TALBOT TALK: EMBEDDING CALCULUS, THE LITTLE DISKS OPERAD AND RATIONAL HOMOLOGY OF SPACES OF EMBEDDINGS

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Abstract. These are the notes for a talk at the Talbot workshop on functor calculus. This talk discusses how under pretty general conditions the embedding calculus Taylor can be written for in terms of a derived mapping space of modules over the little disks operad. As an application we discuss two proofs that the rational homology of the space of reduced embeddings of a manifold M into a Euclidean space of sufficiently high dimension depends only on the rational homology of M.

In this talk we discuss the results of [ALV07] and [AT11]. To keep these notes concise and focused on the statements that we want to prove, we will not recall all of the background in functor calculus and operad theory needed to understand these results. For that we refer to the notes of the other talks at the Talbot workshop or the previously given references.

1. Introduction and overview

Embedding calculus studies a particular class of functors

$$F : \mathcal{O}(M)^{\mathrm{op}} \to \mathsf{D}$$

where $\mathcal{O}(M)$ is the poset of open subsets of a fixed manifold M and D is a nice model category (closed monoidal combinatorial will suffice). In these notes D will always be one of the following four model categories: pointed spaces Top_*, spectra Sp, rational chain complexes $\mathsf{Ch}_{\mathbb{Q}}$ or $H\mathbb{Q}$-module spectra $\mathsf{Sp}_{H\mathbb{Q}}$.

The class of functors we are interested in are the good isotopy functors. Here *isotopy functor* means that F sends inclusions that are isotopy equivalences to homotopy equivalences and *good* means that F sends filtered unions to homotopy limits.

We want to approximate such a functor by its value on open subsets homeomorphic to a disjoint union of finite number of open balls and the relations between these under inclusions. If we fix an integer $k \geq 0$, the relations under inclusions between open subsets of M that are homeomorphic to a disjoint union of at most k balls are encoded by the subposet $\mathcal{O}_k(M)$ of $\mathcal{O}(M)$ consisting of such open subsets. To isolate the values of F on these we just consider the restriction $F|_{\mathcal{O}_k(M)}$. The best possible approximation of the F in terms of the value of F on elements $\mathcal{O}_k(M)$ is given by the left (homotopy) Kan extension

$$T_k F(M) = \operatorname*{holim}_{U \in \mathcal{O}_k(M)} F(U)$$

This is called the k'th Taylor polynomial of F. The reason for phrasing the definition of the Taylor tower in this way, stressing the relations between the open subsets, is to remind you of the little n-disks operad. This operad similarly encodes the relations between open balls under inclusion.

Definition 1.1. The *little n-disks operad* $\mathbb{B}_n$ is the operad in Top with spaces

$$\mathbb{B}_n(k) = \mathrm{sEmb}(\coprod_k D^n, D^n)$$

where D^n is the standard disk in $\mathbb{R}^n$ and sEmb denotes the standard embeddings: these are on each connected component a composition of dilation and translation.

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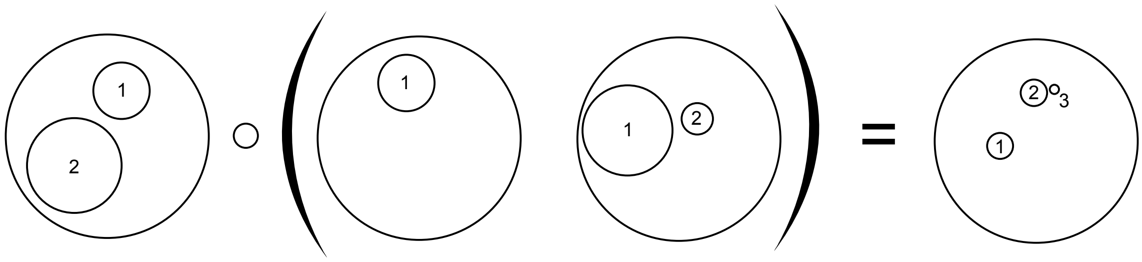

FIGURE 1. The composition of an element of $\mathcal{D}_2(2)$ with elements in $\mathcal{D}_2(1)$ and $\mathcal{D}_2(2)$.

The unit is the identity map $\mathrm{id} : D^n \to D^n$. The composition is given by composition of standard embeddings, which are clearly closed under composition. See figure 1 for an example.

These notes have a two-fold goal.

(1) Under the conditions that M is an open submanifold of $\mathbb{R}^m$ and F is a so-called context-free functor, we make this vague relationship between the Taylor tower and the little disks operad precise. In particular, we will get an expression of $T_k F$ in terms a space of (derived) maps of right modules over $\mathbb{B}_m$. This is theorem 2.11.
(2) We want to apply this to the example
$$F(-) = H\mathbb{Q} \wedge \mathrm{hofib}(\mathrm{Emb}(-, V) \to \mathrm{Imm}(-, V))$$
where V is an Euclidean space. We will often denote this homotopy fiber by $\overline{\mathrm{Emb}}(-, V)$. Note it requires a basepoint, i.e. an embedding $\alpha : M \to V$, to actually be a functor instead of a functor up to homotopy.

The punchline, which can be reached by two related but independent methods, will be that if $\dim V$ is sufficiently large in comparison to $\dim M$, then $H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V)$ only depends on $H\mathbb{Q} \wedge M$. Hence every rational homology equivalence $M_1 \to M_2$ between two manifolds induces a rational homology equivalence $\overline{\mathrm{Emb}}(M_2, V) \to \overline{\mathrm{Emb}}(M_1, V)$ for V sufficiently large. We'll use this to find $H_*(\overline{\mathrm{Emb}}(\mathbb{R}P^{2n}, \mathbb{R}^k); \mathbb{Q})$ for sufficiently large k, though we invite the reader to find this themselves.

Remark 1.2. There is an elegant proof of these results in the enriched setting in [BdBW12], which we recommend as a complement to these notes.

Remark 1.3. There is a formally very similar theory for compactly supported embedding calculus and the spaces $\overline{\mathrm{Emb}}_c(\mathbb{R}^m, \mathbb{R}^n)$ of compactly supported embeddings (i.e. equal to the identity outside a compact subset). This theory generalizes results like Sinha's theorem about the relation of compactly supported embedding calculus to the Vasilliev spectral sequence for $\overline{\mathrm{Emb}}_c(\mathbb{R}, \mathbb{R}^k)$, also known as the space of long knots, Lambrechts and Volic's result on the collapse of this spectral sequence if $k \geq 4$ and Volic's calculations of the E_2-term in terms of the Hochschild homology of the Poisson operad. A good start for this story is [Sin05].

2. WRITING THE TAYLOR TOWER FOR CONTEXT-FREE FUNCTOR IN TERMS OF MODULE MAPS

From now on we suppose that $M \subset \mathbb{R}^m$ is an open submanifold. This gives us additional structure on the open subsets of M, as they are also open subsets of $\mathbb{R}^m$. In particular among the open subsets homeomorphic to open balls we can find open subsets that actually are open balls.

Definition 2.1. A *standard ball* in M is a ball in $\mathbb{R}^m$, i.e. a subset of the form $\{x \in \mathbb{R}^m \mid ||x - x_0|| < r\}$, that is contained in M.

Hence it might be worthwhile to consider a smaller subposet of $\mathcal{O}(M)$

Definition 2.2. Let $\mathcal{O}^s_k(M)$ for $k = 0, 1, \ldots, \infty$ be the subposet of $\mathcal{O}(M)$ consisting of open subsets that are a disjoint union of at most k standard balls in M.

Note that for each $k = 0, 1, \ldots, \infty$ there is an inclusion functor

$$\mathcal{O}_k^s(M) \hookrightarrow \mathcal{O}_k(M)$$

where $\mathcal{O}_k(M)$ is the subposet of $\mathcal{O}(M)$ consisting of open subsets of M homeomorphic to a disjoint union of at most k balls. In an intuitive sense the restriction of F to $\mathcal{O}_k^s(M)$ contains the same amount of information as the restriction of F to $\mathcal{O}_k(M)$: every open subset homeomorphic to a disjoint union of at most k balls is isotopic in M to k standard balls and inclusion of such subsets can similarly be isotoped to be an inclusion of standard balls. This intuition is made precise by the following theorem.

Theorem 2.3. *The natural map of homotopy limits induced by the inclusion* $\mathcal{O}_k^s(M) \hookrightarrow \mathcal{O}_k(M)$

$$\operatorname*{holim}_{U \in \mathcal{O}_k(M)} F(U) \to \operatorname*{holim}_{U \in \mathcal{O}_k^s(M)} F(U)$$

is a weak equivalence.

Sketch of proof. Call the right hand side $T_k^s F$ for the moment. We use the techniques of [Wei99, section 3] to get description of $T_k F$ and $T_k^s F$ as a totalization of a cosimplicial space coming with levels encoding the value of F on exactly p balls. These levels can then be understood more easily. Consider the following diagram, whose terms will be explained later:

$$\begin{tikzcd}
\mathrm{holim}_{\mathcal{I}_k\mathcal{O}_k} F \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & \mathrm{holim}_{\mathcal{O}_k} F = T_k F \arrow[ddd, dotted, "\simeq"] \\
\mathrm{Tot}(p \mapsto \mathrm{holim}_{\mathcal{I}_k\mathcal{O}_k(p)} F) \arrow[d, "\simeq"'] & \\
\mathrm{Tot}(p \mapsto \mathrm{holim}_{\mathcal{I}_k^s\mathcal{O}_k^s(p)} F) \arrow[d, "\simeq"'] & \\
\mathrm{holim}_{\mathcal{I}_k^s\mathcal{O}_k^s} F \arrow[r, "\simeq"'] & \mathrm{holim}_{\mathcal{O}_k^s} F = T_k^s F
\end{tikzcd}$$

Here $\mathcal{I}_k\mathcal{O}_k$ is the double category with the same objects and horizontal morphisms as $\mathcal{O}_k(M)$, but vertical morphisms only the isotopy equivalences in $\mathcal{O}_k(M)$. The double category $\mathcal{I}_k^s\mathcal{O}_k^s$ is similar, using $\mathcal{O}_k^s(M)$ instead of $\mathcal{O}_k(M)$. The category $\mathcal{I}_k\mathcal{O}_k(p)$ for $p = 0, 1, \ldots$ has as objects functors $[p] \to \mathcal{O}_k$, i.e. sequences of morphisms of length $p+1$, and as morphisms maps of sequences with all arrows isotopy equivalences.

The horizontal maps are induced by the inclusion of the horizontal category $\mathcal{O}_k$ into $\mathcal{I}_k\mathcal{O}_k$ (and its standard variant). These are weak equivalences because F sends all isotopy equivalences to weak equivalences. Of the left vertical arrows, the top and bottom ones are weak equivalences as a consequence of the way we calculute homotopy limits over a double category.

This leaves the middle vertical arrow, which is the only part of the proof containing geometric content. Indeed, all the other manipulations were just done to reduce to a situation where it suffices to prove that for all p the natural map $\mathrm{holim}_{\mathcal{I}_k\mathcal{O}_k(p)} F \to \mathrm{holim}_{\mathcal{I}_k^s\mathcal{O}_k^s(p)} F$ is a weak equivalence. Because F takes the morphisms of these category to weak equivalences, both homotopy limits are weakly equivalent to a space of sections associated to the quasifibration over the geometric realisation of the indexing category. Both the fibers and the geometric realisation of the indexing category can inductively be found, as Weiss does in [Wei99, section 3], and are then seen to be the same up to weak equivalence. After this it is not hard to check that the natural map indeed does what one expects and is a weak equivalence.

This implies that all the solid arrows are weak equivalences and hence the dotted one is, completing the proof. $\square$

Since we can now exclusively deal with standard balls it seems more likely that we can connect to $\mathbb{B}_m$, which also deals with standard balls. Our next goal will be to relate $\mathcal{O}_k^s(M)$ to a Grothendieck category constructed from a right module over a category related to $\mathbb{B}_m$. So let's recall some operad theory.

An operad is a symmetric sequence together with composition and unit maps, satisfying unit, associativity and equivariance axioms. Associated to an operad $\mathcal{O}$ in D, a closed symmetric monoidal category with all coproducts, there is a category $\mathsf{F}(\mathcal{O})$ enriched over D encoding the full structure[1] of $\mathcal{O}$.

Definition 2.4. The category $\mathsf{F}(\mathcal{O})$ (for "$\mathcal{O}$-labelled forests") has as objects finite sets A and morphism objects in D as follows

$$\mathrm{hom}_{\mathsf{F}(\mathcal{O})}(B, A) = \coprod_{f : B \to A} \left(\bigotimes_{a \in A} \mathcal{O}(f^{-1}(\{a\})) \right)$$

The composition of morphisms is defined using the operad maps and the identity maps come from the identity maps of the operad.

Examples of operads include the $\mathbb{B}_m$'s as operads in Top and the commutative operad Com as an operad Set (or essentially anything tensored over Set). The latter has as underlying symmetric sequence $\mathrm{Com}(A) = *$, which completely determines the composition maps.

Similarly, recall that a (weak) right module over an operad $\mathcal{O}$ is a symmetric sequence M in D (or more generally anything enriched, tensored and cotensored over D) with composition maps

$$- \circ_a - : M(A) \otimes \mathcal{O}(B) \to M(A \cup_a B)$$

where $A \cup_a B$ is by definition the set $(A \backslash \{a\}) \sqcup B$. These composition maps must satisfy appropiate unit, associativity and equivariance axioms.

As a first example we have that any operad is a right module over itself. The following example is more relevant to these notes and slightly less trivial.

Example 2.5. Let M be an open submanifold of $\mathbb{R}^m$. Then

$$M(A) = \mathrm{sEmb}(A \times D^n, M)$$

is a right module over $\mathbb{B}_m$.

Let's relate this to our category $\mathsf{F}(\mathcal{O})$.

Lemma 2.6. *There is an equivalence of categories between (1) right modules over $\mathcal{O}$ and morphisms between these, and (2) contravariant enriched functors $\mathcal{M} : \mathsf{F}(\mathcal{O}) \to \mathsf{D}$ and natural transformations between these.*

Proof. **From module to functor:** We first construct a contravariant enriched functor $\mathcal{M}$ from a right module M as follows. On a finite set A we set $\mathcal{M}(A) = M(A)$. Because our codomain category is closed symmetric monoidal to define $\mathcal{M}$ on morphisms it suffices to produce maps

$$\mathcal{M}(A) \otimes \bigotimes_{a \in A} \mathcal{O}(f^{-1}(\{a\})) \to \mathcal{M}(B)$$

for all maps $f : B \to A$ of finite sets. To do this we use the operations $- \circ_a -$ for each of the sets $f^{-1}(\{a\}) \subset B$ in the tensor product. Associativity tells us that the order in which we do this doesn't matter. This is a functor using the unit, associativity and equivariance axioms of M: to be precise the unit axiom gives that it has the correct value on identity morphisms and the associativity and equivariance axioms make it functorial on collapses of a subset to a point and bijections respectively.

From functor to module: Given a contravariant enriched functor $\mathcal{M}$ we construct a module M as follows. The underlying symmetric sequence is given by $M(A) = \mathcal{M}(A)$. The

[1] It was remarked during Talbot that a similar statement is not true for operads with more than one colour without serious modifications to the construction.

operations $- \circ_a -$ are defined by looking at the value of $\mathcal{M}$ on the morphism of sets $g : A \cup_a B \to A$ give by collapsing B to a as follows:

$$\mathcal{M}(A) \otimes \mathcal{O}(B) \overset{\text{add units}}{\longrightarrow} \mathcal{M}(A) \otimes \bigotimes_{a' \in A} \mathcal{O}(g^{-1}(a')) \overset{\mathcal{M}(g)}{\to} M(A \cup_a B)$$

Functoriality implies that this M satisfies the axioms of a right module.

It is not hard to see that these constructions are mutually inverse, and furthermore that a morphism of right modules induces a natural transformation and vice-versa, in a mutually inverse way. $\square$

Recall that we are interested the standard balls in M with their inclusions. Equivalently we can consider them as standard balls in M with inclusions considered as standard balls in $\mathbb{R}^m$. Such an interweaving of two structures can be encoded by a Grothendieck construction.

Definition 2.7. If C is a category and $F : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ is an enriched functor, then $\mathsf{C} \ltimes F$ is the category with objects

$$\mathrm{Ob}(\mathsf{C} \ltimes F) = \coprod_{c \in \mathrm{Ob}(\mathsf{C})} F(c)$$

and morphisms

$$\mathrm{Mor}(\mathsf{C} \ltimes F) = \coprod_{c, c' \in \mathrm{Ob}(\mathsf{C})} \mathrm{Map}_{\mathsf{C}}(c, c') \times F(c')$$

This has an obvious generalization to topologically enriched category C and enriched functors F. However we will only need the discrete case, essentially because $\mathcal{O}(M)$ is discrete. We can now define the Grothendieck construction associated $\mathrm{sEmb}^\delta(-, M) : \mathcal{F}(\mathbb{B}_m^\delta)^{\mathrm{op}} \to \mathsf{Set}$, where a superscript δ means that we are consider the underlying sets. In other words δ stands for "discrete". We define a functor

$$\mathrm{ev}^\delta : \mathcal{F}(\mathbb{B}_m^\delta) \ltimes \mathrm{sEmb}^\delta(-, M) \to \mathcal{O}_\infty^s(M)$$

For concreteness we describe this functor on objects and morphisms. An object in $\mathcal{F}(\mathbb{B}_m^\delta) \ltimes \mathrm{sEmb}^\delta(-, M)$ is a pair (A, α) of a finite set A and a standard embedding $\alpha : A \times D^m \to M$. This is sent to $\mathrm{im}(\alpha)$ by ev^δ. A morphism is a sequence (A, B, f, η, α) of finite sets A, B, a map of sets $f : A \to B$, a corresponding element $\eta \in \prod_{b \in B} \mathbb{B}_m(f^{-1}(\{b\}))$ (i.e. an standard embedding of some balls into a ball labelled by b) and an embedding $\alpha : B \times D^m \hookrightarrow M$. The target of this morphism is (B, α), while the source is $(A, \alpha \circ \eta)$ where $\alpha \circ \eta$ is given by $A \times D^m \to B \times D^m \to M$ using η on the first component in the first map. This morphism is mapped by ev^δ to the inclusion $\mathrm{im}(\alpha \circ \eta) \subset \mathrm{im}(\alpha)$.

Given our previous description as the standard open balls in M as standard open balls in M with inclusions as standard open balls $\mathbb{R}^m$ the following lemma should not be surprising.

Lemma 2.8. *The functor* ev^δ *is an equivalence of categories.*

Proof. We give an inverse functor (up to natural isomorphism)

$$I : \mathcal{O}_\infty^s(M) \to \mathcal{F}(\mathbb{B}_m^\delta) \ltimes \mathrm{sEmb}^\delta(-, M)$$

We define I on objects by setting $I(U)$ to be the pair $(\pi_0(U), \upsilon)$, where υ is the unique standard embedding $\pi_0(U) \times D^n \hookrightarrow M$ with image U that is the identity on π_0. It is not hard to extend the definition to morphisms, but a bit cumbersome to write down.

We have that $\mathrm{ev}^\delta \circ I$ is the identity functor on $\mathcal{O}_\infty^s(M)$ and there is a natural isomorphism between the identity functor on $\mathcal{F}(\mathbb{B}_m^\delta) \ltimes \mathrm{sEmb}^\delta(-, M)$ and $I \circ \mathrm{ev}^\delta$. It is given on an object (A, α) by the morphism $(A, \pi_0(\mathrm{im}(f)), A \cong \pi_0(\mathrm{im}(f)), \mathrm{id}, \hat{\alpha})$, where $\hat{\alpha} : \pi_0(\mathrm{im}(f)) \times D^m \to M$ is embedding obtained from α by relabelling. It is not hard to see that this is an isomorphism in $\mathcal{F}(\mathbb{B}_m^\delta) \ltimes \mathrm{sEmb}^\delta(-, M)$ and natural. $\square$

By restricting to the full subcategory on sets of cardinality less than or equal to k, we similarly get equivalences of categories

$$\mathrm{ev}^\delta : \mathcal{F}(\mathbb{B}_m^\delta)_{\leq k} \ltimes \mathrm{sEmb}^\delta(-, M) \to \mathcal{O}_k^s(M)$$

Since an equivalence of indexing categories $J \to I$ induces a weak equivalence on homotopy limits, we can equivalently calculate $T_k F$ as follows:

$$T_k F(M) = \underset{(A,\alpha)\in\mathcal{F}(\mathbb{B}_m^\delta)_{\leq k}\ltimes \mathrm{sEmb}^\delta(-,M)}{\mathrm{holim}} F(\mathrm{im}(\alpha))$$

A general functor F depends on both components in the Grothendieck construction. A context-free functor is one where the $\mathrm{sEmb}^\delta(-, M)$-component doesn't matter. Recall that there is a projection functor

$$\pi : \mathsf{C} \ltimes F \to \mathsf{C}$$

sending $(c, x) \in \mathrm{Ob}(\mathsf{C} \ltimes F)$ to $c \in \mathrm{Ob}(\mathsf{C})$.

Definition 2.9. A functor $F : \mathcal{O}_\infty^s(M)^{\mathrm{op}} \to \mathsf{D}$ is called *context-free* if there exists a functor F' such that the following diagram commutes up to natural weak equivalence

$$\begin{tikzcd}
(\mathcal{F}(\mathbb{B}_m^\delta) \ltimes \mathrm{sEmb}^\delta(-, M))^{\mathrm{op}} \arrow[rr, "\mathrm{ev}^\delta"] \arrow[dr, "\pi"'] & & (\mathcal{O}_\infty^s(M))^{\mathrm{op}} \arrow[r, "F"] & \mathsf{D} \\
& (\mathcal{F}(\mathbb{B}_m^\delta))^{\mathrm{op}} \arrow[urr, "F'"'] & &
\end{tikzcd}$$

We will often not distinguish between F and F'.

Suppose from now that our F is context-free and consider the homotopy limit giving us the k'th Taylor approximation. For concreteness let's think about the simpler case $\lim_{\mathsf{C}\ltimes F} G \circ \pi$ where $F, G : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ are both ordinary functors.

$$\begin{tikzcd}
(\mathsf{C} \ltimes F)^{\mathrm{op}} \arrow[rr, "G\circ\pi_C"] \arrow[dr, "\pi"'] & & \mathsf{Set} \\
& \mathsf{C}^{\mathrm{op}} \arrow[ur, "G"'] &
\end{tikzcd}$$

Lemma 2.10. *We have a natural isomorphism (in F and G)*

$$\lim_{\mathsf{C}\ltimes F} G \circ \pi = \underset{\mathsf{C}}{\mathrm{Nat}}(F, G)$$

Sketch of proof. An element of the limit is given by choice of $y \in G(c)$ for all $c \in \mathrm{Ob}(\mathsf{C})$ and $x \in F(c)$. We define a natural transformation η by $\eta(x) = y$. The compatibility conditions in the limit exactly are exactly those of a natural transformation. $\square$

The homotopy version is now not hard to believe. Say we are given functors $F, G : \mathsf{C}^{\mathrm{op}} \to \mathsf{Top}$, then

$$\underset{\mathsf{C}\ltimes F}{\mathrm{holim}} G \circ \pi = \underset{\mathsf{C}}{\mathrm{hNat}}(F, G)$$

where hNat is defined as enriched natural transformations from a cofibrant replacement of F to a fibrant replacement of G in the category of C^{op}-diagrams.

As a corollary we get that for context-free functors F we have a natural weak equivalence for $k = 0, 1, \ldots, \infty$:

$$T_k F(M) \simeq \underset{\mathcal{F}(\mathbb{B}_m^\delta)_{\leq k}}{\mathrm{hNat}} (\mathrm{sEmb}^\delta(-, M), F(-))$$

Note that we implicitly identified F' as in the definition of context-free with F here. Rewriting this using the correspondence between right modules and contravariant functors out of $\mathcal{F}(\mathbb{B}_m)^\delta$ we achieve our first goal in the form of the following theorem.

Theorem 2.11. *Let F be context-free, then we have a natural equivalence*

$$T_k F(M) \simeq \underset{\mathbb{B}^\delta_m}{\mathrm{hRmod}_{\leq k}}(\mathrm{sEmb}^\delta(-, M), F(-))$$

for $k = 0, 1, \ldots, \infty$. In the case $k = \infty$ we get actual (derived) right module maps. In the case of finite k we only have truncated right module maps.

3. The rational homology of the space of reduced embeddings

Recall that given a manifold M and a Euclidean space V we can define a space $\mathrm{Imm}(M, V)$ of *immersions* as the smooth functions having injective derivative everywhere. This has a subspace $\mathrm{Emb}(M, V)$ of *embeddings*, coonsisting of smooth functions that are additionally a homeomorphism onto their image.

$$\mathrm{Emb}(M, V) \hookrightarrow \mathrm{Imm}(M, V)$$

If we pick a basepoint in $\mathrm{Emb}(M, V)$ we automatically get a basepoint in $\mathrm{Imm}(M, V)$ and then we can define the *reduced embeddings* as the homotopy fiber of the inclusion of embeddings into immersions:

$$\overline{\mathrm{Emb}}(M, V) = \mathrm{hofib}(\mathrm{Emb}(M, V) \hookrightarrow \mathrm{Imm}(M, V))$$

More geometrically, these are embeddings with an isotopy through immersions to a fixed embedding (since our basepoint was actually an embedding).

Example 3.1. Let's take $M = \coprod_k D^n$ and $V = \mathbb{R}^n$. Then we have

$$\mathrm{Emb}(M, V) \simeq \left(\prod_k GL(n)\right) \times C(k, V) \qquad\qquad \mathrm{Imm}(M, V) = \left(\prod_k GL(n)\right) \times V^k$$

where $C(k, V)$ is the configuration of k points in V. Both maps are given by sending an embedding or immersion to the images of the center of each disk and the derivative there. This tells us that the map between $\mathrm{Emb}(M, V)$ and $\mathrm{Imm}(M, V)$ is induced by the inclusion $C(k, V) \hookrightarrow V^n$ and hence we conclude that

$$\overline{\mathrm{Emb}}(M, V) \simeq C(k, V)$$

In other words, we have subtracted of the irritating and easy to understand "tangential" part of the embedding.

Why are we interested in reduced embeddings?

(1) We know Imm already, it is easy and thus uninteresting. To be precise, being $T_1\mathrm{Emb}$ it is linear and hence homotopy equivalent to the space of sections of the bundle of injective linear maps $TM \to V$.
(2) We need to get $T_0 \simeq *$, $T_1 \simeq *$ to get $H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V)$ to converge. More concretely these conditions on the first two Taylor approximations go into Weiss' theorem getting convergence result for $J \wedge F$ where J is any 0-connected spectrum from convergence results for F.
(3) The functor $\overline{\mathrm{Emb}}(-, V)$ is context-free. This also holds for $\mathrm{Emb}(-, V)$ in the circumstances given later (M open subset of $\mathbb{R}^m$), but it is still good to know.

Let us get the remarks made in (2) out of the way. A hard result in embedding theory is the Goodwillie-Klein theorem on convergence of $\mathrm{Emb}(-, N)$.

Theorem 3.2. *The functor* $\mathrm{Emb}(-, N)$ *is* $(n-2)$*-analytic with excess* $3 - n$*. In particular, if* $\dim M < n - 2$ *then the embedding calculus tower for* $\mathrm{Emb}(-, N)$ *converges.*

It is not hard to see that this implies that $\overline{\mathrm{Emb}}(-, V)$ is also $(n-2)$-analytic with excess $3 - n$. We want to get convergence of $H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(-, V)$ from this. To do this we use Weiss' theorem [Wei04].

Theorem 3.3. *Suppose that F is ρ-analytic with excess c and J is a* 0*-connected spectrum, then for $J \wedge F$ we have the following convergence results.*

- *If $c \geq 0$, then $J \wedge F$ is ρ-analytic with excess* 0*.*

- *If $c < 0$ and $T_k F \simeq *$ for $k \leq r$, then $J \wedge F$ is $(\rho + \frac{c}{r})$-analytic with excess 0.*

The following is easy algebra now.

Corollary 3.4. *We have that $\overline{\mathrm{Emb}}(-, V)$ is $\frac{n-1}{2}$-analytic with excess 0.*

This gives our first restriction on $n = \dim V$. It must be sufficiently large in the sense that $m < \frac{n-1}{2}$ or equivalently $2m+1 < n$, to make the Taylor tower of $H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(-, V)$ converge. Under this restriction we can study the Taylor approximations $T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(-, V)$ to learn something about the homology of reduced embedding spaces, and since the immersion part is easy, of embedding spaces in general.

The slogan is now the following

$$\begin{gathered}\text{embedding tower} \; + \; \text{Kontsevich formality} \\ \Downarrow \\ \text{“collapse” of tower into pieces depending on } H_*(M; \mathbb{Q}) \text{ only}\end{gathered}$$

Kontsevich formality tells us that the little m-disks operad is (real) stably formal as an operad. This means that its real homology is a model for its real chains via a zig-zag of quasi-isomorphims compatible with the operad structure. We will actually a slightly more advanced statement known as relative Kontsevich formality, proven in [LV11].

Theorem 3.5 (Relative Kontsevich formality)**.** *Let $2m+1 < n$, then there exist zig-zags of quasi-isomorphisms of operads in chain complexes over $\mathbb{R}$*

$$\begin{tikzcd} C_*(\mathbb{B}_m; \mathbb{R}) \arrow[d] & \mathcal{D}_m \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d] & H_*(\mathbb{B}_m; \mathbb{R}) \arrow[d] \\ C_*(\mathbb{B}_n; \mathbb{R}) & \mathcal{D}_n \arrow[l, "\simeq"] \arrow[r, "\simeq"'] & H_*(\mathbb{B}_n; \mathbb{R}) \end{tikzcd}$$

where the left and right vertical maps are induced by the inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$.

Remark 3.6. The proof of this very interesting, originally given in [Kon99] and having widespread applications, most importantly in deformation quantization [Kon03]. We recommend [LV11], though the notes on Kontsevich formality accompanying these notes might be useful as well.

There are two closely related but essentially distinct methods we can put our slogan in practice for $M \subset \mathbb{R}^m$ open and $\dim V = n > 2m + 1$.

Operadic approach: The first approach rewrites our expression of theorem 2.11 for $H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V)$ in terms of module maps to

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V) \simeq \underset{\mathrm{Com}}{\mathrm{hRmod}_{\leq k}}(C_*(M^-; \mathbb{Q}), H_*(\mathbb{B}_n; \mathbb{Q}))$$

Here we have used the Quillen equivalence between rational chain complexes and $H\mathbb{Q}$-module spectra to freely move between these two categories depending on which notation is more convenient.

The main reason we can move to Com and simplify the domain of the module maps is that Kontsevich formality allowed us replace chains by homology in the codomain. The conclusion that $H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V)$ only depends on $H\mathbb{Q} \wedge M$ now follows from the convergence and the fact that M only appears in the guise of $C_*(M^-; \mathbb{Q})$, which is weakly equivalent to a tensor product of copies of $C_*(M; \mathbb{Q})$. This approach is found in [AT11].

Collapse of the orthogonal tower: The second approach is to use the expression as module maps (or at least something equivalent to it) to prove collapse of the orthogonal calculus tower. This type of functor calculus applies if we fix M (or an open subset of it) and consider $H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V \oplus -)$ as a continuous functor $\mathcal{J} \to \mathsf{Sp}$, where $\mathcal{J}$ is the topologically enriched category with objects finite dimensional subspaces of $\mathbb{R}^\infty$ with inner product and morphism spaces the linear maps preserving the inner product. If we denote the layers of orthogonal calculus by D_i, then one can prove a weak equivalence

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V) \simeq \prod_{i \geq 0} D_i T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V)$$

and by convergence hence weak equivalences

$$H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V) \simeq \prod_{i \geq 0} D_i H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V)$$

where by work of Arone the layers $D_i H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V)$ of the orthogonal tower only depend on $H\mathbb{Q} \wedge M$. This approach can be found in [ALV07].

Since the first part of the notes were concerned with the operadic approach, we will describe only the first of these approaches in more detail. To do this we first check that $\overline{\mathrm{Emb}}(-, V)$ is indeed context-free.

Lemma 3.7. *The functor*[2] $\overline{\mathrm{Emb}}(-, V) : (\mathcal{O}^s_\infty(M))^{\mathrm{op}} \to \mathsf{Top}_*$ *is context free.*

Proof. We are concerned with the diagram

$$\begin{tikzcd}
(\mathcal{F}(\mathbb{B}^\delta_m) \ltimes \mathrm{sEmb}^\delta(-, M))^{\mathrm{op}} \arrow[rr, shift left, "\mathrm{ev}^\delta"] \arrow[dr, "\pi"'] & & (\mathcal{O}^s_\infty(M))^{\mathrm{op}} \arrow[ll, shift left, dotted, "I"] \arrow[r, "\overline{\mathrm{Emb}}(-,V)"] & \mathsf{Top}_* \\
& (\mathcal{F}(\mathbb{B}^\delta_m))^{\mathrm{op}} \arrow[urr, "\mathrm{sEmb}(- \times D^n, V)"'] & &
\end{tikzcd}$$

and we will prove that $\mathrm{sEmb}(- \times D^n, V) : (\mathcal{F}(\mathbb{B}^\delta_m))^{\mathrm{op}} \to \mathsf{Top}_*$ makes the diagram commute up to natural weak equivalence. Since we have the inverse I to ev^δ, it suffices to prove

$$\overline{\mathrm{Emb}}(\coprod_k D^m, V) \simeq \mathrm{sEmb}(\coprod_k D^m, V)$$

naturally in k and V. This is not hard. Generalizing our previous examples we get

$$\mathrm{Emb}(\coprod_k D^m, V) \simeq (\mathrm{Inj}(\mathbb{R}^m, \mathbb{R}^n))^k \times C(k, V) \qquad \mathrm{Imm}(\coprod_k D^m, V) \simeq (\mathrm{Inj}(\mathbb{R}^m, \mathbb{R}^n))^k \times V^k$$

so that $\overline{\mathrm{Emb}}(\coprod_k D^m, V) \simeq C(k, V)$ with map given by sending a reduced embedding to the images of the centers of the disks. It is not hard to see that $\mathrm{sEmb}(\coprod_k D^m, V)$ is homotopy equivalent to $C(k, V)$ as well, with map $C(k, V) \to \mathrm{sEmb}(\coprod_k D^m, V)$ given by sending a configuration $(x_1, \ldots, x_k)$ to the embedding that sends the i'th disk to the one in V with center x_i and radius $\frac{1}{2}\min(\{||x_i - x_j|| \mid i \neq j\})$ using the unique standard embedding doing this. The composite of these two homotopy equivalences induces a homotopy equivalence

$$\overline{\mathrm{Emb}}(\coprod_k D^m, V) \simeq \mathrm{sEmb}(\coprod_k D^m, V)$$

and it is easy to see from our description of the maps involved that this is natural. □

So indeed our previous result applies and we can write

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M, V) \simeq \underset{\mathbb{B}^\delta_m}{\mathrm{hRmod}}(\mathrm{sEmb}^\delta(-, M), H\mathbb{Q} \wedge \mathrm{sEmb}(-, V))$$

The next couple of steps concluding the proof will involve simplifying this expression more and more.

[2]The functor is based because we fix a basepoint embedding $M \hookrightarrow V$, which provides a base point for the entire functor.

(1) Note that as functors $\mathcal{F}(\mathbb{B}_m^\delta)^{\mathrm{op}} \to \mathsf{Ch}_\mathbb{Q}$ the chain complexes $C_*(\mathrm{sEmb}(-,V);\mathbb{Q})$ and $C_*(\mathbb{B}_n(-);\mathbb{Q})$ are clearly equivalent. Here $\mathbb{B}_n$ is seen as right module over $\mathbb{B}_m$ by the inclusion $\mathbb{R}^m \hookrightarrow \mathbb{R}^n$ on the first m coordinates. The idea for this is that the only difference is that for sEmb we are allowed to embed into the entire Euclidean space V, while in $\mathbb{B}_n$ we only embed into the unit disk.

As a consequence we get that

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq \underset{\mathbb{B}_m^\delta}{\mathrm{hRmod}}(\mathrm{sEmb}^\delta(-,M), C_*(\mathbb{B}_n(-);\mathbb{Q}))$$

(2) By linear extension and the fact that we have discretized our operad and right module in the module, we can equivalently write this as

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq \underset{C_*(\mathbb{B}_m^\delta;\mathbb{Q})}{\mathrm{hRmod}}(C_*(\mathrm{sEmb}^\delta(-,M);\mathbb{Q}), C_*(\mathbb{B}_n(-);\mathbb{Q}))$$

By Kontsevich formality, the $C_*(\mathbb{B}_m^\delta;\mathbb{R})$-diagram $C_*(\mathbb{B}_n(-);\mathbb{R})$ is formal, i.e. by a zig-zag weakly equivalent to $H_*(\mathbb{B}_n(-);\mathbb{R})$. Since both $C_*(\mathbb{B}_m^\delta;\mathbb{R})$ and $H_*(\mathbb{B}_n(-);\mathbb{R})$ are of finite type, the same is true over $\mathbb{Q}$. Using this in our expression for the Taylor tower we obtain

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq \underset{C_*(\mathbb{B}_m^\delta;\mathbb{Q})}{\mathrm{hRmod}}(C_*(\mathrm{sEmb}^\delta(-,M);\mathbb{Q}), H_*(\mathbb{B}_n(-);\mathbb{Q}))$$

(3) The next important step is to note that the all the elements of $\mathbb{B}_m(k)$ act in the same way on $H_*(\mathbb{B}_n;\mathbb{Q})$. Thus as the module structure on $H_*(\mathbb{B}_n;\mathbb{Q})$ factors (considering its right module as a contravariant functor) as $\mathcal{F}(C_*(\mathbb{B}_m;\mathbb{Q}))^{\mathrm{op}} \to \mathcal{F}(C_*(\mathrm{Com};\mathbb{Q}))^{\mathrm{op}} \to \mathsf{Ch}_\mathbb{Q}$. Equivalently $H_*(\mathbb{B}_n;\mathbb{Q})$ is the restriction of a module over the commutative operad (using the functor $\mathrm{res}_{C_*(\mathrm{Com};\mathbb{Q})\downarrow C_*(\mathbb{B}_m^\delta;\mathbb{Q})} : \mathrm{RMod}(C_*(\mathrm{Com};\mathbb{Q})) \to \mathrm{RMod}(C_*(\mathbb{B}_m^\delta;\mathbb{Q}))$, shortened to res) and hence we can write

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq \underset{C_*(\mathbb{B}_m^\delta;\mathbb{Q})}{\mathrm{hRmod}}(C_*(\mathrm{sEmb}^\delta(-,M);\mathbb{Q}), \mathrm{res}\, H_*(\mathbb{B}_n(-);\mathbb{Q}))$$

(4) But restricting has a right adjoint in the form of induction (given by the homotopy induction functor $\mathrm{hind}^{C_*(\mathrm{Com};\mathbb{Q})\uparrow C_*(\mathbb{B}_m^\delta;\mathbb{Q})}$, shortened to hind). This is given by a homotopy left Kan extension

$$\begin{tikzcd}
\mathcal{F}(C_*(\mathbb{B}_m;\mathbb{Q}))^{\mathrm{op}} \arrow[r, "C_*(\mathrm{sEmb}^\delta(-,M);\mathbb{Q})"] \arrow[d] & \mathsf{Ch}_\mathbb{Q} \\
\mathcal{F}(C_*(\mathrm{Com};\mathbb{Q}))^{\mathrm{op}} \arrow[ur, "\mathrm{hind}(C_*(\mathrm{sEmb}^\delta(-,M);\mathbb{Q}))"'] &
\end{tikzcd}$$

and the properties of adjoints exactly tell us that we can rewrite the Taylor polynomials as

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq \underset{C_*(\mathrm{Com};\mathbb{Q})}{\mathrm{hRmod}}(\mathrm{hind}\, C_*(\mathrm{sEmb}^\delta(-,M);\mathbb{Q}), H_*(\mathbb{B}_n(-);\mathbb{Q}))$$

(5) So to achieve our final result it suffices to calculute the left Kan extension used in the explicit construction of the induction functor. Note that $C_*(\mathrm{Com};\mathbb{Q})$-modules and Com-modules are the same thing by linear extension. Assuming the next proposition we are done.

Proposition 3.8. *We have a weak equivalence of* Com*-modules*

$$\mathrm{hind}\, C_*(\mathrm{sEmb}^\delta(-,M);\mathbb{Q}) = C_*(M^-;\mathbb{Q})$$

We delay the proof of this proposition for the moment in order to draw two important corollaries. As a first corollary we achieve our final expression for the Taylor polynomials.

Corollary 3.9. *We have that, under the standing assumption that $n > 2m+1$, there is a weak equivalence*

$$T_k H\mathbb{Q} \wedge \overline{\mathrm{Emb}}(M,V) \simeq \underset{\mathrm{Com}}{\mathrm{hRmod}}(C_*(M^-;\mathbb{Q}), H_*(\mathbb{B}_n(-);\mathbb{Q}))$$

Since $C_*(M^-;\mathbb{Q})$ is naturally weakly equivalent to $C_*(M;\mathbb{Q})^{\otimes -}$ as a right Com-module, we have that our expression for the Taylor tower depends only on $C_*(M;\mathbb{Q})$ or equivalently $H\mathbb{Q}\wedge M$.

Remark 3.10. We assumed that M was an open subset of $\mathbb{R}^m$. This is not necessary, we only need that M embeds into $\mathbb{R}^m$. For then we can replace it with a tubular neighborhood $\tilde{M}$. We claim that

$$\overline{\mathrm{Emb}}(M,V)\simeq\overline{\mathrm{Emb}}(\tilde{M},V)$$

To see this note that $\mathrm{Emb}(\tilde{M},V)$ is homotopy equivalent to a bundle over $\mathrm{Emb}(M,V)$ with fiber over an embedding ι the sections of the bundle of injective maps of vector bundles $\nu_\iota\to V$. Similarly $\mathrm{Imm}(\tilde{M},V)$ is homotopy equivalent to a bundle over $\mathrm{Imm}(M,V)$ with the same fiber. Taking the homotopy fiber of $\mathrm{Emb}(\tilde{M},V)\to\mathrm{Imm}(M,V)$ kills off the fiber direction.

Corollary 3.11. *Let $f: M_1\to M_2$ be a map of manifolds of dimension n inducing a rational homology equivalence. If $n>2m+1$ then we have an isomorphism $H_*(\overline{\mathrm{Emb}}(M_1,V);\mathbb{Q})\cong H_*(\overline{\mathrm{Emb}}(M_2,V);\mathbb{Q})$.*

Proof. By the previous corollary the Taylor polynomials are weakly equivalent for both sides. This is induced by f in a way compatible with the structure maps of the Taylor tower (given by truncating the morphism of right modules). Hence the homotopy limits of the Taylor towers are weakly equivalent. Finally our restrictions on the dimensions imply that the Taylor towers both converge, so these homotopy limits coincide with $H\mathbb{Q}\wedge\overline{\mathrm{Emb}}(M_i,V)$. □

Example 3.12. We will find $H_*(\overline{\mathrm{Emb}}(\mathbb{R}P^{2n},\mathbb{R}^k);\mathbb{Q})$. For sufficiently large k our results apply. To be precise k must be bigger than $2m+1$, where m is large enough such that $\mathbb{R}P^{2n}$ embeds in $\mathbb{R}^m$. We have that $\mathbb{R}P^{2n}$ and D^{2n} have the same rational homology and hence their reduced mapping spaces have the same rational homology. But for D^{2n} we have that $\overline{\mathrm{Emb}}(D^{2n},\mathbb{R}^k)\simeq *$, so this homology is trivial. We conclude that $H_*(\overline{\mathrm{Emb}}(\mathbb{R}P^{2n},\mathbb{R}^k);\mathbb{Q})$ is $\mathbb{Q}$ in degree zero and trivial in all positive degrees.

Let's conclude with the proof of proposition 3.8.

Proof of 3.8. Since C_* commutes with homotopy colimits and a homotopy left Kan extension is constructed using homotopy colimits, it suffice to calculute $\mathrm{hind}\,\mathrm{sEmb}^\delta(-,M)$ where the induction is from $\mathbb{B}_m^\delta$- to Com-modules. If $M=U$ is an object of $\mathcal{F}(\mathbb{B}_m^\delta)$, then we get out $(\pi_0U)^-\simeq U^-$.

In general we have that $\mathrm{hocolim}_{U\in\mathcal{O}^s_\infty(M)}\mathrm{sEmb}^\delta(-,U)$ is $\mathrm{sEmb}^\delta(-,M)$. Since homotopy left Kan extension commutes with homotopy colimits, we get that

$$\mathrm{hind}\,\mathrm{sEmb}^\delta(-,M)\simeq\mathrm{hocolim}_{U\in\mathcal{O}^s_\infty(M)}\mathrm{hind}\,\mathrm{sEmb}^\delta(-,U)=\mathrm{hocolim}_{U\in\mathcal{O}^s_\infty(M)}U^-\simeq M^-$$

□

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