1 Introduction and overview

General bla bla provided by Greg or Michael.

2 Polynomial and analytic functors

Most of the material for this talk is in [Goo90] and [Goo92]. [Kuh07] is also a good reference for a slightly more general perspective.

[Goo90, 1.1-1.3]: Begin by defining what a linear functor is. Give some examples, in different contexts. The identity functor from Spectra to Spectra is linear, but the identity functor from Spaces to Spaces is not. A (homogeneous) linear functor is essentially the same thing as a generalized homology theory.

[Goo92, 1.3-1.4,§3], [Kuh07, §4]: Introduce cubical diagrams and define what it means for a cubical diagram to be (strongly) homotopy (k)-(co)-cartesian. Define n-excisive functors and give some examples. The identity functor from Spaces to Spaces is an important non-example.

[Goo92, 4.1-4.5,2.3]: Define stably excisive and analytic functors. State the generalized Blakers-Massey theorem (extra credit: explain the idea of the proof). Show that it is equivalent to saying that the identity functor is analytic. Analytic functors are closed under finite homotopy limits and colimits, but not under arbitrary homotopy limits. Illustrate this through a few examples of functors that are analytic and some that are not.

3 Constructing the Taylor tower

This talk covers [Goo03, 1.1-1.19]. Also see [Kuh07, §5]. The case $n = 1$ can be found in [Goo90, 1.10-1.15].

Describe Goodwillie's constructions T_n and P_n (preferably using Kuhn's more general setting). Roughly speaking, if F is a homotopy functor, then T_nF is a new functor, that is slightly closer than F to being polynomial of degree n. P_nF is obtained by an infinite iteration of T_n.

P_nF is the best possible approximation to F by a polynomial functor of degree n, and we think of it as the n-th Taylor polynomial of F. The proof that P_nF always has degree n is interesting and quite unexpected. Charles Rezk [Rez] found a short proof of the key technical result needed for this theorem.

It is worth mentioning that the operators P_n commute with finite homotopy limits, but not with general homotopy limits. In some cases they commute with homotopy colimits.

There are natural transformations

$$\cdots \longrightarrow P_nF \longrightarrow P_{n-1}F \longrightarrow \cdots .$$

Thus the functors $\{P_nF\}$ fit into a tower. This is the "Taylor tower" of F.

[Goo03, 1.15-1.19]: A functor F is called homogeneous of degree n if it is polynomial of degree n and $P_{n-1}F \simeq *$. That is, the $n-1$-st Taylor polynomial of F vanishes. For example, the homotopy fiber of the natural map $P_nF \longrightarrow P_{n-1}F$ is homogeneous of degree n. These homotopy fibers are the "homogeneous layers" in the Taylor tower of F. They play the role of the summands in the classic Taylor expansion of a function. One of the reasons for the theory's success is that homogeneous functors can be understood quite thoroughly. This is the subject of the next lecture (not this one).

If F is analytic, then its Taylor tower has especially good convergence properties. Namely, if F is a ρ-analytic functor and X is $\rho+1$-connected, then the connectivity of the natural map $F(X) \longrightarrow P_nF(X)$ goes to ∞ with n.

4 Homogeneous functors

Homogeneous functors are analyzed thoroughly in [Goo03, 2-6]. The lecture should cover the following points

[Goo03, 2.1]: An important theorem says that homogeneous functors that a-priori take values in Spaces, in fact take values in infinite loop spaces. This is obvious (but worth mentioning) for linear functors, but far from obvious for homogeneous functors of degree greater than one.

[Goo03, 3.5,discussion following 5.2]: There is a correspondence between homogeneous functors of degree n, symmetric multi-linear functors of n-variables, and (families of) spectra with an action of Σ_n. This motivates the concept of derivatives of a functor. The n-th derivative of a functor (at a point) is a spectrum with an action of Σ_n. The n-th derivative determines the n-th layer in the Taylor tower of F.

For the sake of clarity, it might be good to focus mostly on the special case of homogeneous functors on the category of pointed spaces (rather than the more general category of spaces over a fixed space). This means that we would only be considering the derivatives of a functor at the one point space, rather than at general spaces. However, we will need to use the derivatives at a general space when we get to applications to K-theory, so it would be good to include at least a brief discussion of this as well. In particular [Goo92, Theorem 5.3] is of key importance. It says, roughly speaking, that if two analytic functors have the same first derivative, then the difference between them is locally constant.

[Goo03, 6.1]: The derivative can be calculated via the multilinearized cross-effect. This gives an effective way to describe the layers in the Taylor towers of specific functors.

5 First examples

[Goo03, §7], [Aro99], [AK02]: Let K be a finite complex. Consider the functor $X \mapsto \Sigma^\infty \operatorname{Map}(K, X)$. There is a simple formula for the derivatives of this functor. More unusually, we have a nice explicit model for the entire Taylor tower of this functor. It can be described in terms of maps of modules over the (non-unital) commutative operad. Another way to describe it, which some people may like, is in terms of sections of a sheaf over the Ran space, which arises in the study of chiral homology. Goodwillie's theorem guarantees that the Taylor tower will converge if the connectivity of X is greater than the dimension of K. One wonders idly if one can get better convergence results using the contractibility of the Ran space.

As a variation, one may replace Σ^∞ with another spectrum. In this way one obtains a spectral sequence for calculating any generalized homology of the space of maps $\operatorname{Map}(K, X)$ (the spectral sequence will converge if the spectrum is connective).

A related example is the functor from Spectra to Spectra $E \mapsto \Sigma^\infty\Omega^\infty E$. The derivatives of this functor are easy to describe. They can be identified with the commutative cooperad, which will play an important role in the lecture about operad an module structure.

The Taylor tower of the functor $\Sigma^\infty\Omega^\infty$ gives a spectral sequence expressing the homology of $\Omega^\infty E$ in terms of the homology of E (at least when E is a connected spectrum).

Another interesting example, that is related to $\Sigma^\infty\Omega^\infty$ is the identity functor on the category of commutative ring spectra (S-algebras). See [Kuh07, Section 6.3].

6 The derivatives of the identity functor

The examples in the previous lecture are of functors that take values in Spectra. Functors that take values in Spaces are, in general, harder to understand. The basic example is the identity functor. Recall that it is not n-excisive for any n, but is 1-analytic. The Taylor tower converges for simply-connected spaces.

The derivatives of the identity functor were first described by Brenda Johnson [Joh95], using Goodwillie's machinery of stabilized cross-effects. We suggest that the talk will describe Brenda's work and then show that the spaces that Brenda introduced can also be described in terms of the category of partitions [AM99, 2.1].

There's a summary of all this in [Goo03, §8].

7 Operad and module structures on derivatives

This talk is motivated by two questions: (1) What structure do the derivatives of a functor possess, beyond that of a symmetric sequence? (2) What is the chain rule for higher derivatives (the analogue of the Faa di Bruno formula)?

In attempting to answer these questions, one finds out the following story: The derivatives of the identity functor on based spaces have a natural operad structure. This operad is Koszul dual to the derivatives of the comonad in spectra $\Sigma^\infty\Omega^\infty$, and there is a good reason for that. The derivatives of $\Sigma^\infty\Omega^\infty$ happen to be the commutative cooperad in Spectra, so the derivatives of the identity are a topological version of the Lie operad.

For a general functor F, the derivatives of F have a bimodule structure over the derivatives of the identity functor. It turns out that this is exactly the structure that one needs to write down a chain rule for higher derivatives. Given two composable functors F and G, the $\partial_*(FG)$ is equivalent to a suitably defined composition product of ∂_*F with ∂_*G, over the derivatives of the identity.

The Koszul duality between the derivatives of the identity and of $\Sigma^\infty\Omega^\infty$ leads to a bar-cobar duality between the derivatives of F and of $\Sigma^\infty F$. This can be useful for practical computations. At least, one can use this idea to recover Brenda's calculation of the derivatives of the identity functor.

The main reference for this talk is [AC11]. Some background on operads and Koszul duality will be needed from [Chi05].

8 Classification of polynomial functors

In this lecture we pursue further the question "what structure do the derivatives of a functor possess?" The ultimate goal is to describe enough structure to be able to recover the Taylor tower. This would extend Goodwillie's classification of homogeneous functors to a classification of all polynomial functors.

The bimodule structure described in the previous lecture is a step in this direction, but it does not tell the whole story. In this talk we describe a refinement of the bimodule structure, which answers the problem.

Since this is a talk about a paper that is still being written, it should probably be given by one of the authors...

9 Orthogonal Calculus 1: theory

[Weiss95]: Let J be the topological category of finite-dimensional Euclidean spaces and linear isometric inclusions between them. Orthogonal calculus is concerned with continuous functors from J to Spaces (or more generally from J to a topological model category). The idea is that such functors are generally more tractable at high-dimensional spaces, so we want to do Taylor expansion at infinity.

The lecture will define polynomial functors in this contex and go over the construction of Taylor approximations, derivatives and the layers of the tower. I would be nice to structure the talk in a way that would make clear the many formal similarities between this theory and Goodwillie's homotopy calculus, and also point out some of the salient differences.

10 Orthogonal Calculus 2: examples

Here are a few examples of functors to which one may apply orthogonal calculus with some profit:

[Aro02]: $V \mapsto \Sigma^\infty \mathrm{mor}(V_0, V)_+$. Here V_0 is a fixed Euclidean space and $\mathrm{mor}(V_0, V)$ stands for space of linear isometric inclusion (a Stiefel manifold). The role of this functor in orthogonal calculus is sort of analogous to the role of the functor $X \mapsto \Sigma^\infty \mathrm{Map}(K, X)$ in homotopy calculus. Interestingly, their Taylor towers look formally similar.

$V \mapsto \mathrm{BO}(V)$ and $V \mapsto \mathrm{BU}(V)$ (the latter requires a "unitary" version of orthogonal calculus). The Taylor tower of starts with BO (or BU) and converges to $\mathrm{BO}(V)$ (or $\mathrm{BU}(V)$). The Taylor towers of these functors are related to the Taylor tower of the identity functor (from Spaces to Spaces) in various ways.

[Aro09]: The configuration space functor. Fix an integer i and consider the functor $V \mapsto \Sigma^\infty \mathrm{Emb}(\{1, \ldots, i\}, V)_+$. This is (the suspension spectrum of) the space of ordered i-tuples of distinct points in V. As everyone knows, these spaces arise all over the place. In particular, understanding the Taylor tower of this functor is a step to understanding the Taylor tower of more general embedding spaces. Of course, one may replace Σ^∞ with another spectrum. In particular, the functor $V \mapsto \mathrm{H}\,\mathbb{Z} \wedge \mathrm{Emb}(\{1, \ldots, i\}, V)_+$ represents the homology of configuration space. A curious coincidence: the Taylor tower of this functor coincides with the Postnikov tower. This plays a role in the study of homology of spaces of embeddings.

The list of examples can be continued according to the time available and the interest of the speaker. For example, a discussion of the recent work of Weiss and Reis would be nice, but I am not sure that one can fit all of this into one talk.

11 Introduction to Embedding Calculus

[Weiss99, GKW01]: Fix a dimension m and consider a category of m-dimensional manifolds and codimension zero embeddings between them. Embedding Calculus is concerned with contravariant isotopy functors from this category to the category of Spaces (or more generally, a topological model category, or any category in which "one can do homotopy theory"). A prototypical example of such a functor is $M \mapsto \mathrm{Emb}(M, \mathbb{R}^n)$, hence the name.

Polynomial functors and analytic functors are defined in this theory in essentially the same way as in Goodwillie's homotopy calculus. The construction of Taylor approximations in embedding calculus is different from homotopy calculus and orthogonal calculus. In some sense the construction is simpler here, since it does not involve stabilization. The talk will describe the construction of the Taylor tower, the universal property that characterizes the polynomial approximations, and the layers in the tower. There is a convergence theorem for analytic functors [GKW01, Theorem 4.2.1]

To illustrate the general theory, discuss the functor $\mathrm{Emb}(-, \mathbb{R}^n)$. Show that the linear approximation to this functor is given by the immersion functor. Describe the higher layers in the embedding tower of this functor.

12 Multiple disjunction lemmas

[GKW01], which provides more detailed references: Multiple disjuntion lemmas tell us that the embedding functor (and other functors related to it) is analytic. State the "easy" and "hard" multiple disjunction lemmas and what they imply about the convergence of the tower. For warm-up, show the proof of "easy" multiple disjunction lemma for embeddings [Section 3.1]. Then give an overview of the proof of the "hard" multiple disjunction lemma.

13 Embedding calculus, the little disks operad, and spaces of embeddings

[AT11, ALV07]: There is a close relationship between Embedding Calculus and the theory of modules over the little disks operad. The Taylor tower in embedding calclus can, in many cases, be presented as a space of maps between right modules (or, in some cases, weak bimodules).

After explaining the general idea, you may want to show how it applies to the functor

$$M \mapsto \mathrm{H}\,\mathbb{Z} \wedge \mathrm{hofiber}\left(\mathrm{Emb}(M, \mathbb{R}^n) \longrightarrow \mathrm{Imm}(M, \mathbb{R}^n)\right)$$

and show that the operadic model for the Taylor tower has an especially nice form. Moreover, there is a spectral sequence for calculating this Taylor tower whose E^2 term is a homotopy functor of M. Kontsevich's formality theorem implies that the spectral sequence collapses (assuming that the codimesnion is high enough). It follows that the rational homology of the homotopy fiber is a homotopy functor of M (in fact, a rational homology functor).

There also is a version of all this for spaces of embeddings with compact support $\mathrm{Emb}_c(\mathbb{R}^m, \mathbb{R}^n)$. In the case $m = 1$ this specializes to the work of Sinha, Lambrechts, Volic, Turchin and others on space of long knots.

14 Factorization homology

My (vague) idea for the overall structure of this talk is something like this: Embedding calculus is the study of contravariant functors on the category of m-dimensional manifolds and codimension zero embeddings. Factorization homology, at least in rudimentary form, is the study of covariant functors on the same category. Embedding calculus and factorization homology have close connection to, respectively, the theories of right modules and algebras over the little disk operad. Thus there are formal similarities between the two theories [explain], but there also are important differences [explain].Interesting examples are [fill in the blanks]...

... I am hoping that someone who knows more than I about this will figure this one out.

15 Applications to algebraic K-theory I

Let $A(X)$ be Waldhausen's A-theory of X. Applications to this functor were the original motivation for developing the theory of calculus. The first derivative of A is identified in [Goo90], using the connection between $A(X)$ and manifold theory.

Once one has the calculation of the first derivative of A, one may proceed as follows. Let $L(X)$ be the the suspension spectrum of the free loop space of X. There is an S^1-equivariant trace map $A(X) \longrightarrow L(X)$. It factors through a map $A(X) \longrightarrow L(X)^{hS^1}$. This map does *not* induce an equivalence between the derivatives of these functors. It does induce an equivalence

$$\partial_n A \longrightarrow (\partial_n L)^{hS^1}$$

(the key to proving this assertion is the calculation of the first derivative of both sides). This can be used to calculate the derivatives of $A(X)$ as a symmetric sequence and indeed as a right module over the "topological Lie operad". Except for the Lie-module structure, which is done in [AC11], all this is outlined in [Goo03, Section 9], and further references are provided there.

[BHM93, BCCGHM96]: The functor $L(X)^{hS^1}$ can be replaced with a more subtle functor called $TC(X)$ (topological cyclic homology). There is a natural map (the cyclotomic trace) $A(X) \longrightarrow TC(X)$, and this map does induce an equivalence on first derivatives (after profinite completion). It follows that the difference between A and TC is "locally constant".

16 Applications to algebraic K-theory II

[DM94, McCar97, Dun97]: The previous lecture can be recast in more general K-theoretic terms. $A(X)$ is the K-theory of the ring spectrum $\Sigma^\infty \Omega X_+$. $L(X)$ is the topological Hochschild homology (THH) of the same ring spectrum. The trace can be defined as a natural transformation $K(-) \longrightarrow THH(-)$. The main result of [DM94] says that this map is, in a certain sense, a linearization. In fact, THH with coefficients in a bimodule can be thought of as the derivative of K theory.

This enables one to construct the cyclotomic trace $K(R) \longrightarrow TC(R)$ and prove that the difference is locally constant in the general setting of ring spectra [McCar97, Dun97].

The functor TC is considerably more accessible to calculation than A, and indeed this circle of ideas was used successfully to carry out difficult calculations in K-theory.

17 Calculus of functors and chromatic homotopy theory

N. Kuhn's paper [Kuh07], especially Section 7 is the basic reference for this talk. It is a survey article and it provides further references.

Overview some relevant bits of chromatic homotopy theory: Bousfield localization, v_n-periodic homotopy, the Bousfield-Kuhn functor. Tell us about Kuhn's two theorems: one about the L_n-local splitting of a Taylor tower from Spectra to Spectra [Theorem 7.6] (Extra credit: Explain how it fits with the general classification of Polynomial functors), and one about the Morava K-theory of infinite loop spaces [Theorem 7.14].

18 Taylor tower of the identity functor, part 2

[AD01, AM99, Kuh07]: The talk will focus on the special things that happen when the Taylor tower of the identity functor is evaluated at a sphere.

You may start with a brief review of the theory of equivariant homology approximation. Then show how this theory leads to the following two results. Suppose that we evaluate the Taylor tower at an odd sphere and localize everything at a prime p. Then "most" of the layers in the tower are trivial: the only layers of the tower that are not trivial are the ones numbered by powers of p. Moreover, the layers that are not trivial have a small model, which substitutes the Tits building for the space of partitions.

The connection with Tits building tells us that the layers in this tower are related to spectra much studied by homotopy theorists (Mitchell, Kuhn, Priddy) in the 1980-ies. This in turn has consequences about the behavior of the Taylor tower in v_n-periodic homotopy. The final conclusion is the following theorem: When the Taylor tower of the identity is evaluated at an odd-dimensional sphere, it has only $n+1$-non-trivial stages in v_n-periodic homotopy. This theorem reduces the problem of calculating the unstable v_n-periodic homotopy groups of an odd sphere to $n+1$ (admittedly very difficult) calculations in stable homotopy theory (plus extension problems).

For $n = 0$, this theorem essentially recovers the classical fact that the Hurewicz homorphism is a rational isomorphism for odd spheres. For $n = 1$ it recovers (more or less) some theorems of Mahowald and Thompson.

Optional: tell about the connection with the symmetric powers filtration and the Whitehead conjecture (old and recent work of Kuhn + recent work of Behrens).

Extra credit: Tell us something about Behrens's calculations in his recent AMS memoir on the subject.

19 Where do we go from here?

An informal discussion of loose ends, open questions, etc.

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