to arbitrary n by Kuhn, is that $L_{T(n)}X$ is completely determined by the zeroth space of X. More precisely:

Theorem 3.7. *Let $n \geq 1$, then there exists a functor $\Phi_n : \mathbf{Top} \to \mathbf{Sp}$ factoring through $T(n)$-local spectra and such that the following diagram commutes*

$$\begin{array}{ccc} \mathbf{Sp} & \xrightarrow{L_{T(n)}} & \mathbf{Sp} \\ {\scriptstyle \Omega^\infty}\downarrow & \nearrow{\scriptstyle \Phi_n} & \\ \mathbf{Top}. & & \end{array}$$

Sketch of construction. By Theorem 2.2, there exists a type n resolution $F(1) \to F(2) \to \cdots$ of S^0. Let $f_i : \Sigma^{d_i}F(i) \to F(i)$ be a v_n-self-map of $F(i)$, which exists in light of the periodicity theorem 2.1. Define the functor $\Phi_i : \mathbf{Top} \to \mathbf{Sp}$ by sending a space X to the spectrum with kd_ith space $\mathrm{Map}(F(i), X)$ and the natural structure maps induced by f_i. One can then check that

$$\Phi_n = \lim_i \Phi_i$$

has the required properties. □

We note some immediate corollaries.

Corollary 3.8. *If $\Omega^\infty X \simeq \Omega^\infty Y$, then $L_{T(n)}X \simeq L_{T(n)}Y$.*

Corollary 3.9. *The counit map $\epsilon : \Sigma^\infty\Omega^\infty X \to X$ has a natural section after $T(n)$-localization.*

Proof. Take $\Phi_n(\eta(\Omega^\infty X))$, where $\eta : Id \to \Omega^\infty\Sigma^\infty$ is the unit map. □

In the next section, we will study the interaction of the Bousfield-Kuhn functors with the calculus of functors.

4. Splitting of the localized Goodwillie tower

Let us start with a motivating example of the kind of results we have in mind.

Example 4.1. The Goodwillie tower of the functor Q gives in particular a cofibration sequence $\mathcal{P}_2(X) \to X \xrightarrow{\delta} \Sigma(X \wedge X)_{h\Sigma_2}$, which specialized to $X = S^{-1}$ can be identified, after a shift, with

$$\mathbb{R}P^\infty_{-1} \to \mathbb{R}P^\infty_0 \to S^0$$

This can be shown to split after $T(n)$-localization, whereas the connecting map $\delta : S^0 \to \Sigma\mathbb{R}P^\infty_{-1}$ is non-zero in mod 2 homology.

First of all, we will need a general lemma about the interaction between Bousfield localization and the calculus of functors.

Lemma 4.2. (i) *If $F : \mathbf{Sp} \to \mathbf{Sp}$ is finitary and $f : X \to Y$ is an E_*-equivalence, then so are $\mathcal{D}_dF(f)$ and $\mathcal{P}_dF(f)$.*
(ii) *If $f : F \to G$ is a natural transformation which is a pointwise E_*-equivalence, then so are $\mathcal{D}_df$ and $\mathcal{P}_df$.*

We are now ready to state Kuhn's splitting theorem.

Theorem 4.3. *Let $n \geq 1$ and $F : \mathbf{Sp} \to \mathbf{Sp}$, then the map*

$$p_d(X) : \mathcal{P}_dF(X) \to \mathcal{P}_{d-1}F(X)$$

has a natural section (up to homotopy) after $T(n)$-localization.

Remark 4.4. Observe that, in general, one cannot expect the localized tower to converge, as L_E does usually not preserve products. We will study a particular and important example of this in the next section.

We note two immediate consequences.

Corollary 4.5. (i) $\mathrm{holim}_d L_{T(n)}\mathcal{P}_d F(X) = \prod_d L_{T(n)}\mathcal{D}_d F(X)$.
(ii) *If* E_n *denotes Morava E-theory, then* $E_n^*(\mathcal{P}_d F(X)) \simeq \bigoplus_{c=0}^d E_n^*(\mathcal{D}_c F(X))$.

Remark 4.6. Kuhn also deduces that $L_{T(n)}\Sigma^\infty BG_+$ is self-dual in the category of $T(n)$-local spectra.

The key ingredient in the proof of the theorem is a vanishing result for Tate spectra. Recall that, if Y is a G-spectrum for G a finite group say, we can construct homotopy orbit and fixed point spectra, namely $Y_{hG} = (EG_+ \wedge Y)/G$ and $Y^{hG} = Map(EG_+, Y)^G$. Its Tate spectrum Y^{tG} is then defined by the following cofiber sequence:

$$Y_{hG} \to Y^{hG} \to Y^{tG}$$

where the first map is the norm map. By a result of Klein, this natural transformation can be uniquely characterized by the property that it is an equivalence whenever Y is a finite free G-spectrum. Moreover, note that, if R is a ring spectrum with trivial G-action and $M \in \mathbf{Mod}_R$, then R^{tG} is a ring spectrum and $M^{tG} \in \mathbf{Mod}_{R^{tG}}$. Tate spectra enters the calculus of functors in the following way.

Proposition 4.7. *If* $F : \mathbf{Sp} \to \mathbf{Sp}$ *is any functor, for any* d *and* X *there exists a pullback diagram*

$$\begin{array}{ccc} \mathcal{P}_d F(X) & \longrightarrow & (\Delta_d F(X))^{h\Sigma_d} \\ \downarrow & & \downarrow \\ \mathcal{P}_{d-1} F(X) & \longrightarrow & (\Delta_d F(X))^{t\Sigma_d} \end{array}$$

Sketch of proof. Without loss of generality we can assume that F is d-excisive. In this case, the horizontal maps are induced by the composite map $\alpha^d(X) : F(X) \xrightarrow{\Delta} F(X)^{h\Sigma_d} \to (\Delta^d F)(X)^{h\Sigma_d} \xleftarrow{\sim} (\Delta_d F)(X)^{h\Sigma_d}$, constructed using McCarthy's dual calculus. Here we have to use the key observation that $(\Delta_d F)^{t\Sigma_d}$ is $(d-1)$-excisive to identify

$$\mathcal{D}_d((\Delta_d F)^{h\Sigma_d}) \to \mathcal{P}_d((\Delta_d F)^{h\Sigma_d}) \to \mathcal{P}_{d-1}((\Delta_d F)^{h\Sigma_d})$$

with the norm sequence for $\Delta_d F$. □

Remark 4.8. This proposition says, in colloquial terms, that the classification of n-excisive functors is controlled by the Tate spectrum $(\Delta_d F(X))^{t\Sigma_d}$. To be more precise, the data of a reduced n-excisive functor F is equivalent to the following:

- a reduced $(n-1)$-excisive functor $\mathcal{P}_{n-1}(F)$,
- an n-homogeneous functor $\mathcal{D}_n F$, as classified by Goodwillie, and
- a natural transformation $\alpha^n : \mathcal{P}_{n-1}(F) \to (\Delta_n F)^{t\Sigma_n}$, where $\Delta_n F$ is the diagonal of the multilinearization of the n-th cross effect of F.

The functor F can then be obtained by forming the pullback $F \simeq \mathcal{P}_{n-1}(F) \times_{(\Delta_n F)^{t\Sigma_n}} (\Delta_n F)^{h\Sigma_n}$

Proof of Theorem. To shorten notation, let's write $L = L_{T(n)}$ from now on. Using the natural transformation $Id \to L$ and Proposition 4.7, we obtain a commutative diagram

$$\begin{array}{ccccc} \mathcal{D}_d F(X) & \longrightarrow & \mathcal{D}_d LF(X) & \xrightarrow{\sim} & (\Delta_d LF(X))_{h\Sigma_d} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{P}_d F(X) & \longrightarrow & \mathcal{P}_d LF(X) & \longrightarrow & (\Delta_d LF(X))^{h\Sigma_d} \end{array}$$

in which the upper and lower left horizontal maps are $T(n)$-equivalences by Lemma 4.2. To construct a homotopy section as desired, it suffices to show that the right vertical map becomes an isomorphism after applying L. Since the localized cofiber of this map is a module over $L_{T(n)}(L_{T(n)}S^0)^{tG}$, the proof follows from the next theorem. □

Theorem 4.9. *For all finite groups* G *and all* $n \geq 1$, $L_{T(n)}(L_{T(n)}S^0)^{tG} \sim *$.

First, Kuhn shows that the theorem follows from the case $G = \mathbb{Z}/p$, using the next result.

Proposition 4.10. *Let R be a ring spectrum, E_* a homology theory. If $E \wedge R^{t\mathbb{Z}/p} \sim *$ for all primes p, then M^{tG} is also E-acyclic for all $M \in \mathbf{Mod}_R$ and all finite groups G. The assumption holds in particular if R_* is uniquely p-divisible.*

Idea of proof. First reduce to p-groups via the theory of Mackey functors, then use induction on the order of the group. □

It is therefore enough to prove the following.

Theorem 4.11. $L_{T(n)}(L_{T(n)}S^0)^{t\mathbb{Z}/p} \sim *$

Proof. The idea of the proof goes like this. To start with, present $S^{t\mathbb{Z}/p}$ as a limit of Thom spectra models. In order to show that the localized Tate spectrum $L_{T(n)}(L_{T(n)}S^0)^{t\mathbb{Z}/p}$ vanishes, Kuhn proves that its unit factors through the limit of certain connecting maps associated with the localized Goodwillie tower of $\Sigma^\infty\Omega^\infty$. This limit can then be seen to be null in virtue of the next lemma, which uses the Bousfield-Kuhn functors. □

Lemma 4.12. *The natural map* $\mathrm{holim}_k \Sigma^k L_{T(n)}\mathcal{P}_p(\Sigma^{-k}X) \to L_{T(n)}X$ *has a section (up to homotopy).*

Sketch of proof. Use Corollary 3.9; explicitly, $\mathrm{holim}_k\Sigma^k(L_{T(n)}e_p(\Sigma^{-k}X) \circ \eta_n(\Sigma^{-k}X))$ provides a section. □

We finish this section with a couple of applications and a remark.

Remark 4.13. Theorem 4.9 is equivalent to $T(n)^{tG} \sim *$ versions of which have been appeared in the literature before Kuhn's work. In particular, Greenlees and Sadofsky proved the $K(n)$-analogue in 1996.

The theorem has the following interesting consequences.

Corollary 4.14. *The natural sequence $Y_{hG} \to Y^{hG} \to Y^{tG}$ splits for all naive G-spectra Y.*

Corollary 4.15. *$L_{T(n)}\Sigma^\infty_+ BG$ is self-dual in the $T(n)$-local category.*

5. Periodic homology of infinite loop spaces

The above splitting theorem 4.3 comes with a slight flaw: In general, one cannot expect the localized tower to converge, see Remark 4.4. Our goal in this section is to study the localized tower of the functor $\Sigma^\infty\Omega^\infty$ in more detail.

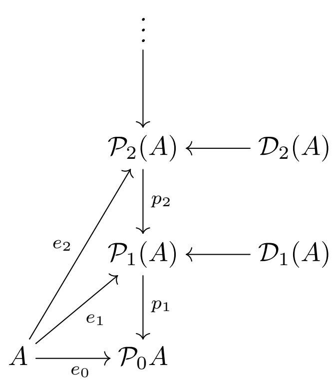
5.1. **Motivating sample theorem.** As a prototypical example for the results to come, we first consider the functor $Q = \Omega^\infty\Sigma^\infty$ and its stable splitting. Recall that for any spectrum X, $\Sigma^\infty_+\Omega^\infty X$ is an augmented commutative algebra; in fact, there is an adjunction (up to homotopy) $\Sigma^\infty_+\Omega^\infty : \mathbf{Sp}^{\geq 0} \rightleftarrows \mathbf{Alg} : gl_1$. By the universal property of the free algebra functor on a spectrum, $\mathrm{Sym} = \bigvee_d D_d$, for any space Z, we obtain a morphism

$$s(Z) : \mathrm{Sym}(\Sigma^\infty Z) \to \Sigma^\infty_+\Omega^\infty\Sigma^\infty Z$$

induced by the unit of the adjunction $\Sigma^\infty \dashv \Omega^\infty$.[2]

We claim that $s(Z)$ induces an equivalence for any connected space Z and also is an E_*-equivalence for any homology theory E_*. In order to reinterpret this statement in terms of the calculus of functors, we need to consider the Goodwillie tower of the identity on $\mathbf{Alg}$; recall the following facts about it.

Theorem 5.1. *For any $A \in \mathbf{Alg}$, the Taylor tower of the identity functor*



[2]We need to be slightly careful about basepoints here; the map is really induced by $\Sigma^\infty Z \xrightarrow{\Sigma^\infty(Z)} \Sigma^\infty QZ \xleftarrow{\sim} I(\Sigma^\infty_+ QZ)$.

has the following properties.

(i) e_0 *is the natural augmentation map* $A \to R$*, and* $I(e_1)$ *can be identified with the canonical map* $I(A) \to \mathrm{TAQ}(A)$*, where* I *denotes the augmentation ideal functor.*
(ii) *More generally, for any* $r \geq 1$, $\mathcal{D}_r(A) \simeq D_r(\mathrm{TAQ}(A)) = \mathrm{TAQ}(A)^{\wedge r}_{h\Sigma_r}$
(iii) *If* $I(A)$ *is 0-connected, then* e_r *is* r*-connected. In particular, in this case* $A \xrightarrow{\sim} \hat{A}$*, where the latter object is the (homotopy) limit of the tower.*

By climbing up the tower, we thus obtain:

Corollary 5.2. *If* $f : A \to B$ *is a map of* R*-algebras such that* $\mathrm{TAQ}(f)$ *is an equivalence, then so is* $\mathcal{P}_r(f)$*. In particular, this implies that* $\hat{f}$ *is an equivalence, and that there exists a commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{can} & \hat{A} \\ & {\scriptstyle f}\searrow \quad \nearrow{\scriptstyle (\hat{f}^{-1})\circ can} & \\ & B & \end{array}$$

In the cases we are interested in, we can be more explicit.

Example 5.3. (1) Evaluating the tower on $\mathrm{Sym}(X)$, the map $I(e_1) : I(\mathrm{Sym}(X)) \to \mathrm{TAQ}(\mathrm{Sym}(X))$ can be identified with $\bigvee_{r=1}^{\infty} \mathcal{D}_r X \to \mathcal{D}_1 X = X$, and $\mathrm{Sym}(X) \to \widehat{\mathrm{Sym}}(X)$ is $\bigvee_{r=0}^{\infty} \mathcal{D}_r X \to \prod_{r=0}^{\infty} \mathcal{D}_r X$. Note that the last map is an E_*-isomorphism for any homology theory E_*.
(2) Similarly, for $\Sigma^\infty_+ \Omega^\infty X$, $I(e_1)$ is just the counit $\epsilon : \Sigma^\infty \Omega^\infty X \to X\langle -1 \rangle$.

Theorem 5.4. *There exists a commutative diagram*

$$\begin{array}{ccc} \mathrm{Sym}(\Sigma^\infty Z) & \xrightarrow{can} & \widehat{\mathrm{Sym}}(\Sigma^\infty Z) \\ & {\scriptstyle s(Z)}\searrow \quad \nearrow{\scriptstyle t(Z)} & \\ & \Sigma^\infty_+ QZ & \end{array}$$

with the following properties:

(i) $s(Z)$ *is an equivalence if* Z *is connected*
(ii) $E_* s(Z)$ *is monic for any homology theory* E_*.

Proof. By Corollary 5.2, we are reduced to showing that $\mathrm{TAQ}(s(Z))$ is an equivalence. But using the above computations, we see that we have the following diagram,

$$\begin{array}{ccc} \Sigma^\infty Z & = & \Sigma^\infty Z \\ \downarrow\sim & & \downarrow\sim \\ \mathrm{TAQ}(\mathrm{Sym}\Sigma^\infty Z) & \xrightarrow[\mathrm{TAQ}(s(Z))]{} & \mathrm{TAQ}(\Sigma^\infty_+ QZ) \end{array}$$

the commutativity of which is checked easily. □

Remark 5.5. Compare with classical stable splittings, see [Kuh01].

5.2. **The localized splitting.** Let's write L for Bousfield localization at $T(n)$. Corollary 3.9 gives rise to a map of commutative LS^0-algebra for any $X \in \mathbf{Sp}$

$$s_n(X) : L\mathrm{Sym}(X) \to L\Sigma^\infty_+ \Omega^\infty X$$

which induces an equivalence on D_1 by an argument similar to the one in Section 1. In fact, Kuhn shows:

Theorem 5.6. *For any spectrum* X*, the map* $s_n(X)$ *induces an equivalence on (localized) completions and thus fits into a commutative diagram*

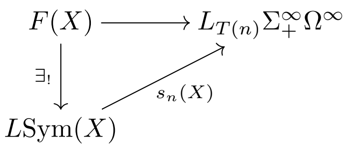
$$\begin{array}{ccc} L_{T(n)}\mathrm{Sym}(X) & \xrightarrow{can} & \hat{L}_{T(n)}\mathrm{Sym}X = \prod_{r=0}^{\infty} L_{T(n)} D_r X \\ & {\scriptstyle s_n(X)}\searrow \quad \nearrow{\scriptstyle t(X)} & \\ & L_{T(n)}\Sigma^\infty_+ \Omega^\infty X & \end{array}$$

Furthermore, it has the following properties:

(1) *If* $\mathcal{C}_{T(n)} \subset \mathcal{C}_E$*, then* $E_* s_n(X) : E_*(\mathrm{Sym}X) \to E_*(\Omega^\infty X)$ *is monic*

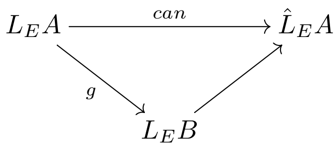
(2) *If X is suitably connected*[3] *and $X \in \mathcal{C}_{T(i)}$ for $1 \leq i \leq n-1$, then $s_n(X)$ is an equivalence. In this case, $X \in \mathcal{C}_{K(i)}$ for all $1 \leq i \leq n-1$.*

(3) *s_n is universal among natural transformations from functors $F : \mathbf{Sp} \to \mathbf{Sp}$ invariant under $T(n)$-equivalence to $L_{T(n)}\Sigma^\infty_+\Omega^\infty$:*



Sketch of proof. The proof is, by virtue of the next proposition, completely analogous to the motivating example above, using the natural transformation η_n coming from the Bousfield-Kuhn functor. ☐

Proposition 5.7. *If $f : L_E A \to L_E B$ is a map in $L_E\mathbf{Alg}$ such that* $\mathrm{TAQ}(f)$ *is an equivalence, then $\hat{g} : \hat{L}_E A \to \hat{L}_E B$ is an equivalence; therefore, the following diagram commutes:*



where the unlabeled arrow is $\hat{g}^{-1} \circ can$.

Remark 5.8. Note that, in general, $\hat{L}_E A \neq L_E \hat{A}$, due to the same convergence problem.

As an important instance of the above theorem, we can take $E = K(n)$; using work of Hopkins, Ravenel and Wilson, who show that Morava K-theory doesn't see k-invariants in Postnikov towers of double loop spaces, Kuhn then proves:

Corollary 5.9. *If $X \in \mathcal{C}_{T(i)}$ for $1 \leq i \leq n-1$, then there exists a short exact sequence of Hopf algebras over $K(n)_*$:*

$$K(n)_* \to K(n)_*\mathrm{Sym}X \to K(n)_*\Omega^\infty X \to \bigotimes_{j=0}^{n+1} K(n)_*K(\pi_j X, j) \to K(n)_*$$

Note that, in some sense, this corollary measures to what extend the localized tower fails to converge. Other applications: Calculate $E_n^*(\Omega^\infty X)$, telescope conjecture, and speculations of Arone-Ching.

Remark 5.10. Kuhn moreover shows that the conclusion of the corollary implies that $X \in \mathcal{C}_{K(m)}$ for $0 < m < n$. Following ideas of May, this could be used to find counterexamples to the telescope conjecture. If Z is a connected space, then the above sequence can be rewritten as

$$K(n)_*\mathrm{Sym}\Sigma^\infty Z \to K(n)_*\mathrm{Sym}\Sigma^\infty Z \to \bigotimes_{j=0}^{n+1} K(n)_*K(\pi_j\Sigma^\infty Z, j)$$

So if Z is $K(n-1)$-acyclic but this sequence does *not* split, then by the corollary there must exist an $m < n$ such that $Z \notin \mathcal{C}_{T(m)}$, thereby violating the telescope conjecture.

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[3] see below