Model Cat /
Simplicial
Presheaves

I. Model Categories

$(\underset{\text{Top}}{\mathcal{C}}, \underset{\xrightarrow{\sim}}{\mathcal{E}})$ want to study $\mathcal{C}[\mathcal{E}^{-1}] := \mathrm{Ho}(\mathcal{C})$

Def: A model cat is a category M w/ 3 classes of distinguished morphisms

$$\underset{\text{w.eq}}{\cdot \xrightarrow{\sim} \cdot} \qquad \underset{\text{cof}}{\hookrightarrow} \qquad \underset{\text{fib}}{\twoheadrightarrow}$$

Satisfying:

M1: All finite limits/colimits

M2: 2 of $\{f, g, fg\}$ are w.eq. then so is the third

M3: each class is closed under retracts

M4: (lifting)

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"'] & C \arrow[d, two heads, "p"] \\ B \arrow[r] \arrow[ur, dashed, "\exists h"] & D \end{tikzcd}$$

$\exists h$ if i or p is acyclic

MC5: $f: X \to Y$ can be factored as a ~~cf pair~~

$X \overset{\sim}{\hookrightarrow} \twoheadrightarrow Y$ or $X \hookrightarrow \overset{\sim}{\twoheadrightarrow} Y$

<u>Def</u>: X is fibrant if $X \twoheadrightarrow *$

cofibrant if $\emptyset \hookrightarrow X$

<u>Def</u>: A cofibrant replacement for X

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[dr, hook] & X \\ & QX \arrow[u, two heads, "\sim"'] \end{tikzcd}$$

fibrant replacement for X

$$\begin{tikzcd} X \arrow[r, two heads] \arrow[dr, hook, "\sim"'] & * \\ & RX \arrow[u, two heads] \end{tikzcd}$$

<u>Exercise</u>: Weak equivalences and (cof) fibrations determine fibrations (fib)

<u>Homotopy Category</u>: There exist 2 notions of homotopy for maps $f, g : X \to Y$

via cylinder objects $\rightsquigarrow$ "$X \times I$"

path objects $\rightsquigarrow$ "$\mathrm{Map}([0,1], Y)$"

<u>Claim</u>: For objects which are cofibrant/fibrant these notions are equiv and form equivalence relation

<u>Def</u>: $Ho(M)$ obj $=$ obj M

$\hom_{Ho(C)}(X, Y) = [RQX, RQY]$

Derived functors

If $F : \underset{\uparrow \atop MC}{M} \longrightarrow A \ni$ weak equivalences between cof $\hookrightarrow$ isomorphisms

then $\exists\, LF : Ho(M) \longrightarrow A$ $\Big\}$ $\boxed{\text{explain}}$
"left derived functor"

If A is a model category, F sends w eq between cof objects to w eq

then $\exists$ a total left derived functor
$LF : Ho(\mathcal{C}) \longrightarrow Ho(A)$

Homotopy limits and colimits
Point: limits of diagrams are not homotopy invariant

I be a finite category, eg I $\left(\begin{tikzcd} & \cdot \arrow[d] \\ \cdot \arrow[r] & \cdot \end{tikzcd} \right)$

Form a diagram category $M^I := Func(I, M)$
$ob M^I$ - diagrams $\left(\begin{tikzcd} & X \arrow[d] \\ Z \arrow[r] & Y \end{tikzcd} \right)$

Claim: Sometimes there is a model structure on M^I

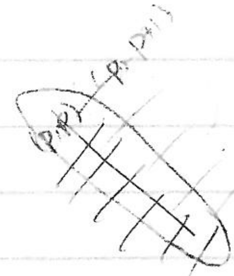
° w. eq. are pointwise w. eq.

$$\begin{tikzcd} & X \arrow[r, "\sim"] \arrow[d] & X' \arrow[d] \\ Z \arrow[r] \arrow[dr, "\sim"] & Y \arrow[r, "\sim"] & Y' \\ & Z' \arrow[ur] & \end{tikzcd}$$

In this $\lim_I : M^I \longrightarrow M$ has a total right derived functor.

<u>holim</u>: $Ho(M^I) \longrightarrow Ho(M)$

$$\begin{tikzcd} M_{pd} \arrow[d, "h"] \arrow[dr, "r"] & \\ \mathbb{C} & h^{-1}(0) \end{tikzcd}$$



$$H^*(M_{pd}, \mathbb{Q}) \to H^*(h^{-1}(0), \mathbb{Q})$$

Higher Non-Abelian Hodge Theory

Cohomology w/ coefficents :

X a space ; $\mathcal{L}$ a local system — rep $\pi_1(X,x) \to GL_n$

of rank n cmplx vector spaces — locally constant sheaf

Q: What do you need on X to show equivalent

$H^i(X;\mathcal{L})$ π_1 in post tower acts on π_n $n \geq 2$ when X not simply connected

in algebraic geometry this comes up in $\mathcal{L}$.S.S.

$X \xrightarrow{f} Y$ smooth, proper

$H^i(X;\mathbb{C}) = ?$

$$H^j(Y, \underbrace{R^k f_* \mathbb{C}}_{L}) \Rightarrow H^{i+k}(X, \mathbb{C})$$

L will be a variation Hodge structure

* {will combine w/ H.S. Y can converge to H.S. on X}

1980's — L can vary in a family (dim will not always be the same)
then we expect to get some kind of family of these
$H^i(X,\mathcal{L})$ ← dim will vary w/ $\mathcal{L}$

Look at DR point of view

$M_{DR}(X, m)$ = moduli stack of (E, ∇)

E is a rank m vb
∇ integrable connection) alg geo world

Artin stack of finite type and infact

$R_{DR}(X, x, m) = \{ \underbrace{(E, \nabla, p) \mid p: E_x \to}_{\text{add framing}} \}$

Then this is a $\overset{\text{quasi-proj}}{\text{scheme}}$ which is a fine moduli space

$M_{DR} = R_{DR}(X, x, m) / Gl(n)$

we have an integrable family $(\mathcal{E}, \nabla)$ on

$$\begin{array}{c} X \times M_{DR} \\ \downarrow \\ M_{DR} \end{array}$$

$\mathcal{E}$ v.b. on $\begin{array}{c} X \times M_{DR} \\ \downarrow \\ M_{DR} \end{array}$

DR point of view for a simple L

$L \leftrightarrow (\mathcal{E}, \nabla)$ then

$H^i_B(X^{top}, L) \cong \mathbb{H}^i(E \to E \otimes \Omega^1_X \to E \otimes \Omega^2_X \to \quad)$

$X \times M_{DR}$
$\downarrow p$ instead of hyper cohomology take
M_{DR} hyper direct image

$R^\bullet p_* : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{M_{DR}} \to \cdots \to \mathcal{E} \otimes \Omega^k_{M_{DR}} \to$

complex of sheaves on M_{DR} ?
$\mathcal{O}$-mod ↖ stack

GL_n-equivariant resolution of M_{DR}

This is a complex of quasi-coh sheaves $\mathcal{O}_M$-modules on M_{DR} $\mathcal{U}$ $\mathcal{U}_\bullet = \{U_{i_1} \cap \cdots \cap U_{i_n}\} \xrightarrow{i} X$

① $X = \bigcup U_i$ an affine Zariski open covering

② $i_* \check{C}(\mathcal{U}_{X \times M}, \mathcal{E} \otimes \Omega^\bullet_{M_{DR}}, d_\nabla)$ $\boxed{\text{Exercise}}$

double complex $\check{C}$ direction
$\Omega^\bullet$ direction

double complex of quasi coh sheaves on $X \times M_{DR}$ (the vertical differentials are diff are diff operators d_∇ on $\mathcal{O}_X$-linear
They are $\mathcal{O}_{M_{DR}}$-linear)

Fact: $Rp_*(\mathcal{E} \otimes \Omega^\bullet_{M_{DR}}) = p_*(i_* \check{C}(\quad))$ on M_{DR}
This is complex of quasi-coherent sheaves on M_{DR}

$\boxed{\mathcal{O}_{M_{DR}}\text{-flat}}$

$R^i_{p*}(\ ) = \mathbb{H}^i(\mathbb{R}p_*(-))$ These are coherent sheaves
Crowert + $\underbrace{\text{the spectral sequence}}$
the vertical differentials are differential operators

Mumford $\Rightarrow$ how to get a perfect complex
Recall: A perfect complex is a bounded complex of finite rank vector bundles

Remark: If Z is a scheme
Then (Mumford) $Z \xrightarrow{\phi} M_{DR}$ then
$\phi^*(\mathbb{R}p_* \mathcal{E} \otimes \Omega^{\bullet}_{M_{DR}})$ is
quasi isomorphic to
a perfect complex on Z
(not canonical) so $Z = \underbrace{R_{DR}}_{\text{framed}}(X, x, n)$ it wont
necessarily be Gl_n-equivariant on the nose

perfect complex on a stack?

~~on the complex~~ is a perfect complex of equivariant vb
on the framed space.

Q: Understand this better

$C^{\bullet}\mathbb{R}p_*(-)$ finite length complex
$C^0 \to C^1 \to$
H^m is a coherent sheaf

locally on Z (the base) choose $\underbrace{\overset{V=}{\mathcal{O}_Z^n}}_{v.b.} \to C^m$

s.t. $V \twoheadrightarrow H^m$

$$C^{m+2} \to \ker(V \oplus C^{m+1} \twoheadrightarrow C^m) \to 0 \quad := B^\circ$$

$$B^\circ \to V[-m] \to 0 \qquad \text{quasi iso to } C^\circ$$

by induction B° is q. iso. to Q° = perfect complex

Idea: Now we have a perfect complex (modulo issue of being equivariant) of $\mathcal{H}^\circ$ on M_{DR} with the property that if $p \in M_{DR}$

$$\mathcal{H}_p^\circ = \text{complex } H^\circ(X, \mathcal{L}_p)$$

then over some open set $U \subseteq M_{DR}$
the dim will all be constant
$H^\cdot(U)$ will be the family of $H^\cdot(X, \mathcal{L}_p)$

First case of this in terms of NAHT is work of $\underbrace{\text{Green - Lazarsfeld}}_{\text{rank 1 - local systems}}$ jump loci

Goal: Understand the NAHT on these higher homotopy theory objects

$=$

MHS

On some results of R. Hain

Motivation: If X cmplx alg variety (smooth, proper /$\mathbb{C}$) then we can try to understand the moduli of X:

(1) Find a topological invariant $H^k(X;\mathbb{Z})$

(2) Put a Hodge filtration and see how the filtration varies w/ complex structure of X
- for that we look at periods of integrals

Q: What do we do if we want to study a pointed variety (X, x_0)?

- $\pi_1(X, x_0)$ not abelian
- $\mathbb{Z}[\pi_1(X, x_0)]$ abelian but not fintely generated
- $\mathbb{Z}[\pi_1]/I^k$ truncate I^k is some power of augmentation ideal

Q: What is the substitute for the Hodge filtration

Claim: Consider iterated integrals

$$\int_\gamma \omega_1 \omega_2 \ldots \omega_k = \int_{0 \le t_1 \le t_2 \le \ldots \le 1} \cdots \int f_1(t_1) f_2$$

$$f = \gamma^* \omega_1$$

Chen's π_1 Theorem: Let $H^0(I_X)$ be the iterated integrals depending on the homotopy class of

(1) $H^0(I_X)_s \cong \mathrm{Hom}(\mathbb{Z}\pi_1/I^{s+1}, \mathbb{Z})$

rank at most s

$s=1$ just get closed 1-forms

(2) $H^0(I_X) \cong \underbrace{\mathcal{O}(\mathcal{U})}$ Hopf alg isomorphism

cordinate ring of unipotent completion of π_1

Thm (H) Let X be smooth cmplx alg variety and $x_0 \in X$. Let V be an admissible VHS on X and let $S = \mathrm{Aut}(V_0, \langle , \rangle)$. Suppose $\rho: \pi_1(X, x_0) \to S$ is Zariski dense and let $\tilde{\rho}: \pi_1(X, x_0) \to G = G(X, x_0)$ the completion of π_1 be the ρ. Then $\mathcal{O}(G)$ carries a MHS with weight ≥ 0 such that the antipode multiplication and comultiplication on MHS w inclusion

$\mathcal{O}(S) \hookrightarrow \mathcal{O}(G)$

A Hopf algebra, H (over k) is a k-algebra $H \otimes H \to H$ which has a compatible comultiplication $\Delta : H \to H \otimes H$ and a counit ($H \to k$) and an antipode $H \to H$ (w/ some properties)

Prototypical example $\mathcal{O}(G)$ some group G
Δ = dual to mult   counit = evaluation at e
antipode: inverting the argument

Fact: G - affine alg group, we can
$\mathfrak{g} = \mathrm{Lie}(\mathcal{G})$ from $\mathcal{O}(\mathcal{G})$

The Malcev completion

Data: π - abstract group
S - linear algebraic group / k of char 0
$\rho : \pi \to S$ of Zariski dense

look at diagrams of this form

$$\begin{array}{ccccccccc} 1 & \to & U & \to & E & \to & S & \to & 1 \\ & & & & \hat{\rho}\uparrow & \nearrow_{\rho} & & & \\ & & & & \pi & & & & \end{array}$$

(E - local algebraic)

where U is unipotent

The completion of π relative to ρ is $G = \varprojlim E$

i.e. G satisfies universal property

If $\tilde{G}$ is an extension

$$\begin{tikzcd} 1 \arrow[r] & U \arrow[r] & \tilde{G} \arrow[r] & S \arrow[r] & 1 \\ & & \pi \arrow[u, "\tilde{\Phi}"] \arrow[ur, "\Phi"'] & & \end{tikzcd}$$

$\exists ! \, u$

$$\begin{tikzcd} & G \arrow[d, "\exists ! u"] \\ \pi \arrow[ur] \arrow[r] & \tilde{G} \end{tikzcd}$$

Construction:

$\hat{k\pi}$ completion of group alg of π

$\Delta : \hat{k\pi} \longrightarrow \hat{k\pi} \oplus \hat{k\pi}$

$G = \{$ grouplike elements of $\hat{k\pi} \}$

$= \{ X \text{ s.t. } \Delta(X) = X \otimes X \text{ and } \varepsilon(X) = 1 \}$

Q: "the ring of functions on group ring or on dual?"

Hain What can we put MHS on?

Homotopy Lie Alg of a space X

- If X is not nilpotent, $\mathfrak{g}(X) = 0$

  π_1 is nilpotent acts unip. on higher π_n's

- If X is nilpotent $\mathfrak{g}_\bullet(X) = \mathfrak{g}_\bullet(\pi_1(X))$

  $=$ M. Lie alge

$k > 1 \quad \mathfrak{g}_k(X) = \pi_k(X;\mathbb{Q})$

$[\ ,\ ]$ is whitehead bracket

$S^k \times S^\ell$ has a decomp w/ 4 cells

$\alpha : S^{k+\ell-1} \longrightarrow S^k \vee S^\ell$ attaching map

$x \in \pi_k(X) \qquad [x,y] \in \pi_{k+\ell-1}(X)$

$y \in \pi_\ell(X)$

Thm: The homotopy Lie alg has a MHS

X - smooth complex variety

DeRham homotopy of cmplx alg varieties
$\underline{\mathcal{Y}}$

- $H^{\circ}(\pi X)$ has MHS
- $f: X \longrightarrow Y$

$$\begin{array}{c} E_f = \text{ho fib} \\ \downarrow \\ X \xleftrightarrow{\sim} X' \longrightarrow Y \end{array}$$

X, Y cmplx varieties

1) E_f is connected
2) Y nilpotent

then $H^{\circ}(E_f)$ has a MHS

X as before
$\hat{X} \to X$ connected normal cover
cover acts unipotently
Then $H^{\circ}(\hat{X})$ has a MHS

Idea: If we have a nice model for $H^*(X, \mathbb{Q})$ under favorible conditions we can construct a model for

$$\left.\begin{array}{l} H^*(\pi X, \mathbb{Q}) \\ H^*(E_f; \mathbb{Q}) \end{array}\right\} \text{ via Bar constructn.}$$

If X is a space we construct a MH complex
where cohomology is MHS

might admit a quasiiso to DR complex of X

multiplicative

If C° is a MHC = mixed Hodge complex
then M° and N° are C°-modules
then $\exists$

$$B(M^\circ; C^\circ; N^\circ)$$

is also a MHC

<u>Thm:</u> (Hain) The DR to Homotopy Hodge theorem
Let X be a pointed space whose Betti numbers are finite and has DR MH complex

Then:

(a) The homotopy Lie alg of X has a MHS in which filtered pieces are Lie alg ideals

(b) The completed group ring of π_1 has MHS

Thm (Deligne) Any smooth complex variety has DR MH complex

(MHS is an invariant of Rational homotopy type)

Functorality of MHS
delicate – not generally functors

$$\begin{array}{ccc} E_f & \longrightarrow & X \\ & & \downarrow f \\ & & Y \end{array}$$

$$\longrightarrow \pi_{k+1}(Y) \xrightarrow{\partial} \pi_k(E_f) \xrightarrow{i_*} \pi_k(X) \xrightarrow{f_*} \pi_k(Y)$$

∂, i_*, f_* morphisms of mixed Hodge structures

Do there a notion of formality for MH Complexes?

$\exists$ examples of varieties which have iso of MHS on cohomology but different on homotopy

$$\begin{array}{ccc} {}^{HS}\ X & \xrightarrow{\text{lift}} & \text{MH Complex} \\ & & \downarrow \text{Bar} \\ & & \Omega X \end{array}$$

Non-Abelian de Rham coh degree 1

$$H^1(X_{DR}, G) = \underline{\mathrm{Hom}}(X_{DR}, K(G,1))$$

$$H^n(X_{DR}; T) = \underline{\mathrm{Hom}}(X_{DR}, T)$$

n-stacks

$\pi_0(T) = *$

$\pi_1(T) = G$ affine alg group

$\pi_i(T) =$ represented by f. dim. v.sp

Derived AG - Lurie

Non-Abelian Cohomology

$$H^n(X;A) = [X, K(A,n)]$$

$$\pi_i K(A,n) = \begin{cases} A & i=n \\ 0 & o.w. \end{cases}$$

Let G be non-abelian group

$$H^1(X,G) = [X, \underset{K(G,1)}{\underset{\parallel}{BG}}]$$ = isomorphism classes of G-torsors / principle G-bundles on X

$= \mathrm{Hom}(\pi_1 X, G)/\text{conjugation}$

For higher N.A.C.: For any space Y we could write $H(X;Y) = [X,Y]$

Def: A prestack is a functor X: com ring $\longrightarrow$ sSets
simplicial presheaf on affine schemes

$\cup$
Kan complex
$\simeq^?$ top space

Ex: X scheme

$$X(R) = \mathrm{Hom}_{\text{scheme}}(\mathrm{Spec}(R), X) = R\text{-pts}$$

sets $\subseteq$ sSets

Ex: G is an algebraic group

$$\overbrace{BG}^{\text{stack}}(R) = \text{(nerve of) groupoid of } G\text{-torsors on } \mathrm{Spec}(R)$$

$$= \text{ordinary stack} = \text{nerve of groupoid}$$

or $\quad X(R) = G(R) = R\text{-points of } G$

$$\underbrace{B(G(R))}_{\text{pre stack}} = \text{trivial } G\text{-torsors}$$

Def: Let V be a com alg group (vector space)

$$K(V,n)(R) = \underbrace{K(V \otimes R, n)}_{\text{prestacks}}$$

X a scheme $\rightsquigarrow$ prestack

$[X, BG]$ = set of iso classes of algebraic G torsors on X

$$[X, K(V,n)] = H^n(X; \mathcal{O}_X \otimes V)$$

We say that a prestack X is a stack if X satisfies descent for flat topology

Means: $\mathcal{U}$ an affine open cover of Spec(R)

$$\begin{tikzcd} \cdots \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathcal{U} \times_{\operatorname{Spec} R} \mathcal{U} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{U} \arrow[r] & \operatorname{Spec}(R) \end{tikzcd}$$

$$\begin{tikzcd} X(R) \arrow[r, shift left, "\sim"] & \operatorname{holim} X(\mathcal{U}) \arrow[l, shift left] \arrow[r, shift left] \arrow[r, shift right] & X(\mathcal{U} \times_{\operatorname{Spec} R} \mathcal{U}) \end{tikzcd}$$

(and also for hypercoverings)

faithfully flat descent

Ex: Descent in a more concrete case

$$\begin{tikzcd} X(\operatorname{Spec} R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner \sim"] & X(U) \arrow[d] \\ X(V) \arrow[r] & X(U \cap V) \end{tikzcd} \qquad U \cup V = \operatorname{Spec}(R)$$

X a scheme $\mathcal{F}$ a q.coh. sheaf on X

$$\begin{tikzcd} K_X(\mathcal{F}, n)(R) \arrow[d] \\ X \end{tikzcd}$$

Let X be a pointed prestack

X : com-ring $\longrightarrow$ pointed simplicial sets

Can define homotopy groups

$\pi_i^{\text{niave}}(X)$: ℓ-algs $\longrightarrow$ $\begin{cases} \text{alg groups} & \geq 2 \\ \text{groups} & 1 \\ \text{sets} & 0 \end{cases}$

$\pi_1^{\text{niave}}(X)(R) = \pi(X(R))$

$=$

$\pi_i(X)$ = sheafification of presheaves $\pi_i^{\text{niave}}(X)$

Ex: $\pi_0^{\text{naive}}(BG)$ = iso classes

$\pi_0(BG) = *$

Def: A pointed prestack X is a <u>higher gerbe</u> if

(1) X is a stack

(2) $\pi_0(X) \simeq *$ (any 2 pts of X after some cover of R are in the same comp)

(3) $\pi_1(X)$ is represented by an affine group scheme

(4) $\pi_i(X)$ representable by affine unipotent group schemes

$$\left(\begin{array}{c} R \longmapsto Hom_{\mathbb{C}}(V, R) \\ \text{for some vector } \mathbb{C} \end{array} \right)$$

(very presentable n-stack)

In this case $X \to$ lim of tower

X is a stack

Postnikov tower

$$\begin{array}{ccc} K(\pi_3 X, 3) & \longrightarrow & \tau_{\geq 3} X \\ & & \downarrow \\ K(\pi_2 X, 2) & \xrightarrow{\text{hofib}} & \tau_{\geq 2} X \\ & & \downarrow \\ \text{if } X \text{ higher gerbe} & & \tau_{\geq 1} X = B(\pi_1 X) \end{array}$$

classifies torsors for π_1 of X

$$K(\pi_2 X, 2)(R) = K(\underbrace{\pi_2 X(R)}, 2)$$

$\pi_2(X)$ is a functor of the form $R \longmapsto Hom(X, R)$

Prestacks are coefficients in a cohomology theory

Q: What is a Quasi-coherent sheaf on a prestack X?

Case 1: X is a scheme

~~Def~~: A quasi coherent sheaf on X is a rule which assigns to each R-point $\eta \in X(R)$ an R-module M_η (functorial on η)

$$\begin{tikzcd} & \mathrm{Spec}\, R \arrow[r, "\eta"] & X \\ \mathrm{Spec}\, R' \arrow[ur] \arrow[urr, "\eta'"'] & & \end{tikzcd}$$

(Some cocycle condition?)

$$M_{\eta'} \simeq R' \otimes_R M_\eta$$

$$QCoh(X)^\heartsuit = \underset{\substack{\longleftarrow \\ \eta \in X(R)}}{\mathrm{holim}} \,(R\text{-modules})$$

makes sense for any prestack

$QCoh(BG)^\heartsuit =$ cat of algebraic representations of G

$QCoh(K(V,2))^{B}$ = complex vector spaces

Analogies

Spaces	Prestacks
covering spaces	quasi coh sheaves
$G \circlearrowright$ Set	BG linear rep of G

dg categories / $\mathbb{C}$

Ex: $R = \mathbb{C}$-alg

$Ch(R)$ = chain complexes of R-modules

$\cup |$

$Ch(R)^{cof}$

dg-cat form a model cat

$Hom_{he}(X,Y) = H_0(Hom(X,Y)_\bullet)$

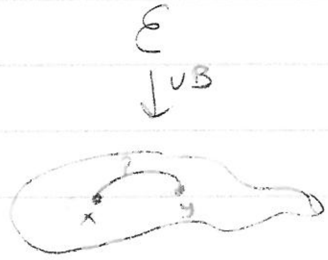
$QCoh(X) = \underset{\eta \in X(R)}{\text{holim}}\, Ch(R)^{cof}$

Thm: (Tannaka Duality) if X is any prestack, Y is

Higher Stack as coefficient system for Non-abelian Coh

$[X, BG]$ = G-bundles on X
(algebraic)

Q: What does it mean to have a flat connection?



$E_x \to E_y$ by connection
only depends on homotopy type
of P by flat

If X is an alg variety, we say that two
points $x, y \in X(R) = \mathrm{Hom}(\mathrm{Spec}\,R, X)$
are infinitesimally close if x, y have the same
image in $X(R/\mathrm{nil}(R))$.

Def (Grothendieck) An algebraic ~~vector bundle~~ quasi coh sheaf
w/ connection on X (smooth variety $/\mathbb{C}$) is
a quasi-coherent sheaf $\mathcal{F}$ + data of an
iso $x^*\mathcal{F} \simeq y^*\mathcal{F}$ whenever x, y are inf close

Claim: TFAE$_{(v.b)}$

(1) A quasi coherent sheaf in the sense of Grothendieck

(2) A quasi coherent$^{(v.b)}$ sheaf $\mathcal{F}$ w/ connection $\nabla \cdot \mathcal{F} \longrightarrow \mathcal{F} \otimes \mathcal{L}$ satisfying ∇^2, Leibniz rule

(3) A sheaf on X w/ an action on the sheaf Dx acts on X which is quasi coherent (coherent) as an $\mathcal{O}_X$-module

$$X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$$
$$\mathcal{D}_X = \mathbb{C}[t, \partial/\partial t]$$

(4) If X is projective then this is equivalent to smooth v.b. w/ connection (GAGA theorems)

Reformulate (1)

$$X_{DR} : \mathbb{C}\text{-alg} \longrightarrow \text{Sets} \subseteq \text{sSets}$$

$$X_{DR}(R) = X(R/\text{nilpotents})$$

X_{DR} is smooth $= X(R)/\text{inf. closeness}$

$\text{Qcoh}(X_{DR})^{\heartsuit}$ = quasi coherent sheaves w/ connection.

Non Ab Coh

$[X_{DR}, BGL(n)]$ = flat vector bundles of rank n on X

$[X_{DR}, BG]$ = flat G-bundle on X
= holomorphic G-bundles w/ flat connection

$[X_{DR}, K(\mathbb{A}^1, n)] = H^n_{DR}(X)$

<u>Question</u>: How to modify to get Dol cohomology?

$$\begin{array}{l} X(R) \qquad\qquad 0\text{-simp} \\ \uparrow\ \uparrow \\ \widehat{X \times X}(R) \subseteq (X \times X)(R) = X(R) \times X(R) \end{array}$$

formally completing along diagonal

Nerve $(X(R)/\text{inf closeness})$

$X_{DR}(R)$ = this simplicial set homotopy equiv to original construction

$X^{n+1}(R) = \mathrm{Hom}(\mathrm{Spec}(R) \times \{0,1,\dots,n\}, \quad X)$

$n \longmapsto \{0,\dots,n\}$
cosimplicial scheme

$\mathrm{Hom}(\mathrm{Spec}(R) \times \{0,1,\dots,n\}, X) =$
simplicial scheme

We want to generalize:

$T^\bullet$ $\qquad\qquad$ $T^n = \mathbb{C}$-scheme
Spec $\mathbb{A}_n$ = is $n+1$ dim
as vect space over $\mathbb{C}$

$$\begin{tikzcd}
\mathbb{A}_0 = \mathbb{C} \\
\mathbb{A}_1 = \text{some 2d } \mathbb{C}\text{-algebra} \arrow[u, shift left, "d_0"'] \arrow[u, shift right, "d_1"] \\
\mathbb{A}_2 \arrow[u] \arrow[u, shift left=2] \arrow[u, shift right=2]
\end{tikzcd}$$

$A_1 = \mathbb{C} \oplus \underbrace{\ker(d_0)}_{I}$

$$\begin{tikzcd}
d_1 : A_1 \arrow[r] & \mathbb{C} \\
I \arrow[u, phantom, "\cup" description] \arrow[ur, "\lambda"']
\end{tikzcd}$$

Any time we have the above data, get a 1-dim $\mathbb{C}$-vsp I + map $\lambda: I \to \mathbb{C}$

- ∃ a "unique" object which can look like this

Whats mult in A_1?

Choose a generator $x \in I$

$$x^2 = \mu x$$

$$\lambda(x)$$

$$x^2 \to \lambda(x^2) \quad \text{ring homomorphism}$$

$$\mapsto \mu \lambda(x)$$

suggests that $\mu = \lambda(x)$ only way that we can satisfy these equations

this defines A_1 as an algebra

$$A_1 \times_{\mathbb{C}} A_1 \to A_1 \leftarrow \text{groupoid?}$$

$A_2 = A_1 \times_{\mathbb{C}} A_1$ on ideal just given by addition

$A_n = A_1 \times_{\mathbb{C}} A_1 \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} A_1$

Only two options

$\lambda = \text{iso} \Rightarrow X_{DR}$

$\lambda = 0\text{-map}$

Ex: $\lambda = 0$ WOLG $I = \mathbb{C}$

$$\mathbb{C} \leftleftarrows \mathbb{C}[\varepsilon]/(\varepsilon^2) \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \mathbb{C}[\varepsilon]/(\varepsilon^3) \times_{\mathbb{C}} \mathbb{C}[\varepsilon]/(\varepsilon^2)$$

make simplicial scheme

If X is a smooth alg variety makes sense to $\mathrm{Hom}(-, X)$

$$\begin{array}{ccc} & TX \times_X TX & \\ \widehat{TX} & TX & \\ \downarrow\downarrow & \downarrow\downarrow & \\ X & X & \end{array}$$

analog of previous but we need to complete along diag

R point of this is a point w/ tangent vector infintessimally close to identity

functor $\mathbb{C}$-alg $\longrightarrow$ simplicial sets
called X_{Dol}

$$X_{Dol}(R) = \coprod_{\eta \in X(R)} B(\underbrace{Nil(R) \otimes_R \underbrace{T_{X/\eta}}_{\text{project R-mod rank n}}}_{\text{abelian group}})$$

Q: What is a quasi coh sheaf on X_{Dol}

A: A rule which assigns to each point $\eta \in X(R)$
an R-mod acted on by the group
$Nil(R) \otimes_R T_{X/\eta}$

Quasi coherent sheaves on X w/ an action
of $\hat{T}_X$
(acted on by
some formal group)

$X = \mathbb{A}^1$ $\quad T_X \simeq X \times \mathbb{A}^1 \quad V = vsp$

if we formally
complet $\hat{T}_X$
we don't need
polynomial
any power series
will do

action $G \to Aut(V)$
algebraic action $\Rightarrow$
should be algebraic
map

$1 + tE + t^2E^2/2 + \cdots$

$$\begin{array}{ccc} G & \longrightarrow & Aut(V) \\ \| & & \\ \mathbb{A}^1 & \longrightarrow & e^{tE} \\ t & \longmapsto & \end{array} \qquad E = End(V)$$

polynomial iff E is
nilpotent

$$\begin{array}{ccc} G & \longrightarrow & \mathrm{Aut}(V) \\ \| & \nearrow & \\ \hat{A}_1 & \longrightarrow & e^{tE} \\ t & \mapsto & \end{array} \qquad E = \underbrace{\mathrm{End}(V)}_{\text{any endomorphism}}$$

$\hat{T}_X = X \times \hat{V}$

representation of formal completion of $\hat{V}$ is an action of $\mathrm{Sym}^* V$

Quasi coherent sheaves on X_{Dol} $\simeq$ quasi coherent sheaves on X w/ an action of $\mathrm{Sym}^* T_X$

$\simeq$ quasi coherent sheaves on cotangent complex T_X^*

$\simeq$ Higgs sheaves

$\theta : F \longrightarrow F \otimes \Omega^1_X$

$\theta^2 = 0$

Ex : X affine $\mathrm{Sym}^* B$ = ring of functions on B^*

~~Quasi-coherent~~

$QCoh(X_{Dol})^{\heartsuit}$ = Higgs sheaves

$[X_{Dol}, G]$ = G-Higgs bundles

$[X_{Dol}, K(\mathcal{O}_X, n)] \simeq H^n_{Dol}(X)$

Def: For any stack Y

$$H_{Dol}(X;Y) = [X_{Dol}, Y] \qquad \lambda = 0$$

$$H_{DR}(X;Y) = [X_{DR}, Y] \qquad \lambda = 1$$

Q: How are these related?

Assume Y is a <u>higher gerbe</u>

$$H_{Dol}(X;Y) \cong \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \text{dg-cat complexes of Higgs sheaves})$$

$$H_{DR}(X;Y) \cong \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \text{category algebraic D-modules})$$

How are these related

NAHT $\Rightarrow$ big pieces of these dg categories are related

$QCoh(X_{Dol})$ = complexes of Higgs sheaves

$\cup|$

Stable Higgs bundles w/ vanishing chern classes

what does $Map(\mathcal{E},\mathcal{F})_0 = A^*(\underbrace{\underline{Hom}(\mathcal{E},\mathcal{F})}_{\mathcal{E}^* \otimes \mathcal{F}}, \bar{\partial} + \theta)$

Harmonic Bundles

$Map(\mathcal{E},\mathcal{F})_0 = A^*(\overbrace{\underline{Hom}(\mathcal{E},F)}^{\mathcal{E}^* \otimes \mathcal{F}}, \nabla^{\text{flat conn}}_{\text{given diff}})$

semi-simple vector bundles w/ connection

$\cap|$

$QCoh(X_{Del})$ = complexes of D_X-modules

Harmonic bundles

$(\mathcal{E}, D = \partial + \bar{\partial} + \theta + \bar{\theta}, h)$

$D'' = \bar{\partial}, \theta$

Idea for $\mathcal{E}, \mathcal{F}$ in Harmonic Bundles try to define a complex $Map^{har}(\mathcal{E},\mathcal{F})$ which is quasi-iso to both complexes

$\rightsquigarrow$ harmonic forms

$A^*(\mathrm{Hom}(\mathcal{E},\mathcal{F}))$

multiplication of forms does not respect harmonicity

$\ker(D'' : A^*(\underline{\mathrm{Hom}}(\mathcal{E},\mathcal{F})) \longrightarrow A^*\mathrm{Hom}(\mathcal{E},\mathcal{F}))$
$\overset{\shortparallel}{\partial+\theta}$ w/ differential D

* every semisimple v.b admits a harmonic metric
* every polystable Higgs bundle admits a harmonic metric

Extrapolates formally to an equivalence

Complexes of Higgs sheaves

$\cup$

complexes built from polystable Higgs bundles closed ~~sms~~ contains all colimits

equivalent

Smallest full subcategory of s.s. v.b w/ connection closed under ho colimits

$\boxed{Y = \text{higher gerbe}}$

$$H_{DR}(X;Y) \longleftrightarrow H_{Bet}(X;Y)$$

$H_{DR}(X;Y) = \pi_0(\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y),$ complex of Higgs sheaves$))$

$H_{Bet}(X;Y) = \mathrm{iso}\ \underbrace{\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y)}, \text{D mod}))$

we don't know this will factor through poly stable Higgs

factors through the D mod build from S.S. v.b. w/ connection

if $\pi_1(Y)$ is assumed to be unipotent then get a bijection

Schematic Homotopy Types

Goal: X variety (smooth, proj) $/\mathbb{C}$

define $(X \otimes \mathbb{C})^{sch}$ a stack

$\mathbb{C}^*$ action on this stack

Thm It recovers the Hodge structure

(Affine Stacks - Toën)

The schematization problem

- Pursuing Stacks (Toën)

A abelian group object in Ho(Stacks)

BA classifying stack

abelian group stack

Eilenberg Maclane stacks

$K(A, i) = BK(A, i-1)$

$K(A, 0) = A$

Grothendieck: $K(G_m, i)$ are fundamental examples of schematic homotopy types. This are to be stable by homotopy limits

Def: $C \subseteq Ho(k)$ a schematic homotopy category if it contains $K(\mathbb{C}, i)$ and it is stable by holim

$Ho(k) \xrightarrow{R\Gamma} Ho(sSet)$

$R\Gamma|_C : C \longrightarrow Ho(sSet)$

schemetization problem : Find a sht ,
s.t.

- $R\Gamma|_C$ has a left adjoint $- \otimes k$
- $- \otimes k : Ho(sSet) \longrightarrow C$ is fully faithful when restricted rational connected finite type homotopy types.

II <u>Affine stacks</u> k field
Alg_k - category of com k-alg
$Sch_{/k}$ - schemes over k

$Spec : Alg_k \longrightarrow Sch_k$
$A \longmapsto Spec\, A$

$Alg_k \hookrightarrow Alg_k^{\Delta} \longleftarrow$ model structure

$Sch_k \hookrightarrow$ Simplicial $Pr(k)$ over
$A \otimes_k Spec$

Model Structure

Alg_k^{Δ} ← csim algebras

w.eq. $A \to B$ such that

$H^\circ(N(A)) \to H^\circ(N(B))$

is an iso $\forall i \geq 0$

fib $A \to B$ are surjections levelwise

$A_n \to B_n$ is surj

c levelwise

cof LLP w/ r.t. $\overset{\sim}{\twoheadrightarrow}$

Using local model structure on simplicial presheaves

$B \in Alg_k$

$$\mathrm{Spec} : (\underset{A}{\overset{\cup}{Alg_k^{\Delta}}})^{op} \longrightarrow SP_r(k)$$

$\mathrm{Spec}(A)(\mathrm{Spec}\, B) = \underline{Hom}(A, B)$

↑simplicial hom in Alg_k°

$\underline{Hom}(A, B)_n = Hom(A_n, B)$

Fact: Spec is right Quillen

$\mathbb{R}\mathrm{Spec} : Ho(Alg_k^{\Delta}) \to Ho(k)$

Prop: (Toën) $\mathbb{R}\mathrm{Spec}$ fully faithful

Def: $F \in SPr(k)$ is affine if $F \simeq \mathbb{R}Spec(A)$

Examples: $K(\mathbb{G}_a, i) = Spec(S(i))$

$S(i) = Free(S^i_k)$

cosimplicial alg

S^i_k = cosimplicial k-module $0 \to k \to 0$

Prop: All affine stacks are homotopy limits of $K(\mathbb{G}_a, i)$

?

F is affine iff it is subaffine and $\mathcal{O}$-local

$F = h_X$ X simplicial affine submodule

local w/ $\mathcal{O}$-equivalence

$F \in Ho(k)$

there is affination of F

as something universal

mapping into affine stacks

$$\begin{array}{c} F \\ \downarrow \\ (F \otimes k)^{uni} \end{array}$$

$(- \otimes k)^{uni}$ provides an answer to the schematization problem

$(F \otimes k)^{uni} = \mathbb{R}Spec\,\mathcal{O}(F)$ $\mathcal{O}$ is right adjoint to Spec

Replace $\mathbb{Q}$-equivalence by P-equivalences
morphisms which induce iso in cohomology
in $K(A, V, n)$
A affine group scheme
V linear

A schematic homotopy type is pointed connected
stack F s.t.
$B\pi_* F$ is affine
F is local

Schematic Homotopy types
Rmk: Hodge filtration $H^n(X, \mathbb{C})$
$\Leftrightarrow$ $\mathbb{C}^*$ action on $H^n(X, \mathbb{C})$
$y \in H^{p,q}$ $\lambda \in \mathbb{C}^*$ $\lambda(y) = \lambda^p(y)$

X a smooth proj variety / $\mathbb{C}$

$(X \otimes \mathbb{C})^{sch}$ a special stack
(1) $H^*((X \otimes \mathbb{C})^{sch}; \mathbb{G}_a) \cong H^*(X, \mathbb{C})$ X simply connected
(2) $\pi_i(F, *) \cong \pi_i(X, x) \otimes \mathbb{C}$ $i > 1$
(3) $\pi_1(F, *) \cong \pi_1(X, *)^{alg}$

true for any X top space

$\boxed{\mathbb{C}^* \text{ action } \circlearrowright (X \overset{top}{\otimes} \mathbb{C})^{sch}}$

Morgan '78 Mixed Hodge Structure on
$\pi_i(X,*) \otimes \mathbb{C}$ $i > 1$ simply connected

Simpson '93 $\mathbb{C}^*$ action on $\pi_1(X,*)$ pro reductive completion

Thm: $\exists\, \mathbb{C}^* \circlearrowright F$ such that
(1) The induced action on $H^*(F, \mathbb{G}_a) \cong H^*(X, \mathbb{C})$ is the above action

(2) The induced action on $\pi_1(F,x)^{red} \cong \pi_1(X,*)^{red}$

max red quotient of $\pi_1(X,x)^{alg}$

Weight Tower on
$\exists\, F \to F_1 \to \cdots \to F_1 \to F_n = *$
pointed schemes
↳ generalization of weight filtration

Local systems on $\mathbb{P}^1$
"middle convolutions"

I. Local systems on $\mathbb{P}^1 - S$
$S = \{q_1, \ldots, q_n\}$
Betti approach: Rep of $\pi_1(U) \to GL_r$ i.e.
collections of Matrices
$M_1, \ldots, M_n \quad M_1 \ldots M_n = I$
up to some conjugation
DR approach: Meromorphic connections w/ logarithmic singularities
$\mathcal{L}$ = conj classes of local monodromy (basically we fix the local data)

Fix $\mathcal{L}$ consider $M(\mathcal{L}) = \{$local systems w/ local data $\mathcal{L}\}$
Q: When is $M(\mathcal{L})$ non-empty? Deligne-Simpson problem

Q: How can we construct these local systems?
Important special case $\dim M(\mathcal{L}) = *$
- given local data can we calculate the dim of the moduli space?

A local system determined by local data is called rigid. In the rigid case, Katz gave an answer to these questions. Every rigid, irr local system on U could be built from 1-dim local systems.

Where did this come from?

The hypergeometric equation is

$$z(1-z)f'' - [c-(a+b+1)z]f' - abf$$

2d-surface of solutions built from the hypergeometric fun ${}_2F_1(a,b,c;z)$

Local system: Eval of local monodromy

$$\left.\begin{array}{lll} 0: & 1 & e^{2\pi i c} \\ 1: & 1 & e^{-2\pi i(c-a-b)} \\ \infty: & 1\ e^{2\pi i a} & e^{2\pi i b} \end{array}\right\} \text{eigenvalues of local}$$

Thm: The local system ${}_2F_1(a,b,c)$ is rigid and irr and every irreducible rank 2 l.s. on $\mathbb{P}^1 - \{0,1,\infty\}$ is of the form $\underset{\substack{\uparrow \\ \text{rank 1 l.s.}}}{\mathcal{L}} \otimes {}_2F_1(a,b,c)$

for some a,b,c