BACKGROUND MATERIAL FOR TALBOT 2012

Contents

1. Some category theory

1.1. **Closed symmetric monoidal categories.** For a quick review of relevant aspects of symmetric monoidal, enriched, tensored and cotensored categories we suggest section 1 of M. Ching's paper "Bar constructions for topological operads and the Goodwillie derivatives of the identity" [G&T, volume 9] (which you may want to look at anyway, in preparation for the workshop). There also is plenty of material available on the internet. In particular, nLab is a good resource.

A category is *closed* if it has internal hom objects. For example, the category of Abelian groups is closed, because the set of homomorphisms between Abelian groups is again an Abelian group. The category of all groups is not closed. The categories of simplicial sets, chain complexes, topological spaces and spectra are closed, although in the last two cases proving it takes a great deal of work.

A symmetric monoidal structure is a pairing on the category $(X, Y) \mapsto X \odot Y$ that is associative, commutative and unital up to coherent isomorphisms. The three most commonly used symmetric monoidal structures are:

- Categorical sum (topological spaces and disjoint unions, pointed spaces and wedge sums, groups and free products, etc.)
- Categorical product (topological spaces and cartesian products, etc.)
- A left adjoint to an internal hom functor. In this case we say that a category is *closed* symmetric monoidal. Examples are: chain complexes and tensor products, topological spaces and cartesian products, pointed topological spaces and smash product, spectra and smash product.

In some cases, two of these structures coincide. For example, in Abelian Groups, sums and products are the same. In Topological Spaces categorical product is also the left adjoint to internal hom (such categories are called cartesian closed).

Exercise 1.1.1. *Show that the category of chain complexes has a closed symmetric monoidal structure. Work it out both for the category of non-negatively graded and infinitely graded chain complexes.*

1.2. **Enriched, tensored and cotensored categories.** Let $\mathcal{C}$ be a category and $\mathcal{D}$ a monoidal category. We say that $\mathcal{C}$ is *enriched* over $\mathcal{D}$ if $\mathcal{C}$ has hom-objects that are elements of $\mathcal{D}$. We say that $\mathcal{C}$ is *tensored* over $\mathcal{D}$ if for all objects $X \in \mathcal{C}$ and $A \in \mathcal{D}$, we have defined a "tensor product" object $X \otimes A \in \mathcal{C}$ in a manner satisfactorily associative, unital and functorial in both X and A. In this case, we can think of $\mathcal{C}$ as a module in **Cat** over the monoidal category $\mathcal{D}$. Dually, we say $\mathcal{C}$ is *cotensored* over $\mathcal{D}$ if we can functorially form $\mathsf{Hom}(A, X) \in \mathcal{C}$. Then $\mathcal{C}$ is a module over $\mathcal{D}^{op}$.

Here are some standard examples of enriched, tensored and cotensored categories

- Every category is enriched over Sets.
- If a category has direct sums, then taking direct sums of an object with itself can be interpreted as a tensoring over Sets. That is, if X is an object of your category and S is a set, then
$$X \otimes S = \coprod_{s \in S} X.$$
- Similarly, if a category has products, then taking a product of an object with itself can be interpreted as a cotensoring over Sets.
- A category enriched over Abelian Groups is called a pre-additive category. Examples are the category of chain complexes and the stable homotopy category (which we will discuss below).
- The category of non-negatively graded chain complexes is enriched, tensored and cotensored over Simplicial Sets. This uses the Dold-Kan correspondence.
- The category of Spectra is enriched, tensored and cotensored over Topological Spaces.

Exercise 1.2.1. *Let A be an Abelian group and let F be a finite set. Prove that the following formula defines a natural tensoring of Abelian Groups over the category of Finite Sets (with cartesian product for monoidal structure)*
$$A \otimes F := A^F = \prod_{f \in F} A.$$
Note that normally this formula defines a co*tensoring over Finite Sets. The claim is that in the case of Abelian groups it also defines a tensoring.*

Consider the following variant. Let R be a commutative ring with unit and let F be a finite set. Show that the following formula defines a tensoring of the category of commutative, unital rings over Finite Sets.
$$R \otimes F := \bigotimes_{f \in F} R.$$

Would any of these examples work if we dropped the requirement of commutativity?

Exercise 1.2.2. *Show that the category of (non-negatively graded) chain complexes has a natural tensoring over Simplicial Sets, extending the obvious tensoring over Sets. Let C be a chain complex and let S^1 be a simplicial model for the circle. Describe $C \otimes S^1$.*

1.3. **Functors.** Perhaps unsurprisingly, functors play an important role in the Calculus of Functors. Here we will review some basic operations on functors, such as ends and coends, Kan extensions, etc.

We will assume that you are familiar with the categorical notions of functors, limits and colimits. We suggest that you familiarize yourself with (or refresh your memory on) notions of enriched functors, limits and colimits.

1.3.1. *Coends.* Suppose $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $G\colon \mathcal{C}^{op} \longrightarrow \mathcal{D}$ are two functors. The *coend* or "tensor product" of G and F is definted by means of a co-equalizer diagram

$$G \otimes_{\mathcal{C}} F \longleftarrow \coprod_{c\in\mathcal{C}} G(c) \otimes F(c) \Leftarrow \coprod_{c_1\to c_2\in\mathcal{C}} G(c_2) \otimes F(c_1).$$

Informally, we think of F and G as some kind of "left" and"right module" over $\mathcal{C}$ respectively. The coend is a kind of generalized tensor product.

Exercise 1.3.1. *Show that a ring can be thought of as a category with one object, enriched over Abelian Groups. Moreover, a right and a left module is the same thing as a contravariant and a covariant enriched functor from this category to Abelian Groups. Show that in this case the enriched coend of these two functors specializes to the ordinary tensor products of a right and a left module.*

1.3.2. *Representable functors.* Let $\mathcal{C}$ be a category, and let x_0 be an object. The free functor generated by x_0, or the functor represented by x_0 is the functor R_{x_0} from $\mathcal{C}$ to Sets defined by the formula

$$R_{x_0}(Y) = \hom(x_0, y).$$

More generally, if $\mathcal{C}$ is enriched over a category $\mathcal{D}$, then R_{x_0} can be thought of as an enriched functor from $\mathcal{C}$ to $\mathcal{D}$.

1.3.3. *coYoneda lemma.* The basic tool for computing coends is the "coYoneda lemma". It gives a simple formula for the coend of any functor with a representable functor.

Lemma 1.3.1. *Let $G\colon \mathcal{C}^{op} \longrightarrow$ Sets be a functor. Let R_{x_0} be a representable functor. There is a natural isomorphism*

$$G \otimes_{\mathcal{C}} R_{x_0} \cong G(x_0).$$

Exercise 1.3.2. *There is an enriched version of the coYoneda lemma. Figure out what it says, or read up on it.*

One usually computes the coend of two functors by presenting one of the functors as a colimit of representable functors and using the coYoneda lemma, along with the fact that coend commutes with colimits in each variable (why is that?).

Exercise 1.3.3. *Let Spaces$_*$ be the category of pointed spaces. Let $[n] = \{0, 1, \ldots, n\}$ be considered a pointed space (with discrete topology and basepoint being 0). Consider the representable functor $R_{[n]}(X) = X^n$. Let $G\colon Top_*^{op} \longrightarrow Top_*$ be a Top_*-enriched functor. Consider the enriched coend*

$$R_{[n]} \otimes_{Top_*} G.$$

By coYoneda Lemma, it is isomorphic to $G([n])$.

Now consider the functor $\Lambda(X) = X^{\wedge n}$. Represent this functor as a colimit of representable functors (hint: it is a total cofiber of a cubical diagram), and use this to compute the coend

$$\Lambda \otimes_{Top_*} G.$$

This coend is sometimes called the n-th cross-effect of G. Can you see why?

1.3.4. *Ends, or natural transformations.* Now let F, G be two covariant functors from $\mathcal{C}$ to $\mathcal{D}$. The *end* of F and G, or more descriptively, the set of natural transformations from F to G is defined by means of an equalizer diagram

$$\operatorname{nat}(F, G) \longrightarrow \prod_{c \in \mathcal{C}} \hom(F(c), G(c)) \rightrightarrows \prod_{c_0 \to c_1 \in \mathcal{C}} \hom(F(c_0), G(c_1)).$$

There is an enriched version of nat, where everything is enriched, tensored and cotensored over a symmetric monoidal category, and then $\operatorname{nat}(F, G)$ is not just a set, but an object of the background category.

This construction is formally analogous to the coend, and all general properties of the coend have a counterpart here. In particular, there is the Yoneda Lemma, which desribes the natural transformations from a representable functor to an arbitrary functor.

Exercise 1.3.4. *Let $G\colon Top_* \longrightarrow Top_*$ be a Top_*-enriched functor. Let $\Lambda(X) = X^{\wedge n}$ be the functor defined in Exercise 1.3.3. Use the Yoneda lemma, along with the fact that nat converts colimits in the source variable to limits (why?) to calculate*

$$nat(\Lambda, G).$$

1.4. **Kan extensions.** Let $\mathcal{C}_0$, $\mathcal{C}$, $\mathcal{D}$ Suppose $\mathcal{C}_0$ is equipped with a fixed functor $\mathcal{C}_0 \to \mathcal{C}$. You may think of $\mathcal{C}_0$ as a subcategory of $\mathcal{C}$, but this is not necessary.

Suppose $\psi\colon \mathcal{C}_0 \longrightarrow \mathcal{D}$ is a functor. One may ask whether it is possible to extend ψ to a functor $\Psi\colon \mathcal{C} \longrightarrow \mathcal{D}$. This is not always possible. But one can always (well, not really always) find "the best possible approximation" to such an extension. In fact, there are two such approximations: from the left and from the right. They are called the left and the right Kan extensions respectively. We will denote them $L\psi$ and $R\psi$. For an object c of $\mathcal{C}$, $L\psi(C)$ can be defined in good cases as the colimit of $\psi(c_o)$, where c_0 ranges over all objects of $\mathcal{C}_0$ mapping to c. More formally

$$L\psi(c) = \operatorname*{colim}_{c_0 \to c \in \mathcal{C}_0 \downarrow c} \psi(c_0).$$

Dually, the right Kan extension can be defined by the formula

$$R\psi(c) = \lim_{c \to c_0 \in c \downarrow \mathcal{C}_0} \psi(c_0).$$

Exercise 1.4.1. *Let $[\mathcal{C}, \mathcal{D}]$ be the category of functors from $\mathcal{C}$ to $\mathcal{D}$. The functor $\mathcal{C}_0 \to \mathcal{C}$ gives rise to a restriction functor $[\mathcal{C}, \mathcal{D}] \longrightarrow [\mathcal{C}_0, \mathcal{D}]$.*

Prove that the left and right Kan extension are the left and right adjoint respectively to the restriction functor.

Exercise 1.4.2. *Let c be a fixed object of $\mathcal{C}$. Consider the contravariant functor from $\mathcal{C}_0$ to Sets given by the formula $c_0 \mapsto \hom_{\mathcal{C}}(c_0, c)$ (for simplicity, we assume that $\mathcal{C}_0$ is a subcategory of $\mathcal{C}$). Prove that $L\psi$ can be defined as a coend*

$$L\psi(c) = \hom(-, c) \otimes_{\mathcal{C}_0} \psi(-).$$

Find an analogous formula for the right Kan extension, using natural transformations.

1.5. **Derived functors.** (This section ows much to the nLab page on derived functors. Another good introduction to this material is the survey paper of Dwyer and Spalinsky which can be downloaded from Dwyer's web page.) In homotopy theory one works with categories equipped with a special class of maps, called *weak equivalences*, and one often is interested in functors that take weak equivalences to weak equivalences. Functors that have this property are called "homotopy functors". If a functor does not preserve weak equivalences that we may be interested in a best possible approximation to a functor by a homotopy functor. This can be expressed in terms of Kan extensions.

Let $\mathcal{C}$ be a category with weak equivalences (e.g., topological spaces, spectra, chain complexes, etc.). The homotopy category of $\mathcal{C}$, denoted

Ho($\mathcal{C}$), is the category obtained from C by formally inverting weak equivalences. There is a canonical functor $\mathcal{C} \to \mathrm{Ho}(\mathcal{C})$. A functor $\mathcal{C} \to \mathcal{D}$ is a homotopy functor if an only if it factors through Ho($\mathcal{C}$).

Definition 1.5.1. *Let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. The left (resp. right) derived functor of F is the right (resp. left) Kan extension of F along the functor $\mathcal{C} \to Ho(\mathcal{C})$*

(The reversal of handedness is not a typo. Things are arranged so that the left derived functor of F maps to F, while F maps to its right derived functor).

Often $\mathcal{D}$ is itself a category with weak equivalences, and when one says a derived functor of F one often means the derived functor of the composite functor

$$\mathcal{C} \xrightarrow{F} \mathcal{D} \longrightarrow \mathrm{Ho}(\mathcal{D}).$$

In this setting, the pointwise formula for Kan extension given above usually does not apply, because the homotopy category of $\mathcal{D}$ will not, normally, have most limits and colimits. Nevertheless, the Kan extension (and therefore the derived functors of F) exist in many cases even when the pointwise formula is not valid. In practice, $\mathcal{C}$ and $\mathcal{D}$ will usually have a Quillen model structure, and derived functors can often be constructed using fibrant/cofibrant replacement.

Exercise 1.5.2. *Suppose that $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a homotopy functor. Show that in this case both derived functors of F are naturally isomorphic to F.*

1.5.1. *Homotopy limits and colimits.* An important example of derived functors is given by homotopy limits and colimits. Let $\mathcal{C}$ be a small category. Let $\mathcal{D}$ be a category with weak equivalences. Let $[\mathcal{C}, \mathcal{D}]$ be the category of functors. Colimits and limits of functors from $\mathcal{C}$ to $\mathcal{D}$ can be thought of as functors $[\mathcal{C}, \mathcal{D}] \longrightarrow \mathcal{D}$. Limits and colimits do not, usually, preserve weak equivalences of functors. The homotopy colimit is the left derived functor of the colimit functor and the homotopy limit is the right derived functor of the limit functor. Thus homotopy limits and colimits are, in a sense, the bost possible approximations to categorical limits and colimits by a homotopy invariant construction.

If $\mathcal{D}$ is enriched over topological spaces (or simplicial sets), then a concrete construction of homotopy colimits and limits is given as follows. For an object c of $\mathcal{C}$, let $c \downarrow \mathcal{C}$ be the "under category". Its objects are arrows $c \to x$ in $\mathcal{C}$ and its morphisms are commuting triangles. Let $|c \to \mathcal{C}|$ be the geometric realization of this category (or, if you prefer to work simplicially, the simplicial nerve). Then the assignment $c \mapsto |c \to \mathcal{C}|$ defines a contravariant functor from $\mathcal{C}$ to Topological Spaces. Finally, let $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a functor. Then the homotopy colimit of F is given by the following formula

$$\mathrm{hocolim} F = |- \to \mathcal{C}| \otimes_{\mathcal{C}} F(-).$$

(one may also want to pre-arrange things so that F takes values in cofibrant objects).

There is a dual description of homotopy limits, using over categories and natural transformations (figure it out!).

Another popular concrete way to construct homotopy colimits and limits is via simplicial and cosimplicial resolution of a diagram.

Exercise 1.5.3. *Let us suppose we have a diagram*

$$X_1 \longleftarrow X_0 \longrightarrow X_2.$$

Describe the homotopy colimit of this diagram assuming it takes values in (a) Spaces, (b) Pointed spaces, (c) chain complexes. Homotopy colimit of a diagram of this shape is called a homotopy pushout.

Exercise 1.5.4. *Do the same thing with the homotopy limit of the diagram*

$$Y_1 \longrightarrow Y_0 \longleftarrow Y_2.$$

Homotopy limits of this shape are called homotopy pullbacks.

1.6. **Simplicial model categories.** Many categories arising in homotopy theory and algebra have the structure of a *simplicial model category.* These categories are, as a matter of definition, tensored and cotensored over simplicial sets, and they satisfy a suitably enriched version of the model category axioms (that is, they have a good notion of cofibration, fibration and weak equivalence). There is an analogous notion of a *topological model category*, which is a model category enriched, tensored and cotensored over topological spaces, subject to a compatibility condition. For an introduction into model categories we recommend the survey article of Dwyer and Spalinski (can be downloaded from Bill Dwyer's web page) as well as Quillen's original paper *Homotopical Algebra*; for a more general and systematic discussion, see section 4.2 of Hovey's book *Model Categories.*

If $X \in \mathcal{C}$ is an object of a simplicial model category and A a simplicial set, we can give a formula for $X \otimes A$ in terms of homotopy colimits in $\mathcal{C}$. If A is discrete, then regard it as a set; we have

$$X \otimes A = X^{\coprod A}.$$

More generally, recall (or convince yourself) that any simplicial set A can be expressed as the homotopy colimit of a diagram I of discrete simplicial sets: $A = \mathrm{hocolim}\, I$. Then

$$X \otimes A = \mathrm{hocolim}\, X^{\coprod I}.$$

1.6.1. *Derived natural transformations and derived coends.* Suppose $\mathcal{C}$ is a topological (or simplicial) model category. The functor

$$\mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{Top}$$

given by the formula

$$(X, Y) \mapsto \mathrm{map}(X, Y)$$

is not a homotopy functor, but it has a right derived functor. The derived functor is given by the formula

$$\mathrm{hmap}(X, Y) := \mathrm{map}(X^c, Y^f).$$

Here X^c and Y^f are a cofibrant and fibrant replacement of X and Y respectively (we are assuming that $\mathcal{C}$ has functorial cofibrant and fibrant replacements). We call it the derived mapping space.

Now consider the situation when $\mathcal{C}$ is itself a functor category. Let $\mathcal{A}$ be a small category and let $\mathcal{D}$ be a topological (or simplicial) model category. Let $[\mathcal{A}, \mathcal{D}]$ be the functor category. Under favorable circumstances, the functor category will be, again, a topological model category. The most commonly used model structure on the category of functors is the so-called projective model structure. In this model structure, fibrations and weak equivalences are defined objectwise, and the cofibrations are what they are forced to be.

Let F, G be two functors from $\mathcal{A}$ to $\mathcal{D}$. We call the derived mapping space from F to G the space of homotopy natural tranformations, and denote it $\mathrm{hnat}(F, G)$. The following are important properties of homotopy natural transformations

- the bifunctor $(F, G) \mapsto \mathrm{hnat}(F, G)$ preserves homotopy limits in the second variable and converts homotopy colimits in the first variable to homotopy limits.
- the set of path components of $\mathrm{hnat}(F, G)$ is the set of morphisms from F to G in the homotopy category of $[\mathcal{A}, \mathcal{D}]$.
- If F is a representable functor then the natural map

  $$\mathrm{nat}(F, G) \longrightarrow \mathrm{hnat}(F, G)$$

  is an equivalence. In particular, one can apply the Yoneda lemma to the space of homotopy natural transformaitons.

One can use these properties to calculate $\mathrm{hnat}(F, G)$ by writing F as a homotopy colimit of representable functors.

Dually, if F is a contravariant functor from $\mathcal{A}$ to $\mathcal{D}$ and G is a covariant functor from $\mathcal{A}$ to $\mathcal{D}$ then one can define the derived coend of F and G to be the coend of cofibrant replacements of F and G. We leave it to you to figure out the basic properties of this construction.

Exercise 1.6.1. *Let* $\Lambda(X) = X^{\wedge n}$, *as before. Describe the space of homotopy natural transformations* $hnat(\Lambda, G)$, *when* G *is a covariant functor and the derived coend of* Λ *and* G *when* G *is a contravariant functor.*

2. Square diagrams of spaces

Homotopy pushout and pullbacks play an important role in the calculus of functors. Let us review some generalities about them. Suppose we have a diagram of spaces (more generally, the discussion could be held in any category where you can make sense of homotopy limits and colimits)

$$\begin{array}{ccc} X_0 & \longrightarrow & X_1 \\ \downarrow & & \downarrow \\ X_2 & \longrightarrow & X_{12} \end{array} \tag{1}$$

Let O be the homotopy pushout of the diagram

$$X_1 \longleftarrow X_0 \longrightarrow X_1.$$

Let B be the homotopy pullback of the diagram

$$X_1 \longrightarrow X_{12} \longleftarrow X_1.$$

The square diagram gives rise to canonical maps $O \longrightarrow X_{12}$ and $X_0 \to B$.

Definition 2.0.2. *We say that the diagram* (1) *is a homotopy pushout (or "homotopy cocartesian", or just "cocartesian") square if the map* $O \longrightarrow X_{12}$ *is a weak homotopy equivalence. We say that the diagram is a homotopy pullback (or "homotopy cartesian", or just "cartesian") square if the map* $X_0 \longrightarrow B$ *is a weak homotopy equivalence.*

More generally, we say that the square is k-cocartesian (resp. k-cartesian) if the map $O \longrightarrow X_{12}$ (resp. the map $X_0 \longrightarrow B$) is k-connected. Thus (co)cartesian = ∞-(co)cartesian.

Exercise 2.0.3. *Suppose that in diagram* (1) $X_1 = X_2 = *$.
(a) supposing that the square is cocartesian, describe X_{12} *in terms of* X_0.
(b) supposing that the square is cartesian, describe X_0 *in terms of* X_{12}.

It is worth noting that if the square (1) is cocartesian, then the *homology* groups of the four spaces fit into a long exact sequence (the Meyer-Vietoris sequence) (more generally, there will be a Meyer-Vietoris sequence for any generalized homology theory). On the other hand if the square is cartesian then the *homotopy* groups of the spaces fit into an analogous exact sequence.

Exercise 2.0.4. *What can you say about the homology (resp. homotopy) groups of the four spaces if the diagram is k-cocartesian (resp. k-cartesian)?*

The purpose of the following exercise is to illustrate another important point.

Exercise 2.0.5. *Find a square diagram of spaces that is both cocartesian and cartesian.*

Having done this exercise you know that in the category of spaces a diagram is almost never both cartesian and cocartesian. This comes down to the fact that in the category of spaces, colimit (union) is a very different construction from limit (cartesian product). This should be contrasted with the situation in the category of Abelian groups, where sums and products are isomorphic, or more pertinently to us, the category of Spectra, where the two are homotopy equivalent (see section on spectra below).

Nevertheless, there is an important theorem (the Blakers-Massey theorem) that says that if a diagram is homotopy cocartesian then it is homotopy cartesian in a certain range (and vice versa).

Theorem 2.0.6. *Suppose that the square diagram is cocartesian. Suppose also that the maps $X_0 \longrightarrow X_i$ are k_i connected, for $i = 1, 2$. Then the square is $k_1 + k_2 - 1$-cartesian.*

Dually, if the square is cartesian and the maps $X_i \longrightarrow X_{12}$ are l_i-connected, for $i = 1, 2$, then the square is $l_1 + l_2 + 1$-cartesian.

Exercise 2.0.7. *Suppose that in square diagram* (1) $X_1 = X_2 = *$ *and X_0 is k-connected. Assume that the diagram is co-cartesian. Use Blakers-Massey theorem to conclude that the natural map*

$$X_0 \longrightarrow \Omega\Sigma X_0$$

is $2k + 1$-connected. This is the Freudenthal suspension theorem.

Exercise 2.0.8. *Suppose that the diagram* (1) *is a homotopy cocartesian diagram of pointed spaces. Consider the diagram obtained by applying the functor $S^n \wedge -$ (a.k.a n-fold suspension) to* (1). *Prove that the resulting square is $2n - 1$-cartesian.*

Now apply the functor $\Omega^n(S^n \wedge -)$ to (1). Prove that the resulting diagram is $n-1$-cartesian.

Let

$$QX = \Omega^\infty \Sigma^\infty X = \underset{n\to\infty}{colim}\, \Omega^n(S^n \wedge X).$$

Conclude that if you apply the functor Q to any *homotopy pushout square you get a homotopy pullback square. You have proved that Q is a* linear functor *in the sense of Goodwillie calculus. This is closely related to the assertion that stable homotopy is a generalized homology theory.*

3. Spectra

3.1. **The fundamental property.**

Here an important theme is that in Spectra, the distinction between homotopy limits and colimits gets blurred to a considerable extent. For example, in spectra a finite sum of objects is equivalent to a finite product (from this perspective, it is natural to see infinite loop spaces as generalizations of Abelian groups). This is a special case of the following fundamental fact.

Theorem 3.1.1. *In spectra, a square diagram is a homotopy pushout if and only it is a homotopy pullback.*

This is related to the fact that in spectra, homotopy groups behave just like homology groups (indeed in spectra "homotopy groups" are an example of a "generalized homology theory").

Exercise 3.1.2. *Use this to prove that if X is a spectrum then the following natural maps are weak homotopy equivalences*

$$\Sigma\Omega X \longrightarrow X \longrightarrow \Omega\Sigma X.$$

3.2. **The definitions.**

Here is the naïve, old-fashioned definition of a spectrum.

Definition 3.2.1. *A spectrum Y is a sequence of pointed spaces $Y_0, Y_2, Y_2, \ldots,$ equipped with maps $Y_i \to \Omega Y_{i+1}$. We say that Y is an Ω-spectrum if the structure maps $Y_i \to \Omega Y_{i+1}$ are weak homotopy equivalences.*

Exercise 3.2.2. *Let Y be a spectrum. Define a new sequence of spaces Y_i' by the formula*

$$Y_i' := \underset{j\geq 0}{\operatorname{hocolim}}\, \Omega^j X(i+j)$$

Show that this sequence forms an Ω-spectrum (you need the fact that Ω preserves sequential homotopy colimits since S^1 is a compact object.)

The homotopy groups of a spectrum Y are defined by the formula

$$\pi_k Y := \operatorname{colim}_{i\geq 0} \pi_{k+i} Y_i.$$

Exercise 3.2.3. *Show that the natural map $Y \to Y'$ induces an isomorphism on the homotopy groups.*

Roughly speaking, Y' is a fibrant replacement of Y.

There is a self-evident notion of functions between spectra and of what it means for two functions to be homotopic. The *stable homotopy category* is the category whose objects are spectra, and where the set of maps from X to Y, denoted $[X, Y]$, is the set of homotopy classes of functions from X to the Ω-spectrum associated to Y. To be more precise, one also needs to replace X with a cofibrant approximation. Roughly speaking, this means that we need to arrange that the spaces X_k are SW-complexes, and the maps $\Sigma X_k \to X_{k+1}$, adjoint to the structure maps, are inclusions of subcomplexes.

Example 3.2.1. *Let X be a pointed topological space. The* suspension spectrum *of X, denoted $\Sigma^\infty X$, is defined by the sequence $X, \Sigma X, \Sigma^2 X, \ldots$. In particular, the spectrum $\Sigma^\infty S^0$ is called the sphere spectrum.*

Exercise 3.2.4. *Let X_0 be a pointed CW-complex. Let Y be a spectum. Show that the set of maps in the stable homotopy category $[\Sigma^\infty X_0, Y]$ is the same as the set of homotopy classes of maps from X_0 to Y_0', where Y_0' is the zero-th space of the Ω-spectrum associated with Y.*

(Roughly speaking, this exercise says that the zero-th space functor is right adjoint to the functor Σ^∞, and the adjunction passes to homotopy categories.)

Example 3.2.2. *Let A be an Abelian group. The sequence of Eilenberg - Mac Lane spaces $K(A, 0), K(A, 1), \ldots$ forms an Ω-spectrum. This is the Eilenberg - Mac Lane spectrum KA.*

One of the reasons that spectra are so important in homotopy theory is that they represent generalized homology and cohomology theories. Let E be a spectrum, X a pointed space. $E \wedge X$ is the spectrum obtained by taking termwise smash product of the spaces that make up E with X. Define

$$E_*(X) = \pi_*(E \wedge X).$$

$$E^*(X) = [\Sigma^\infty X, E].$$

It is not too hard to show (once you have established the basic properties of the stable homotopy category) that E_* and E^* define a generalized homology and cohomology theory respectively. Morever, *Brown's representability theorem* says that every generalized (co)homology theory is represented in this way by some spectrum E. For example the Eilenberg - Mac Lane spectrum KA represents the "ordinary" singular (co)homology with coefficients in A. At the other extreme, the sphere spectrum represents stable homotopy theory.

3.3. **Highly structured spectra.** The old fashioned definition of spectra is adequate for many purposes, but for modern applications one often needs to work with a more sophisticated notion. For example, one would like there to be a well-behaved notion of smash product of spectra and one can not have it with the old-fashioned definition. At some points during the workshop (specifically in lectures 7 and 8) we will have to use the category of spectra constructed by Elmendorf-Kriz-Mandell-May (popularly known as EKMM). Therefore it would be good to familiarize yourself with the basic properties of "highly structured spectra". For another perspective on modern foundations of stable homotopy theory you may want to check out Stefan Schwede's unfinished book on symmetric spectra (http://www.math.uni-bonn.de/ schwede/SymSpec.pdf).

4. Operads

For a quick introduction to operads that focuses on topics that are relevant to us we suggest, again, the paper of Ching "Bar constructions for topological operads and the Goodwillie derivatives of the identity". Note however that this paper only deals with operads without zero term.

Let $(\mathcal{C}, \otimes, \mathtt{1})$ be a symmetric monoidal category. We may also want to assume that $\mathcal{C}$ is a pointed category. I.e., that it has an object that is both initial and final. We denote this object by $\mathtt{0}$.

A *symmetric sequence* in $\mathcal{C}$ is a sequence of objects of $\mathcal{C}$, $C_0, C_1, \ldots, C_n, ..$ togher with an action, for each n, of the symmetric group Σ_n on C_n.

An *operad* in $\mathcal{C}$ is a symmetric sequence in $\mathcal{C}$ together with structure maps

$$\mathtt{1} \longrightarrow C_1.$$

$$C_i \otimes C_{n_1} \otimes \cdots \otimes C_{n_i} \longrightarrow C_{n_1+\cdots+n_i}$$

that satisfy some axioms that say that the structure maps are associative, symmetric and unital.

Remark 4.0.1. *Quite often we will want to consider operads without a zero term... the name should be self-explanatory enough. An operad without a zero term is essentially the same thing as an operad whose zero term is the zero object.*

One can define an associative composition product on the category of symmetric sequences so that operads are precisely monoids with respect to the composition product. From this point of view we see a natural way to define right and left modules over an operad.

Definition 4.0.2. *Let O be an operad, M a symmetric sequence. We say that M is a right module over O if there is a map of symmetric sequences $M \circ O \longrightarrow M$ that is associative and unital in the evident sense. Similarly, we say that M is a left module if there is an associative and unital map $O \circ M \longrightarrow M$.*

Example 4.0.1. *The commutative operad is the operad that has the unit object $\mathtt{1}$ in every degree. It is self-evident what the structure maps are (indeed, every structure map is the identity map on $\mathtt{1}$). Similarly one may define the commutative operad without zero term, a.k.a the non-unital commutative operad.*

Exercise 4.0.3. *Show that a right module over the commutative operad in $\mathcal{C}$ is the same thing as a contravariant functor from the category of finite sets to $\mathcal{C}$.*

Show that a right module over the non-unital commutative operad is the same thing as a contravariant functor from the category of finite sets and surjective functions between them to $\mathcal{C}$.

Perhaps more widely familiar than right and left modules are the notions of an algebra and a coalgebra over an operad.

Definition 4.0.4. *Let O be an operad and let X be an object of $\mathcal{C}$ (more generally, X can be in any category tensored over $\mathcal{C}$). An O-algebra structure on X consists of maps*

$$X^{\otimes n} \otimes O_n \longrightarrow X$$

that are symmetric, associative and unital.

Dually, an O-coalgebra structure on X consists of maps

$$X \otimes O_n \longrightarrow X^{\otimes n}$$

that are, again, symmetric associative and unital.

Exercise 4.0.5. *Suppose that X is a coalgebra over O. Show that the sequence $\mathtt{1}, X, X^{\otimes 2}, \ldots, X^{\otimes n}, \ldots$ has a natural structure of a right O-module.*

Exercise 4.0.6. *Show that a coalgebra over the commutative operad is the same thing as a contravariant symmetric monoidal functor from the category of finite sets to $\mathcal{C}$.*