TALBOT PRETALK: KONTSEVICH'S FORMALITY OF THE LITTLE N-DISKS OPERAD

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Abstract. This is a talk for Stanford's Student Topology seminar, given as a pretalk for a Talbot talk on the rational homology of embedding spaces. In the calculation of the rational homology of embedding spaces using functor calculus, Kontsevich's formality theorem plays a key role.

In this talk we give a sketch of the proof of Kontsevich's result as given in [LV11], which we motivate by looking at Arnold's proof of the formality of configuration spaces of points in $\mathbb{C}$.

1. What is formality?

If one is given a space X there are several models to compute its cohomology algebra $H^*(X;R)$. Most generally for R an arbitrary ring of coefficients one can use singular cochains or, for X a CW-complex, cellular cochains. There are also more geometric models, which have the additional property that they are strict algebras: for $R=\mathbb{Q}$ and X any space we can use Sullivan's polynomial deRham forms. This is a generalization of the smooth deRham forms that one can define if X a manifold and $R=\mathbb{R}$. In this talk the relevant type will be semi-algebraic forms for X a semi-algebraic space and $R=\mathbb{R}$.

The chains with coefficients in $\mathbb{R}$ determine the stable real homotopy type of X, while by results of Sullivan the cochain algebra with coefficients in $\mathbb{R}$ determine the unstable real homotopy type.

Definition 1.1. A space X is said to be stably formal if there is a zigzag of quasi-isomorphisms of chain complexes

$$C_*(X;\mathbb{R}) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} H_*(X;\mathbb{R})$$

A space X is said to be formal if there is a zigzag of quasi-isomorphisms of CDGA's

$$C^*(X;\mathbb{R}) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} H^*(X;\mathbb{R})$$

Remark 1.2. One should replace every mention of the cochain algebra $C^*(X,\mathbb{R})$ here and later in the notes with a strictly commutative model, for example Sullivan's PL deRham forms $\mathcal{A}^*_{PL}(X)\otimes_{\mathbb{Q}}\mathbb{R}$ or, if X is smooth, ordinary deRham forms. Alternatively, one can work with E_∞-algebras.

Remark 1.3. Every space is stably formal. To see this, consider a complement $E_*\subset C_*(X,\mathbb{R})$ of boundaries $B_*\subset C_*(X;\mathbb{R})$ in the cycles $D_*\subset C_*(X,\mathbb{R})$. We take the zero differential on B_*. Then we have a zigzag

$$C_*(X;\mathbb{R}) \longleftarrow E_* \longrightarrow H_*(X;\mathbb{R})$$

where the left map is given by including E_* into $C_*(X;\mathbb{R})$ and the right map is given by sending the elements of E_* to corresponding homology classes. By construction, both maps are quasi-isomorphisms.

Rational homotopy theory tells us that at least for a simply-connected finite type space X, its rational homotopy type is determined by a rational CDGA model. A similar statement holds for the real homotopy type. So we can rephrase our definition of formality as saying that space is formal if its real cohomology is a real CDGA model for the real homotopy type of the space.

Examples of formal spaces are spheres, complex projective spaces and Kahler manifolds. An example of a non-formal space is the complement of the Borommian link in S^3 (which can be proven using Massey products).

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But in these notes we don't simply care about topological spaces, but actually care about topological operads. Operads encode algebraic structures on topological spaces and are given by a sequence $\{\mathcal{O}(n)\}_{n\geq 0}$ of spaces with an action of Σ_n, together with unit and composition maps

$$1 \to \mathcal{O}(1)$$

$$\mathcal{O}(n) \times \mathcal{O}(k_1) \times \ldots \times \mathcal{O}(k_n) \to \mathcal{O}(\sum_{i=1}^{n} k_i)$$

which satisfy three axioms: (i) the unit is indeed a unit for composition, (ii) composition is associative, (iii) composition is equivariant with respect to the symmetric group actions.

Since most if not all models of chains are symmetric monoidal, applying them to a topological operad gives one an operad in chain complexes. As a consequence we have a similar statement for homology.

Definition 1.4. An operad $\mathcal{O}$ is stably formal if there is a zigzag of morphisms of operads in chain complexes

$$C_*(\mathcal{O};\mathbb{R}) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} H_*(\mathcal{O};\mathbb{R})$$

that are quasi-isomorphisms for each $\mathcal{O}(n)$.

Remark 1.5. This is an interesting nothing, because not all operads are stably formal. Even if we can pick complements to the boundaries in the cycles for each $\mathcal{O}(n)$, there is no guarantee we can do these in a way compatible with the operad structure.

One can also ask about formality of operads. Since cochains are contravariant we have to talk about cooperads instead. In cochain complexes over $\mathbb{R}$ these are a sequence of cochain complexes $\mathcal{O}^*(n)$ with Σ_n-action and composition and counit maps

$$\mathcal{O}^*(1) \to \mathbb{R}$$

$$\mathcal{O}^*(\sum_{i=1}^{n} k_i) \to \mathcal{O}^*(n) \otimes \mathcal{O}^*(k_1) \otimes \ldots \otimes \mathcal{O}^*(k_n)$$

Writing down the properties that these maps have to satisfy is straightforward, except for a slight hitch coming from the fact that many if not all models of cochains are not symmetric comonoidal. That means that even though there is a natural chain map $C^*(X;\mathbb{R}) \otimes C^*(Y;\mathbb{R}) \to C^*(X \times Y;\mathbb{R})$ which is an quasi-isomorphism it is not an isomorphism. One can modify the definition of a cooperad in an ad-hoc way to deal with this.

Definition 1.6. An operad $\mathcal{O}$ is formal if there is a zigzag of morphisms of cooperads in CDGA's

$$C^*(\mathcal{O};\mathbb{R}) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} H^*(\mathcal{O};\mathbb{R})$$

that are quasi-isomorphisms for each $\mathcal{O}(n)$.

Kontsevich's theorem is about the little N-disks operad $\mathcal{D}_N$, which we recall in definition 3.1.

Theorem 1.7 (Kontsevich, Tamarkin)**.** *$\mathcal{D}_N$ is stably formal.*

This theorem was later extended to keep track of the actual real homotopy type, not just the stable real homotopy type. In this lecture we will sketch the proof of this theorem, closely following [LV11].

Theorem 1.8 (Lambrechts-Volic)**.** *$\mathcal{D}_N$ is formal if $N \neq 2$.*

2. Arnold's formality of configuration spaces of points in $\mathbb{C}$

As an example of the techniques used in the proof of the formality of little N-disks, we discuss Arnold's (complex) formality of configuration of points in the complex plane. This is of independent interest and the proof can easily be extended to determine the homology operad obtained by taking the homology of the little 2-disks operad, as $\mathcal{D}_2(n) \simeq C_n(\mathbb{C})$ due to the remarks following definition 3.1. For more details see the excellent article by Sinha [Sin10].

Definition 2.1. The configuration space $C_n(\mathbb{C})$ of ordered n-tuples of distinct points in $\mathbb{C}$ is defined to be $\{(z_1,\ldots,z_n)|z_i \neq z_j \text{ if } i \neq j\}$.

Arnold considered the following complex 1-forms on this space, for $1 \leq j < k \leq n$:

$$\omega_{ij} = \frac{d(z_k - z_j)}{z_k - z_j}$$

Note that this definition extends to $j \neq k$ and then $\omega_{jk} = \omega_{kj}$. A simple computation shows that the ω_{ij} are closed and hence represent complex cohomology classes.

Lemma 2.2. *For $j < k$, the ω_{jk} represent distinct cohomology classes. Furthermore, for distinct j, k, l these one-forms satisfy*

$$\omega_{jk} \wedge \omega_{kl} + \omega_{kl} \wedge \omega_{lj} + \omega_{lj} \wedge \omega_{jk} = 0$$

Proof. Consider ω_{jk} and ω_{lm} for $j < k$ and $l < m$ distinct pairs of integers. We must show there is no function f on $C_n(\mathbb{C})$ such that

$$\omega_{jk} - \omega_{lm} = df$$

To do this it suffices to find a loop such that the integral of the left hand side over that loop is non-zero. Suppose that $j \neq l$ (the other cases are similar). Consider the following loop $\gamma_j : S^1 \to C_n(\mathbb{C})$ (where we use the coordinate θ on $S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$):

$$\gamma_j : \theta \mapsto (1, 2, \ldots, j + (k - j + \frac{1}{2})e^{i\theta}, \ldots, n)$$

We have that

$$\int_{\gamma_j} \omega_{jk} = \int_0^{2\pi} \frac{i(k - j + \frac{1}{2})}{(k - j + (k - j + \frac{1}{2})e^{i\theta})} e^{i\theta} d\theta$$

Up to a non-zero constant this is the integral of the meromorphic function $z \mapsto \frac{1}{k-j+z}$ around a loop that goes around the pole once and hence is non-zero. We conclude that $\int_{\gamma_j} \omega_{jk} \neq 0$. On the other hand $\int_{\gamma_j} \omega_{lm} = 0$, since the loop doesn't go around the pole of $\frac{1}{m-l+z}$. This proves all ω_{jk} are independent.

For Arnold's 3-term relation, we do a straightforward computation:

$$\begin{aligned} \omega_{jk} \wedge \omega_{kl} &= \frac{dz_l \wedge dz_k - dz_l \wedge dz_j + dz_k \wedge dz_j}{(z_k - z_j)(z_l - z_k)} \\ &= \frac{dz_l \wedge dz_k + dz_j \wedge dz_l + dz_k \wedge dz_j}{(z_k - z_j)(z_l - z_k)(z_j - z_l)}(z_j - z_l) \end{aligned}$$

This term is symmetric with respect to cyclic permutations of j, k, l, except in the $(z_j - z_l)$ appearing at the end. Hence adding the three cyclic permutations to obtain $\omega_{jk} \wedge \omega_{kl} + \omega_{kl} \wedge \omega_{lj} + \omega_{lj} \wedge \omega_{jk}$ we get

$$\frac{z_l \wedge dz_k + dz_j \wedge dz_l + dz_k \wedge dz_j}{(z_k - z_j)(z_l - z_k)(z_j - z_k)}(z_j - z_l + z_k - z_j + z_l - z_k) = 0$$

$\square$

Similarly to the graphs that appear in the proof of the formality of little disks, we can encode the 3-term relation in graphs as in figure ... . As a results we have found a subalgebra of $\Omega^*_{dR}(C_n(\mathbb{C}); \mathbb{C})$:

$$A^* := \frac{\bigwedge_{1 \leq i < j \leq n} \mathbb{C} \cdot \omega_{jk}}{\langle \omega_{jk} \wedge \omega_{kl} + \omega_{kl} \wedge \omega_{lj} + \omega_{lj} \wedge \omega_{jk} + \text{ cylic permutations} \rangle}$$

This is a CDGA with trivial differential.

Theorem 2.3 (Arnold)**.** *We have a quasi-isomorphism and isomorphism of CDGA's as follows:*

$$\Omega^*_{dR}(C_n(\mathbb{C}); \mathbb{C}) \overset{\simeq}{\hookleftarrow} A^* \xrightarrow{\cong} H^*(C_n(\mathbb{C}); \mathbb{C})$$

and hence configuration spaces of ordered n-tuples of points in $\mathbb{C}$ are formal (over $\mathbb{C}$).

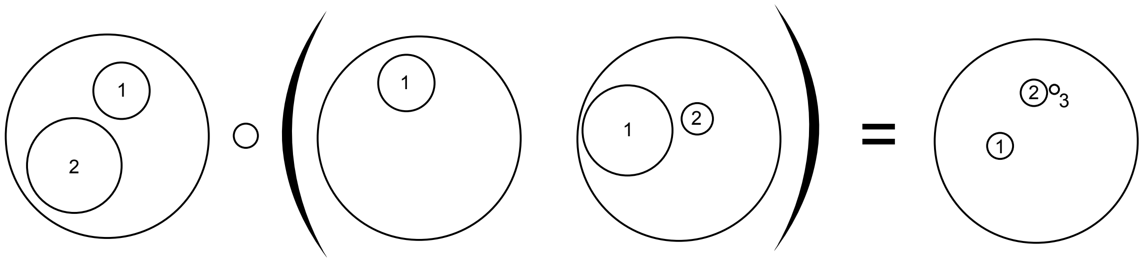

FIGURE 1. The composition of an element of $\mathcal{D}_2(2)$ with elements in $\mathcal{D}_2(1)$ and $\mathcal{D}_2(2)$.

Proof. It suffices to prove that the map $A^* \to H^*(C_n(\mathbb{C}); \mathbb{C})$ is an isomorphism. This will be done by induction using the Serre spectral sequence.

We begin with the case $C_2(\mathbb{C}) \simeq S^1$, where indeed the non-trivial cohomology class in degree 1 is represented by $\omega_{1,2}$. For the induction step, consider the fibration $C_{n+1}(\mathbb{C}) \to C_n(\mathbb{C})$ given by forgetting the last point. The fiber is homotopy equivalent to $\mathbb{C}\backslash\{1, \ldots, n\} \simeq \bigvee_{i=1}^n S^1$. The fundamental group of $C_n(\mathbb{C})$ is a pure braid group on n-strings and it acts trivially on the cohomology of the fiber (this is not easy to see, but can be found in e.g. [CLM76]).

Thus the cohomology Serre spectral sequence for this fibration is relatively simple: in fact the cohomology of the base and fiber are both generated in degree one, so all differentials must vanish. It now follows that $H^*(C_{n+1}(\mathbb{C}); \mathbb{C}) \cong H^*(C_n(\mathbb{C}); \mathbb{C}) \otimes H^*(\bigvee_{i=1}^n S^1; \mathbb{C})$. This means that the complex cohomology of $C_{n+1}(\mathbb{C})$ is generated in degree one by exactly $\binom{n+1}{2}$ classes. But the ω_{ij} are exactly $\binom{n+1}{2}$ cohomologically independent degree one cohomology classes. By looking at the higher degree cohomology, one sees that there are no additional relations to Arnold's 3-term relation. $\square$

3. KONTSEVICH'S FORMALITY OF THE LITTLE N-DISKS OPERAD

As noted before Arnold's theorem in fact brings us very close to the formality of the little 2-disks operad and the idea behind the proof of the formality of the little N-disks operad is closely related to it. Let us recall the definition of these operads. This uses the notion of a standard embedding: an embedding of a subset of $\mathbb{R}^N$ into another subset of $\mathbb{R}^N$ is standard if it is a composition of translations and dilations.

Definition 3.1. The n'th space $\mathcal{D}_N(n)$ of the little N-disks operad is given by the space of all standard embeddings of $\coprod_{i=1}^n D^N$ into D^N. The unit is the identity map $D^N \to D^N$, the composition comes from composing the embeddings and the action of the symmetric group is given by permuting the labels of the disks. See figure 1 for an example.

Let's take a closer look at the homotopy type of the spaces in the little N-disks operad. A standard embedding of a unit disk is uniquely determined by the midpoints and radii of the image disks. For each choice of distinct points as midpoints of images, the choice of acceptable radii is contractible. Using this it is not hard to see that $\mathcal{D}_N(n) \simeq C_n(D^N)$, the configuration space of ordered n-tuples of points in D^N. However this homotopy equivalence does not respect the operad structure and there is in fact no natural way to given an operad structure on configuration spaces. Later we will fix this by defining a compactification of $C_n(D^N)$ which does admit an operad structure compatible with that of $\mathcal{D}_N(n)$.

Example 3.2. In the case $N = 1$ each space of $\mathcal{D}_1$ is homotopy equivalent to a configuration space of points in the interval. These spaces break apart into connected components depending on the ordering of the points and each of these connected components is contractible. Hence $\mathcal{D}_1$ is homotopy equivalent to the associative operad Ass. The operad $\mathcal{D}_1$ is also known as A_∞.

3.1. **Overview.** Let us now give an overview of the proof of the formality of the little N-disks operad. We must produce a zigzag of quasi-isomorphisms of cooperads, starting at the cooperad

$C^*(\mathcal{D}_N;\mathbb{R})$ and ending at the cohomology cooperad $H^*(\mathcal{D}_N;\mathbb{R})$. We could write down a long zigzag doing this, but for convenience we break this up in several steps:

Step 1: The Fulton-MacPherson operad: It turns out to be more convenient to work with a different operad, weakly equivalent to the little N-disks operads, known as the Fulton-MacPherson operad $C_N[-]$. Each of its spaces $C_N[n]$ is obtained as a compactification of the configuration spaces $C_n(\mathbb{R}^N)$ by allowing infinitesimally close points. We will establish a zigzag of weak equivalences of operads

$$\mathcal{D}_N \xleftarrow{\simeq} W\mathcal{D}_N \xrightarrow{\simeq} C_N[-]$$

This induces a zigzag of quasi-isomorphisms of cooperads upon applying C^* or H^*. Hence we have reduced the problem to showing that the Fulton-MacPherson operad is formal.

Step 2: Semi-algebraic forms: We can now try to adapt Arnold's proof by writing down $\omega_{jk} = \theta_{jk}^*(\mathrm{vol}_{S^{N-1}})$, where $\theta_{jk} : C_n(\mathbb{R}^N) \to S^{N-1}$ is the map $(z_1, \ldots, z_n) \mapsto \frac{z_k - z_j}{||z_k - z_j||}$. This definition extends to $C_N[n]$. Now Arnold's 3-term relation no longer holds: the sum of the three terms is cohomologous to zero, but not zero on the nose. Thus additional "higher order correction" forms start playing a role. This can be constructed using integration along the fibers of particular bundles.

Thus we need to be able to work with forms and integrals. Usually one can do this by replacing singular cochains by the quasi-isomorphic algebra of deRham forms. However our configuration spaces are not compact, making it hard to define the integration maps in enough generality. This lack of compactness, together with the need for an operad structure, is the reason for passing to the Fulton-MacPherson compactification. Unfortuantely in doing so we end up up with something that is no longer a manifold.

However, it is still a semi-algebraic space and all operad maps are semi-algebraic maps. We will describe these notions in more detail later. The main consequence of this additional structure is that there exists a relatively small and well-behaved algebra of semi-algebraic forms Ω^*_{PA} which admits a zigzag of quasi-isomorphisms to the singular chains. Hence we have reduced our problem to finding a zigzag of quasi-isomorphisms of cooperads between $\Omega^*_{PA}(C_N[-])$ and $H^*(C_N[-];\mathbb{R})$. This will involve the ω_{jk} defined above.

Step 3 & 4: Admissible diagrams and the Kontsevich integral: The final step in our zigzag of quasi-isomorphisms will be the following zigzag:

$$\Omega^*_{PA}(C_N[-]) \xleftarrow{I} \mathcal{A}^*_N \xrightarrow{J} H^*(C_N[-]:\mathbb{R})$$

where $\mathcal{A}^*_N$ is a cooperad of admissible diagrams similar to the diagrams we used in our description of Arnold's theorem, J is a map defined by hand using the results of calculations of the cohomology of configuration spaces and finally I is Kontsevich's integral map, systematically producing the higher order correction forms described in the previous step. As soon as we have constructed these two maps and have shown they are quasi-isomorphisms, we will have proven the Kontsevich formality theorem.

The argument can be summarized as follows, where the arrows $\longleftrightarrow$ stand for zigzags of arrows.

$$\begin{tikzcd}
C^*(\mathcal{D}_N;\mathbb{R}) \arrow[d, leftrightarrow, "\simeq"'] & & H^*(\mathcal{D}_N;\mathbb{R}) \arrow[d, leftrightarrow, "\simeq"] \\
C^*(C_N[-];\mathbb{R}) \arrow[d, leftrightarrow, "\simeq"'] & & H^*(C_N[-];\mathbb{R}) \arrow[d, equal] \\
\Omega^*_{PA}(C_N[-]) & \mathcal{A}^*_N \arrow[l, "\simeq"] \arrow[r, "\simeq"'] & H^*(C_N[-];\mathbb{R})
\end{tikzcd}$$

Remark 3.3. It is the last step that rules out this proof of the case $N = 2$, though nobody doubts the result holds in that situation as well. The reason for this is that the obvious extension of the grading of the diagrams for higher N to $N = 2$ gives a $\mathbb{Z}$-graded instead of non-negatively graded cooperad of admissible diagrams.

At the end of this section will look at a related result known as relative formality. We look at this result because it has applications in the homology of embedding spaces through functor calculus as in [ALV07] [AT11]. A morphisms of operads is said to be stably formal or formal if there is a zigzag of quasi-isomorphisms between the induced maps on (co)chains and (co)homology. We are interested in the morphism $\iota : \mathcal{D}_M \hookrightarrow \mathcal{D}_N$ for $M \leq N$ of little disks operads given by using the centers and radii of the embeddings of M-disks into D^M as a description for an embedding of N-disks into D^N using the inclusion $D^M \hookrightarrow D^N$ coming from a linear inclusion of $\mathbb{R}^M$ into $\mathbb{R}^N$ as the first M coordinates.

Theorem 3.4. *If $M \geq 1$ and $N \geq 2M + 1$, then the morphism $\iota : \mathcal{D}_M \hookrightarrow \mathcal{D}_N$ is always stably formal, and formal if $M \neq 2$.*

Now that we actually start with the sketch of the proof we fix a $N > 2$ (the case $N = 1$ is slight different and easy to drop by hand) and drop this from the notation of the various operads, cooperads, etc., appearing in our proof.

3.2. **The Fulton-MacPherson operad.** We start by defining the Fulton-MacPherson operad and sketching the proof that it is weakly equivalent to the little N-disks operad. It is more convenient to use a coordinate-free approach to operads, where we have a space $\mathcal{O}(A)$ for every finite set A, which are related by bijections of finite sets. So every $\mathcal{O}(A)$ by identified in a non-unique way with $\mathcal{O}(n)$, and it is not hard to convince oneself that this notion of operad is equivalent to the one previously given.

We will define $C[A]$ as a compactification of a space $C(A)$ of A-labelled normalized configurations in $\mathbb{R}^N$, the precise definition of which is as follows:

Definition 3.5. The group of similarities of $\mathbb{R}^N$ is the subgroup of $\mathrm{Homeo}(\mathbb{R}^n)$ generated by the translation and dilations by positive real numbers.

We let $C(A) = \mathrm{Emb}(A, \mathbb{R}^N)/\mathrm{Sim}(\mathbb{R}^N)$. Equivalently, for $|A| \geq 2$ this is the space of A-labelled configurations of radius 1 and with center of mass at the origin.

This space is homotopy equivalent to $C_{|A|}(\mathbb{R}^N)$: it would be a homeomorphism if not for the loss of the homotopically trivial information of a overall radius and center of mass. There exists a map

$$\iota : C(A) \hookrightarrow \prod_{a \neq b \in A} S^{N-1} \times \prod_{a \neq b \neq c \in A} [0, \infty]$$

where for $x \in C(A)$ the components $\theta_{ab} : C(A) \to S^{N-1}$ are given by the relative direction

$$x \mapsto \frac{x(b) - x(a)}{||x(b) - x(a)||}$$

and the components $\delta_{abc} : C(A) \to [0, \infty]$ are given by the relative distances

$$x \mapsto \frac{||x(a) - x(b)||}{||x(a) - x(c)||}$$

We can completely recover the normalized A-labelled configuration from this data and ι is in fact a homeomorphism onto its image.

Definition 3.6. We defined the space $C[A]$ to be the closure of the image of ι in $\prod_{a \neq b \in A} S^{N-1} \times \prod_{a \neq b \neq c \in A} [0, \infty]$:

$$C[A] := \overline{\iota(C(A))}$$

This is a space of configurations that are allowed to be infinitesimally close: the points lie at the same "macroscopic" point in $\mathbb{R}^N$ but we remember their relative direction and distance. Note that there are different "levels" of being infinitesimally close. These are determined by looking at what it means for two points a, b to be infinitesimally close with respect to a third points c. This exactly happens when $\delta_{abc} = 0$. Thus we get a tree-like set of configurations with the lowest one

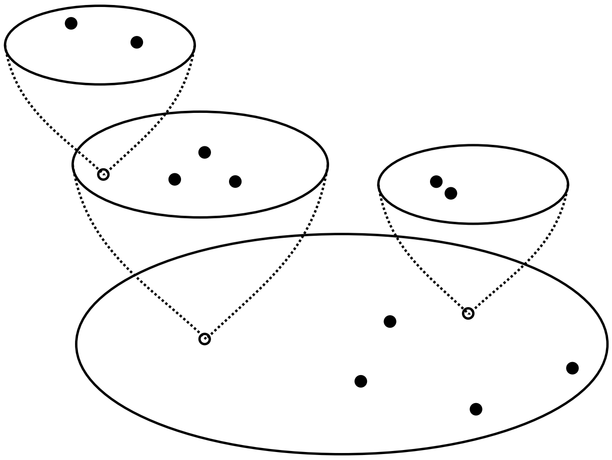

FIGURE 2. An element in $C[11]$ with four macroscopic points and seven microscopic points on two different levels. We have omitted the labels on the points of the configuration.

being the macroscopic levels and the latter ones being "microscopic configurations". See figure 2 for an example.

Let us now define the operad structure on the spaces $C[A]$. The unit map is easy: it picks the unique configuration in $C[*]$ corresponding to a point at the origin. The symmetric group actions are replcaed in the coordinate free framework by homeomorphisms $C[A] \to C[B]$ for every bijection $A \to B$. In our case this homeomorphism simply relabels the points. Finally, we need to define composition maps for each map[1] $f : B \to A$:

$$C[A] \times \prod_{a \in A} C[f^{-1}(a)] \to C[B]$$

This map is given by taking the configurations in $C[f^{-1}(a)]$ and shrinking them to become microscopic configurations to be placed at the point labelled by a in $C[A]$.

Theorem 3.7. *These composition maps make $C[A]$ into an operad.*

We must check that the Fulton-MacPherson operad is weakly equivalent to the little N-disks operad. For the case $N = 2$ one can use the Fiedorowicz recognition principle, which says that any two operads whose universal covers can be given the structure of a free contractible braided operad are equivalent. In general we have the following argument due to Salvatore.

Theorem 3.8. *There is a zigzag of weak equivalences of operads:*

$$\mathcal{D} \xleftarrow{\simeq} W\mathcal{D} \xrightarrow{\simeq} C[-]$$

Proof. The construction of the left-hand map is done using abstract machinery. The operad $W\mathcal{D}$ is given by Boardman-Vogt's W-construction applied to $\mathcal{D}$. It is given by assigning to a set A the space of metric planer rooted trees with edge length in the interval $[0, 1]$, leaves labelled by A and non-leaf vertices v decorated by an element of $\mathcal{D}(\mathrm{in}(v))$ (here $\mathrm{in}(v)$ is the set of incoming edges). We identify a tree with an edge of length 0 with the tree where that edge is collapsed and the corresponding decorations from $\mathcal{D}$ are composed. Composition is by grafting of decorated metric trees. Composing all elements of $\mathcal{D}$ decorating vertices according to the tree gives a map of operads $W\mathcal{D} \to \mathcal{D}$ which is a homotopy equivalence (in fact W is a cofibrant replacement in some model category structure on the category of topological operads, so is always a homotopy equivalence).

The non-abstract part is constructing a map $W\mathcal{D} \to C[-]$. We map a decorated tree to a configuration of points given by the centers of balls in the elements of $\mathcal{D}$ decorating the trees. The

[1]Note that f is not necessarily surjective. $C[\emptyset]$ is the one-point space and using it in a composition will cause us to forget about a point.

length of the edges determines whether a such a configuration should be considered as macroscopic (of a particular size) or microscopic. More precisely if t is the edge-length we scale the configuration by $1-t$ and if $t=1$ we make it microscopic. It is not hard to see this is compatible with the operad structure and is a weak equivalence of operads. □

This completes step 1 of the proof.

3.3. **Semi-algebraic forms.** The Fulton-MacPherson operad is not an operads of manifolds, but of manifolds with corners. Alternatively, it is an operad of compact semi-algebraic sets, the exact definition of which is given as follows:

Definition 3.9. A semi-algebraic set is a subset of $\mathbb{R}^k$ for some k, that can obtained using finite unions, finite intersections and complements from subsets defined by polynomial equations or inequalities. A map from semi-algebraic sets is semi-algebraic if its graph is.

The following is not hard to check using the correct charts on $C[A]$ and essentially follows because the θ_{ab} and δ_{abc} are algebraic.

Proposition 3.10. *The operad $C[-]$ is an operad of semi-algebraic sets.*

Thinking of the Fulton-MacPherson in this way turns out to be fruitful for getting a nice model of forms. We can't use (smooth) deRham forms since our compactification is not smooth, but we can't use polynomial deRham forms either because they don't have a well-behaved theory of fiber integration. We will end up picking a model of forms half-way between these two extremes to get the best of both worlds: a small geometric model of forms with a well-behaved theory of fiber integration.

To do this we note that on semi-algebraic sets there is a distinguished class of semi-algebraic functions and using these we can define semi-algebraic forms. A good reference for this is [HLTV11]. The semi-algebraic are given by a completion of an algebra of minimal forms. The minimal forms are a subalgebra of the dual algebra to the complex semi-algebraic chains, which are given semi-algebraic maps from a semi-algebraic compact manifold into X. We define the minimal forms to be the subalgebra generated by forms of the type

$$\mu = f_0 df_1 \cdots df_k$$

where all f_i are semi-algebraic functions $X \to \mathbb{R}$. The minimal forms on X are denoted by $\Omega^*_{\min}(X)$. They are insufficient as a model of forms for our purposes, as they don't have a Poincaré lemma nor do they have a good notion of fiber integration. To do this, we simply add the required forms. The following definition is made precise in [HLTV11, section 5.4]

Definition 3.11. Let X be a semi-algebraic set, then $\Omega^*_{PA}(X)$ is the subalgebra of the dual of the semi-algebraic chains generated by pushforwards of minimal forms along bundles of semi-algebraic sets.

The two most important properties of semi-algebraic forms are given in the following two propositions.

Proposition 3.12. *If X is a compact semi-algebraic set, then $\Omega^*_{PA}(X)$ and Sullivan's polynomial deRham forms $\mathcal{A}_{PL}(X;\mathbb{R})$ are connected by a zigzag of quasi-isomorphisms.*

Proposition 3.13. *If $\pi : E \to B$ is a bundle of semi-algebraic sets, whose fibers are consistently orientable semi-algebraic sets, then there exists a fiber integration map $\pi_! : \Omega^{*+k}_{\min}(X) \to \Omega^*_{PA}(X)$ having all the expected properties of a pushforward.*

Having completed step 2 of the proof we have now done enough prerequisites to start work on the heart of the theorem: constructing the following zigzag of quasi-isomorphisms:

$$\Omega^*_{PA}(C_N[-]) \xleftarrow{I} \mathcal{A}^*_N \xrightarrow{J} H^*(C_N[-]:\mathbb{R})$$

In the next section we will describe the admissible diagrams $\mathcal{A}^*$ and their cohomology using the map J (step 3), and in the final section we will describe the Kontsevich integral I (step 4).

3.4. **The cooperad of admissible diagrams.** In this section we describe a cooperad $\mathcal{A}^*$ of CDGA's of admissible diagrams due to Kontsevich. These diagrams model the N-forms ω_{jk} generalizing Arnold's forms and all the additional "higher order correction" forms needed for the relations between these. The subalgebra of those diagrams having no internal edges will exactly be the cohomology algebra of the Fulton-MacPherson operad and the quasi-isomorphism $J : \mathcal{A}^* \to H^*(C[-], \mathbb{R})$ will be the projection.

We will only give the definition of the cooperad of admissible diagrams for $N > 2$. The case $N = 1$ can be done by hand, but in the case $N = 2$ the grading leads to difficulties that are apparently difficult to resolve. The way we will approach the definition is by describing the CDGA $\mathcal{A}^*(A)$ of admissible diagrams as a quotient of a larger CDGA $\mathcal{B}^*(A)$ of diagrams. Let's define these guys.

Definition 3.14. An A-labelled diagram Γ consists of a linearly ordered finite set I_Γ of *internal vertices*, a linearly ordered finite set E_Γ of (oriented) *edges* and source, respectively target, maps $s_\Gamma, t_\Gamma : E_\Gamma \to A_\Gamma \sqcup I_\Gamma$. We call the set A the *external vertices*.

As a $\mathbb{R}$-vector space we let $\mathcal{B}^*(A)$ be generated by the isomorphism classes of diagrams, where a permutation changing the linear order on the internal vertices acts by the N'th power of the sign representation and a permutation changing the linear order on the edges acts by the $(N+1)$'st power of the sign representations. The degree of a diagram Γ is given by $|E_\Gamma| \cdot (N-1) - |I_\Gamma| \cdot N$.

Next we define the product $\Gamma \cdot \Gamma'$ of diagrams. It is given by the union: the new sets of internal vertices and edges are obtained by taking the disjoint union and ordering them consecutively. Note that A remains the same.

Finally, we define the differential d. It is given by taking the alternating sum over all contractions of edges that are not of the following types: (i) loops, (ii) connecting two external vertices, (iii) a "dead end", i.e. an edge ending on an internal vertex of valence one. These are called contractible edges.

This definition contains several claims. The most important one, that d satisfes the properties of a differential is the content of the next lemma.

Lemma 3.15. *The differential d is well-defined, homogeneous of degree 1 and satisfies the Leibniz rule*

$$d(\Gamma \cdot \Gamma') = d(\Gamma) \cdot \Gamma' + (-1)^{\deg \Gamma} \Gamma \cdot d(\Gamma')$$

and finally satisfies $d^2 = 0$.

Proof. The well-definedness is easy to check. To see that it is homogeneous of degree 1 one notes that upon contracting an edge we lose an edge and a vertex, changing the degree by $-(N-1)+N = 1$.

An edge is contractible in $\Gamma \cdot \Gamma'$ if and only if it is in Γ or Γ'. Say e came from Γ, then $\Gamma/e \cdot \Gamma' \cong (\Gamma \cdot \Gamma')/e$ up to a sign. A similar statement holds for edges coming from Γ'. This makes Leibniz rule clear, except for the signs. Finally, $d^2 = 0$ is essentially a consequence of our sign convention and the fact that e_2 is contractible in Γ/e_1 if and only if e_1 is contractible in Γ/e_2. □

Hence $\mathcal{B}^*(A)$ is indeed a well-defined CDGA. Now we define the non-admissible diagrams. It will be quotient of $\mathcal{B}^*(A)$.

Definition 3.16. A diagram is admissible if it doesn't contain one of the following situations: (i) loops, (ii) double edges, (iii) internal vertices of valence ≤ 2, (iv) components disconnected from externel vertices. See figure 3.

Let $\mathcal{N}^*(A)$ be the submodule generated by those diagrams that are not admissable. Then we define the CDGA of admissible A-labelled diagrams to be

$$\mathcal{A}^*(A) = \mathcal{B}^*(A)/\mathcal{N}^*(A)$$

So we have now defined our CDGA's of admissible diagrams (see figure 4 for an example for multiplication). It remains to fit them into a cooperad structure and exhibit a quasi-isomorphism J of cooperads to $H^*(C[-]; \mathbb{R})$.

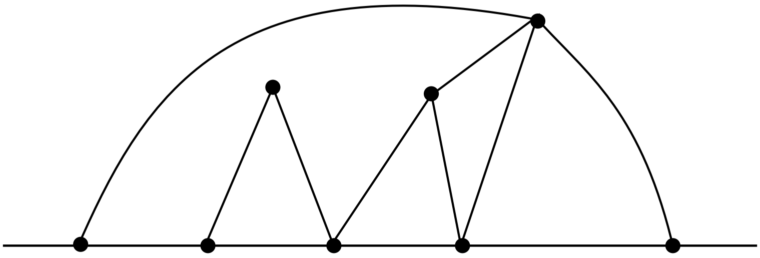

Figure 3. An admissible diagram with three internal vertices and five external vertices (which we picture as lying on a horizontal line that is *not* part of the admissible diagram). We omitted the labelling and ordering of the vertices and the edges.

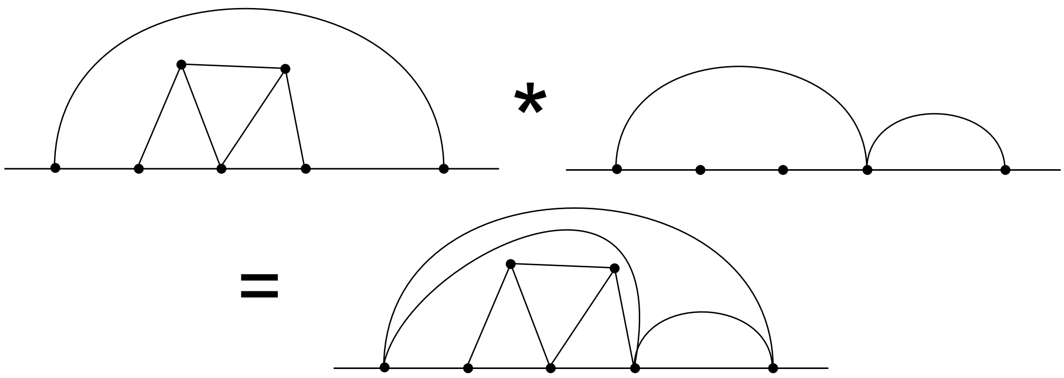

Figure 4. The product of two admissible diagrams.

For the cooperad structure, we define one on $\mathcal{B}^*(A)$ and check it descends to $\mathcal{A}^*(A)$. We do this in the coordinate-free framework for cooperads. Let $f : B \to A$ be a map of sets, then we must define a map

$$\mathcal{B}^*(B) \to \mathcal{B}^*(A) \otimes \bigotimes_{a \in A} \mathcal{B}^*(f^{-1}(a))$$

For a B-labelled diagram $\Gamma \in \mathcal{B}^*(B)$ our map will be a sum over localizations of Γ. These localizations are by definition maps $\lambda : B \sqcup I_\Gamma \to A$ that restrict to f on B. The components Γ_a in the big tensor product are the f-local graphs. They are given by the subgraph containing the vertices v with $\lambda(v) = a$ and the edges both of whose vertices map to a. The component Γ_A is obtained by collapsing the graphs Γ_a to a vertex labelled by a. It turns out that when this procedure is applied to a $\Gamma \in \mathcal{N}^*(B)$, at least one of the terms will not be admissible and hence this cooperad structure descends to one on $\mathcal{A}^* = \mathcal{B}^*/\mathcal{N}^*$.

Finally, we must define the map $J : \mathcal{A}^* \to H^*(C[-]; \mathbb{R})$. The main result we need is F. Cohen's calculation of the cohomology of configuration spaces (to which the $C[n]$ are homotopy equivalent):

Theorem 3.17. *For a finite set A, there are classes $g_{ab} \in H^{N-1}(C[A]; \mathbb{R})$ indexed by distinct $a, b \in A$, such that there is an isomorphism*

$$H^*(C[A]; \mathbb{R}) \cong \frac{\Lambda_{a \neq b} \mathbb{R} \cdot g_{ab}}{\langle \text{3-term relation}, \, g_{ab}^2, g_{ab} = -(-1)^N g_{ba} \rangle}$$

Sketch of proof. The g_{ab} are represented by the analogues of Arnold's classes:

$$\omega_{ab} = \theta_{ab}^*(\mathrm{vol}_{S^{N-1}})$$

where $\theta_{ab} : C[A] \to S^{N-1}$ was the map $x \mapsto \frac{x(b)-x(a)}{||x(b)-x(a)||}$. The proof now proceeds similarly to that of Arnold's theorem. Calculations by hand verify the independence of the classes and the relations.

After this, one uses induction using a Serre spectral sequence to prove that these are indeed all generators and relations. □

Now the map J is given by sending the diagram Γ_{ab} consisting of vertices A and a single edge between a and b to g_{ab}, and extending it additively and multiplicatively.

Lemma 3.18. *J is a quasi-isomorphism between $\mathcal{A}^*(A)$ and $H^*(C[A];\mathbb{R})$ and is compatible with the cooperad structure.*

Sketch of proof. One starts by proving that the subalgebra of $\mathcal{A}^*(A)$ of admissible diagrams without internal vertices is isomorphic to $H^*(C[A];\mathbb{R})$ and surjects onto it. After this it is a matter of checking by calculation that $H^*(\mathcal{A}^*(A))$ is abstractly isomorphic to $H^*(C[A];\mathbb{R})$. □

This completes step 3 of our sketch of the proof.

3.5. **The Kontsevich integral.** Finally we defining the remaining quasi-isomorphism $I : \mathcal{A}^* \to \Omega^*_{PA}(C[-])$ of cooperads. It will be given the classes ω_{ab} when evaluated on the graphs Γ_{ab} with a single edge between $a, b \in A$. This is enough to show it is quasi-isomorphism, because for that it suffices to note that

$$I^* : H^*(\mathcal{A}^*) \to H^*_{PA}(C[-])$$

is an isomorphism by the previous calculations of the real cohomology of $C[A]$ and the cohomology of $\mathcal{A}^*$.

Let's define I now. It is constructed similarly Arnold's forms using maps to spheres. We will define it on $\mathcal{B}^*(A)$ first and then check it descends to $\mathcal{A}^*(A)$. Given a diagram Γ we have a map

$$\theta_\Gamma : C[A \sqcup I_\Gamma] \to (S^{N-1})^{E_\Gamma}$$

with component $\theta_e : C[A \sqcup I_\Gamma] \to S^{N-1}$ given by the map $\theta_{s(e),t(e)}$ defined earlier. Then we set

$$\hat{\omega}_\Gamma = \theta^*_\Gamma(\mathrm{vol}_{E_\Gamma})$$

where the volume form is the product of the (normalized) volume forms of the spheres. This is a form in $\Omega^{|E_\Gamma|\cdot(n-1)}_{PA}(C[A \sqcup I_\Gamma])$ and is in fact minimal.

There is a projection map $\pi_\Gamma : C[A \sqcup I_\Gamma] \to C[A]$ by just forgetting the points labelled by I_Γ. Since the spaces in the Fulton-MacPherson operad are compact semi-algebraic, one can check that this projection map is a bundle with compact semi-algebraic bundle fibers, of dimension $N \cdot |I_\Gamma|$. The definition of semi-algebraic forms tells us that we integrate along this fiber and thus we define

$$\omega_\Gamma = (\pi_\Gamma)_! \theta^*_\Gamma(\mathrm{vol}_{E_\Gamma}) \in \Omega^{|E_\Gamma|\cdot(n-1)-|I_\Gamma|\cdot N}_{PA}(C[A])$$

One can check that if we apply this construction to a non-admissible graph we get 0 and hence this construction descends to a map $I : \mathcal{A}^*(A) \to \Omega^*_{PA}(C[A])$, called the Kontsevich integral. The proof is now completed by the following proposition outlining the properties of I to be checked.

Proposition 3.19. *The Kontsevich integral I has the following properties:*

1. *I is a map of algebras*
2. *I commutes with the differential.*
3. *I is a morphism of cooperads.*

Sketch of proof. The first is a consequence of the compatibility of induced maps and fiber integration with pullback diagrams.

The second is a consequence of a fiberwise Stokes formula, expressing an integral over a boundary as an integral of a fiberwise exact form. Finally, one checks that a form pulled back to a part of the boundary of the Fulton-MacPherson compactification can be expressed as a sum of forms on the different components of the boundary. □

This completes step 4 and hence our sketch of the proof of the Kontsevich formality theorem.

3.6. **Relative formality.** Finally, we make some remarks on relative formality. Recall that the theorem we want to prove says that the map $\iota : D_M \hookrightarrow D_N$ is formal if $M \neq 2$ and $N \geq 2M+1$. This meant that there was a zigzag of commutative squares starting at $C^*(\iota)$ and ending at $H^*(\iota)$ whose horizontal arrows are quasi-isomorphisms of cooperads.

It suffices to prove the statement for the corresponding map on the Fulton-MacPherson compactification $\iota : C_M \hookrightarrow C_N$ induced by the linear inclusion $\mathbb{R}^M \hookrightarrow \mathbb{R}^N$ and using semi-algebraic forms. Hence it suffices to find a map $\mathcal{A}_N^* \to \mathcal{A}_M^*$ making the following diagrams commute

$$\begin{tikzcd}
\Omega^*_{PA}(C_N[-]) \arrow[d] & \mathcal{A}_N^* \arrow[l, "I"'] \arrow[r, "J"] \arrow[d, "\epsilon"] & H^*(C_N[-];\mathbb{R}) \arrow[d] \\
\Omega^*_{PA}(C_M[-]) & \mathcal{A}_M^* \arrow[l, "I"'] \arrow[r, "J"] & H^*(C_M[-];\mathbb{R})
\end{tikzcd}$$

This map $\epsilon : \mathcal{A}_N^* \to \mathcal{A}_M^*$ can be given by mapping every admissible diagram to zero except the unit diagram. The right square commutes because the right-hand map is zero except on the generator in degree zero: there is no place for the generators of the cohomology to go. The left square commutes because the restriction $N \geq 2M+1$ means that for every admissible diagram the first non-zero forms of positive degree in $\Omega^*_{PA}(C_N[A])$ live in degree higher than $\dim(C_M[A])$ and hence must be mapped to zero.

4. A quick sketch of two applications

In this section we discuss two applications of Kontsevich formality.

4.1. **Deformation quantization.** Kontsevich showed in [Kon99] how the Deligne conjecture and the (stable) formality of the little 2-disks operad imply deformation quantization. This is an alternative proof from the one in [Kon03][2]. From a product $\star : C^\infty(X)[[\hbar]] \to C^\infty(X)[[\hbar]]$ one can recover a Poisson bracket by looking at the first-order term in $\hbar$. Deformation quantization is concerned with reversing this process: for a Poisson bracket on a manifold, can we construct a product on $C^\infty(X)[[\hbar]]$ whose first-order term is that Poisson bracket? The answer is yes:

Theorem 4.1 (Kontsevich)**.** *For every Poisson bracket $\{-,-\}$ we can find a product $\star$ of the form*

$$f \star g = fg + \{f,g\}\hbar + \sum_{n=2}^{\infty} B_n(f,g)\hbar^n$$

with the $B_n(f,g)$ bidifferential operators of degree n. Furthermore, this product $\star$ is unique up to "gauge equivalence".

Kontsevich deduces this from a L_∞-quasi-isomorphism between the Hochschild cohomology complex of the algebra of smooth functions on a smooth manifold and its cohomology, the algebra of polyvector fields on X.

Let's define these things: the Hochschild complex $CH^*(A,A)$ of an algebra A is the cochain complex given in degree n by $\mathrm{Hom}_{\mathrm{Bimod}_A}(A^{\otimes n}, A)$ and differential

$$\begin{aligned} df(a_0 \otimes \ldots \otimes a_n) = &\sum_{i=1}^{n} (-1)^i df(a_0 \otimes \ldots \otimes a_{i-1}a_i \otimes \ldots \otimes a_n) \\ &+ a_0 df(a_1 \otimes \ldots \otimes a_n) + (-1)^{n+1} df(a_0 \otimes \ldots \otimes a_{n-1})a_n \end{aligned}$$

The polyvector fields are sections of the exterior powers of the tangent bundle. Both of these are Lie algebras up to homotopy, known as L_∞-algebras. That there is a quasi-isomorphisms between them is essentialy the HKR theorem, but that there is one which respects in the L_∞ is what makes this imply deformation quantization.

This is the case because it implies that the spaces of formal solutions in $\hbar$ to the Maurer-Cartan equation $dx + [x,x] = 0$ are equivalent (up to gauge-equivalence): in the case of polyvector fields Poisson structures give us solutions to the Maurer-Cartan equations, and in the case of the

[2]The article [Kon03] was actually written and distributed as a preprint before [Kon99].

Hochschild complex the solutions are given by star products (up to gauge equivalence). So any Poisson structures gives us a star product.

So let's see how the Deligne conjecture and (stable) formality of the little 2-disks operad imply the existence of such a L_∞-quasi-isomorphism. First one proves that the case $\mathbb{R}^n$ implies the general case. Now the Deligne conjecture tells us that the cochains of $\mathcal{D}_2$ act on $CH^*(A, A)$ for any algebra A. Hence the cohomology of $\mathcal{D}_2$ acts on $HH^*(A, A)$, but by formality we can also (up to homotopy) consider this as an algebra over the chains of $\mathcal{D}_2$. The L_∞-structure is encoded as part of the action of the chains of $\mathcal{D}_2$.

If A is the algebra of smooth functions on $\mathbb{R}^n$, then $CH^*(A, A)$ are $HH^*(A, A)$ are as desired. By HKR they are quasi-isomorphic and we can compare their L_∞-structures. These are different a priori, but the obstruction to them not being equivalent up to homotopy lies in an obstruction group that is isomorphic to the space of bivector fields on $\mathbb{R}^n$. The obstruction can be made $\mathrm{Aff}(\mathbb{R}^n)$-invariant and no such bivector field exists. So we can in fact find a L_∞-quasi-isomorphism between them.

4.2. **Rational collapse of the Vassiliev spectral sequence.** The ideas of these theorems extends to general theorems about the collapse of Taylor towers to embedding functors, as explained in [LTV10] and [ALV07].

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