Principle Bundles

X compact Kähler Manifold
G reductive Lie group

NonAHT: $\pi_1(X) \longrightarrow GL(n, \mathbb{C})$ Higgs bundles X

Harmonic Bundles

We can replace $Gl(n, \mathbb{C})$

$$\{\pi_1(X) \longrightarrow G\} \xleftrightarrow[\text{Corlette}]{\text{Simpson}} \{G\text{-Higgs bundles}\}$$

G-real?
Stability?

I. $\mathcal{G}$-Higgs Bundles
$\mathcal{G}$ complex reductive group
Ex: $GL(n, \mathbb{C})$
Non Ex: ~~nea~~ upper Δ matrices

$GL(n, \mathbb{C})$ - Higgs bundle

$$E \in H^1(X, GL_n(\mathcal{O}_X)) \qquad \phi \in H^0(X, \underset{\substack{\| \\ ad(P_E)}}{End(E)} \otimes \Omega^1_X)$$

$$\updownarrow$$

$$P_E$$

s.t. $\phi^2 = 0$
$[\phi, \phi] = 0$

Def: A G-Higgs bundle is a holomorphic principal G-bundle $P \to X$ and a Higgs field $\Phi \in H^0(X, ad(P) \otimes \Omega^1_X)$, s.t. $[\phi, \phi] = 0$

eg $G = Sl(n, \mathbb{C})$ G-Higgs is:
- holom rank n vector bundle
- E s.t. $\det E \cong \mathcal{O}_X$
- ϕ is traceless

Real forms:

$GL(n, \mathbb{C})$ - $gl_n(\mathbb{C}) = M_n(\mathbb{C})$
$U(n)$ - $u(n)$
$GL(n, \mathbb{R})$ - $gl_n(\mathbb{R})$

We can ~~relate~~ write $gln(\mathbb{C}) = \underbrace{u(n)}_{\text{compact}} \oplus m$
$= \underbrace{gln(\mathbb{R})}_{\text{non-compact}} \oplus m'$

$U(n)^c = GL(n, \mathbb{R})^c$
$= GL(n, \mathbb{C})$

$u(n)$
$gl_n(\mathbb{R}) = O(n) \otimes m$

Def: A split ~~sep~~ real group G is s.t. every (Cartan) decomposition $g = \underbrace{h}_{\text{max compact}} \oplus \underbrace{m}_{\text{non compact}}$ ~~contains~~ m contains a maximal abelian sub algebra

Ex: $Gl(n,\mathbb{R})$

~~only~~

Non-Example: any compact Lie group

G - reductive, real, K maxim compact $\mathfrak{k}\oplus\mathfrak{m}$

Def: A G-Higgs bundle is a holomorphic principal $K^{\mathbb{C}}$-bundle $P\to X$ equipped w/ $\phi\in H^0(X,\Omega\otimes P(\mathfrak{m}^{\mathbb{C}}))$

$$P(\mathfrak{m}^{\mathbb{C}}) = P\times_{\text{isotropy}}\mathfrak{m}^{\mathbb{C}} = P\times_{K^{\mathbb{C}}}\mathfrak{m}^{\mathbb{C}}$$

$[\phi,\phi]=0$

Examples: $GL(n,\mathbb{C})$, $U(n)$

~~associated~~

$\mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n)\oplus\mathfrak{m}$

$\mathfrak{u}(n)^{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C})$

P is holom $GL(n,\mathbb{C})$-bundle

$\phi\in H^0(X,\mathrm{ad}(P)\otimes\Omega^1_X)$

when you complexify a complement in $Gl(n,\mathbb{C})$ ~~get original~~ Lie alg back

• $U(n)$ $\quad \mathfrak{u}(n)^{\mathbb{C}} = \mathfrak{gl}_n(\mathbb{C})$

· A $U(n)$-Higgs bundle is just a rank n holomorphic vector bundle

$Sl(n,\mathbb{R})$ $\quad \mathfrak{sl}_n(\mathbb{R}) = \mathfrak{so}(n)\oplus\mathfrak{m}$

a holom rank n-vector bundle E, $\det E\cong\mathcal{O}_X$

- $\omega\in H^0(X,S^2E^*)$ nondegenerate
- $\phi\in H^0(X,\mathrm{End}(E)\otimes\Omega^1_X)$ traceless and symmetric

Def: A G-Higgs bundle is ~~stable~~ poly stable if the associated Higgs bundle under some faithful representation of G is poly-stable

$\left\{ \text{Supposed we had vanishing generalized chern classes} \right\}$

$G \to H$
to induce a map
G-Higgs bundle to
H-Higgs bundle
choose decomp of H
in a way to match maximal compact

NAHC: Non Abelian Hodge Correspondence

Thm: (Simpson) $\exists$ an equivalence of moduli spaces (homeomorphism)

$\{ \text{Reps (reductive) } \pi_1(X) \to G \} \longleftrightarrow$ polystable G-Higgs bundles w/ vanishing generalized Chern classes

X - Riemann surface
G - connected, ss. reductive Lie group
(E, ϕ) - G-Higgs
$\tau: \Omega^1_X(E(\mathfrak{g}^c)) \longrightarrow \Omega^1_X(E(\mathfrak{g}^c))$
is conjugation on Ω^1_X & $\mathfrak{g}^c$

σ fix a K-structure h

Thm (Hitchin, BGG): $\exists h$ satisfies equation (fix a K-structure)
$\cancel{F_h - [\phi, \phi]}$ $F_h - [\varphi, \tau(\sigma)] = 0$

(get back to here)

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(5) Toledo Invariant: Milnor-Wood
X - Riemann surface
G - Reductive, real max'l K

(E, ϕ) - G-Higgs

$$1 \to \pi_1(K^c) \to \hat{K}^c \to K^c \to 1$$

we get $[X, BK] \to [X, K/\pi_1 K^c)]$

Fact: If G is connected, split real

$$\Rightarrow \pi_1 G = \pi_1 K = \pi_1 K^c = H^2(BK^c, \mathbb{Z})$$
$$= H^2(K(\pi_1 K^c, 2), \mathbb{Z})$$

$\tau(E) \in H^2(X, \mathbb{Z})$ which is pulled back from a class in $H^2(X, \pi_1 G)$

Def: G is Hermitian if G/K is a G-Kähler Manifold (G object in category of Kähler Manifolds)

Def: $\text{rank}(G/K) = $ max'l dim of $T_p G/K$ on which the curvature vanishes

Ⓡ $\text{rank}(G/K) = \text{rank}(G)$ often

Fact: G/K irr $\Rightarrow \pi_1(K^c)/\text{torsion} \cong \mathbb{Z}$

$\Rightarrow$ We can identify $\tau(E)$ w/ an integer and we call it the Toledo Invariant

Prop: G Hermitian w/ G/K irr

then if (E, ϕ) is polystable

$$|\tau(E)| \leq rk(G/K)(g-1)$$

Ex: Max'l reps to $Sl(2,\mathbb{R})$ are the F Reps

Hitchin: For $G = PSL(2,\mathbb{R})$ we can explicitly describe the subspaces of the $M_B(X, G)$ corresponding to each value of $\tau(E)$

Vector bundles over a symmetric

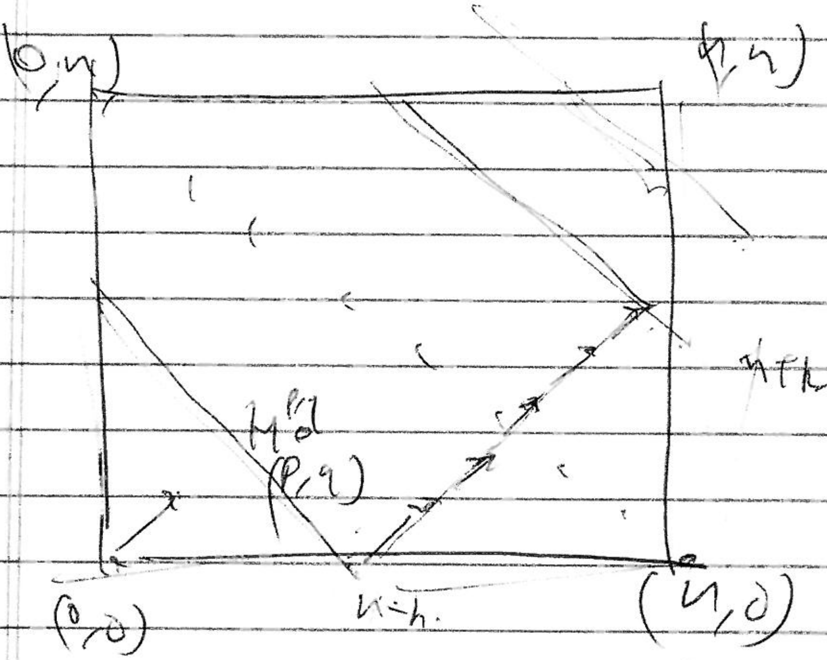
$$\begin{array}{c} V \\ \downarrow \\ \mathrm{Sym}^k \Sigma \end{array}$$

Perverse Filtration

M_{Dol} has a natural hyperKähler metric, an S^1-action by isometries and a projective map $\chi : M_{Dol} \to A$ the Hitchin fibration where the target $A = H^0(C, K_C) \oplus H^0(C, 2K_C)$ and the fibre of χ over a general curve $s \in A$ is an Abelian variety isomorphic to the Jacobian of the spectral curve C_s (branched double cover of C)

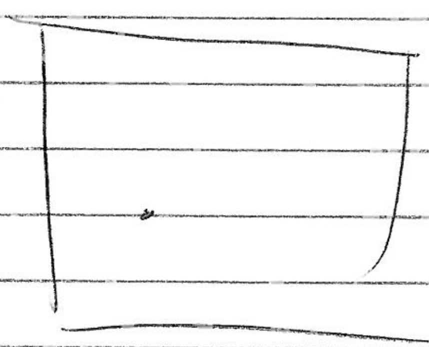
$H^*(M_B)$

$\alpha \in H^2(M_B)$ type $(2,2)$



$\omega \in H^{1,1}$
$= [\text{hyperplane}]$

$$H^{n-k} \xrightarrow[\simeq]{\wedge\, \omega^k} H^{n+k}$$



Simpson

Moduli of Representations:

Non-Abelian Hodge Analogy

$\{\text{Rep } \pi_1 \to G\} = H^1_B(X, G)$ B = Betty

Betty = topological cohomology

$H^i_B(X, \mathbb{C}) = H^i_{sing}(X^{top}; \mathbb{C})$

$\{\text{Higgs bundles}\}/\sim = H^1_{Dol}(X, G)$ Dol = Dolbeault cohomology

~~$H^1_{Dol}(X, \mathbb{C}) = \oplus H^1(X, \Omega$~~

What would $H^1_{DR}(X, G)$ be?

$\left\{ \begin{matrix} \text{flat bundles} \\ (E, \nabla) \end{matrix} \right\}$

algebraic flat bundles

E algebraic vb on X
$\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^1_X$) Zariski topology

Riemann-Hilbert-correspondence $H^1_{DR}(X, G) \cong H^1_B(X, G)$

Today's Theme is Moduli Spaces ~

(taking algebraic powers
then integrating over a
topological cycle.
get some numbers)

periodicity $G/\mathbb{Q}$

If X is defined over a field K

$H^1_B(X, G)$ defined on K

look at $\bar{K}$-points

$H^1_B(X, G)$ depends on $\bar{\mathbb{Q}}$ (or $\mathbb{Q}$)

look at its $\bar{\mathbb{Q}}$-points

eg: if $k \subseteq \bar{\mathbb{Q}}$ we get 2 sets of points

$$H^1_B(X,G)\bar{\mathbb{Q}} \subseteq H^1_B(X,G) \cong H^1_{DR}(X,G) \supseteq H^1_{DR}(X,G)\bar{\mathbb{Q}}$$

<u>Conjecture</u>: The intersection of these two sets of points consists exactly of the motivic points
= monodromy reps of
algebraic ~~varieties of~~ families
of varieties

G_m comes from transcendental number theory

$$\begin{array}{ccc} \mathcal{X} & & X = X_s \\ \downarrow & & \downarrow \\ \mathcal{S} & & s \end{array}$$ smooth family of varieties

we get a family of $H^1_{DR}(X, G)$ -spaces varying in an algebraic way over $s \in \mathcal{S}$

(we also have the Gauss-Manin connection aka isodromic deformation equations)

ex Par VI = this is $SL(2)$ on $\underset{X_s=}{\mathbb{P}^1}$-points

What about the variation of Hodge Structure in this case?

we would like to say that $\underbrace{H^1_{DR}(X_s, G)}_{\text{stack or scheme}}$ has a hodge filtration

so not just a

filtration = a family / $\mathbb{A}^1$ w/ a G_1-equivariance

$$\underbrace{H^1_{Hodge}(X_s, G)}_{\substack{\text{moduli space} \\ \text{of } \lambda\text{-connections}}} \longrightarrow \mathbb{A}^1 \circlearrowleft G_1$$

fiber over $0 = H^1_{Dol}$

$1 = H^1_{DR}$

Hitchin's says $H^1_{Del} \cong H^1_{DR}$ provides 2 complex structures forming part of a family of quaternionic structure (I, J, K)

(preserves a metric $\Rightarrow$ hyperkähler structure)

$\Rightarrow$ twistor space $M \times \mathbb{P}^1 \to \mathbb{P}^1$ complex structure

Deligne's comment : $H^1_{Hodge}\, C = \{\lambda\text{-connections}\}$

$\subset_{open} M \times \mathbb{P}^1$ (can look at M as a weight 1 Hodge structure)

$$\begin{array}{ccc} H^1_{Hodge} = \{\lambda\text{-conn}\} & \subset & M \times \mathbb{P}^1 \\ \mathbb{G}_m \circlearrowleft \downarrow & & \downarrow \circlearrowleft S^1 \\ \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array}$$

"This is the Hodge filtration"

you can use this to interpret the twistor space

you get Griffiths transversality w.r.t. Gauss-Manin

Q: what's the fibre over ∞

A: Hitchin moduli space over

Hitchin Fibration

$$H: Higgs_{\Sigma}(r, d, \omega_{\Sigma}) \longrightarrow B_{\omega_{\Sigma}}$$

Σ = smooth, projective / $\mathbb{C}$

Historically: $H: \underbrace{T^{*}M}$ $\longrightarrow B$

moduli space of stable vector bundles of rank r, deg d

Def: A v.b. E is stable (μ-stable) if for all $F \subseteq E$, $F \neq 0$ $\frac{\deg(F)}{rk(F)} < \frac{\deg(E)}{rk(E)}$ semistable φ = can hold

Def: A stable Higgs bundle: (E, ϕ) is stable (respectively, semi-stable) if the above holds for F preserved by ϕ.

Def: Equivalence on Higgs bundles $(E, \phi) \sim (E', \phi')$ if $\exists$ ϕ, ϕ' invariant filtrations on E, E' by semistable pairs of constant slope w/ isotropic associated graded

(rank 2)?

(if you have 2 extensions of a semistable line bundle they are equivalent in the moduli space)

Fact: There's a course algebraic moduli scheme ~~parameterization~~ parameterizing these equivalence classes.

$\Sigma = \{pt\}$ K vector space (dim 1)

$(E, \phi : E \to E \otimes K)$

Define $H(E, \phi) = \det(I - \phi)$

$$= y^r + \underset{\underset{K}{\cap}}{a_1} y^{r-1} + \dots + \underset{\underset{K^{\otimes r}}{\cap}}{a_r}$$

$H(E, \phi) = (a_1, \dots, a_r) \in \bigoplus_{i=1} K^{\otimes i}$

Σ = smooth projective curve ; K = line bundle

$H : \mathrm{Higgs}_K(r, d, \ldots) \longrightarrow B = \bigoplus_{i=1} H^0(K^{\otimes i})$

$$(E, \phi) \longmapsto \det(I_y - O)$$

expect fibers to be tori
we use "spectro curves"

Spectral Curve: Let $a \in B$ define $p_a(x) = y^r + a_1 y^{r-1} + \dots + a_r$

if we map to a $p_a(x)$
would have to be characteristic polynomial

total space of the line bundle K

$p_a: \mathbb{K} \longrightarrow \mathbb{K}^{\otimes r}$

define spectral curve C_a to be $p_a^{-1}(0)$ - section of $\mathbb{K}^{\otimes r}$

- Generically smooth (for generic point in B smooth)

- $\underbrace{\pi_* \mathcal{O}_{C_a}}_{\text{pushforward of structure sheaf}} = Sym(K^{-1}) / I$

$\cancel{\mathbb{R}}$

$\pi : \mathbb{K} \longrightarrow \Sigma$

relations being of really premimage of structure sheaf

$I =$ ~~genera~~ generated by the image of $u \mapsto (a_1 + \cdots + a_r) \otimes u$

$a_1 \in \mathbb{K} \quad a_r \in \mathbb{K}^{\otimes r}$

$K^{-r} \longrightarrow Sym(K^{-1})$

Prop: (Hitchin)

let $a \in B$, there's a bijection between

$$\left\{ \begin{array}{l} \text{iso classes} \\ \text{of rank 1} \\ \text{torsion-free} \\ \text{sheaves on} \\ C_a \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{iso classes of Higgs} \\ \text{pairs } (E, \phi) \text{ of rank} \\ r \text{ w/ } char(\phi) = p_a(x) \\ \qquad \parallel \\ \qquad H^{-1}(a)^r \end{array} \right.$$

(if C_a is integral) (reduced?)

given $\mathcal{L}$ rank 1 torsion free on C_a, look at $\pi_* \mathcal{L}$ (push forward along bijection)

This is a rank r vector bundle E; with a $\pi_* \mathcal{O}_{C_a}$-alg structure on E $\Rightarrow$
$\phi : E \to K \otimes E$ with $\mathrm{char}(\phi) = p_a$

$\pi_* \mathcal{O}_{C_a} = \mathrm{Sym}(K^{-1}) / I$

$\Rightarrow \pi_* \mathcal{O}_{C_a}$-algebra structure $\Rightarrow$

$$K^{-1} \otimes E \to E$$
$$\Rightarrow \quad E \longmapsto K \otimes E$$

(generic)
fibers of $\mathrm{Higgs}_K(r, d, \Sigma) \to B$ is Jacobian of spectral curve C_a

We can "count dim" of all the objects up to this point

$$\dim M_{Higgs} = 2r^2(g-1) + 2$$

$$\dim \mathrm{Jac}(\tilde{C}) = \dim \bigoplus_{i=1}^{n} H^0 K^i$$

$$= r^2(g-1) + 1$$

The Hitchin map is a maximal independent collection of Poisson commuting holomorphic functions

we will have $r^2(g-1)+1$ functions on the moduli space M_{Higgs} and they all Poisson-commute

- dim depends on degree fixed
bundle is topologically trivial

the generic fibers of this map are isomorphic to abelian varieties

Cor: This is algebraically completely integrable Hamiltonian system.

Thm: This remains true for reductive complex G

Chevalley $K[\mathfrak{h}]^{W \leftarrow S_n} \simeq K[\mathfrak{g}]^G \hookrightarrow k[\mathfrak{g}]$

eg diagonal matrices

degrees of generators are the exponents of $\mathfrak{g}$

$$\mathfrak{h}/W \longleftarrow \mathfrak{g}$$

polynomial map

$H^0(\mathrm{ad}P \otimes K)$

$$\begin{array}{c} \| \\ H^0((P \times_G \mathfrak{g}) \otimes K) \longrightarrow H^0((P \times \mathfrak{h}/W) \otimes K) \\ \qquad\qquad\qquad\qquad\qquad\qquad \| \\ \qquad\qquad\qquad\qquad\qquad\qquad \bigoplus_{i=1}^{M} H^0 K^{d_i} \end{array}$$

← exponents

Ex: $G = Sp_{2m}\mathbb{C}$

- on $Sp_{2m}\mathbb{C}$ - Higgs bundle is

$$(E, \underbrace{\langle\,,\,\rangle}_{\text{skew non-degenerate form}}, \Phi)$$

$\langle \Phi s, t \rangle = \langle s, \Phi t \rangle$

- generically Φ has distinct eigenvalues

$$\langle \Phi v_i, v_j \rangle = -\langle v_i, \Phi v_j \rangle$$

$$\lambda_i \langle v_i, v_j \rangle = -\lambda_j \langle v_i, v_j \rangle$$

Sp_{2m}-M Higgs bundles $\longrightarrow \underbrace{\bigoplus_{i=1}^{m} H^0 K^{2m}}_{\text{smaller}}$

fibers should be subset of Jacobian but remain abelian variety

Note that the form $\langle\,,\,\rangle$ gives an iso $E \simeq E^*$ which changes sign of eigenvalues

$\tilde{C}$ has involution σ multiplies by -1

- the fiber is line bundles L s.t. $\sigma^* L \simeq L^*$

<u>Goal</u>: Construct M_B, M_{Dol}, M
coarse moduli spaces
(want scheme as opposed to stack
for complex analytic structures)

<u>Def</u>: E coherent sheaf on X/G projective
E is a μ-(semi)stable
$\mathrm{slope}(E) = \deg(E)/\mathrm{rk}(E)$
If $\forall F \subset E$ coh. subschemes
then $\mathrm{slope}(F) \leq \mathrm{slope}(E)$

$$\frac{P(F,n)}{\mathrm{rk}(F)} \leq \frac{P(E,n)}{\mathrm{rk}(E)} \qquad P = \text{Hilbert polynomial}$$

$n \gg 0$

E is pure dim d

$$\dim(\mathrm{supp}(F)) = d = \dim(\mathrm{supp}(E))$$

If Y^G is a functor ~~Set~~ Sch $\longrightarrow$ Set
then a scheme Y universally corepresents
Y^G if $Y^G \to Y$ s.t.

$$\begin{tikzcd} V \times_Y Y^G \arrow[r] \arrow[d] & Y^G \arrow[d] \\ V \arrow[r] & Y \end{tikzcd}$$

V corepresents $V \times_Y Y^G$

G ~~reduced alg gr~~ reductive alg group $G \# X$

Have $X^{\#}/G^{\#}$

X scheme

$X^{\#}$ = functor of points

$X^{\#} \longrightarrow Y^{\#}$ is a universal

if Y.

(skipped boundedness of schemes)

GIT says ~~Q~~ $\exists$ a scheme $M(O, P)$ which represents the quotient

same (or similar ideas) give moduli of semi stable χ-modules

χ = sheaf of rings of differentiable operators

χ-module = module / sheaf of rings

Ex: $\Lambda = \mathcal{D}_{X/S}$

I get vector bundles w/ flat connection
$\nabla^2 = 0$

Ex: $\Lambda = \mathrm{Sym}(T(X/S))$
I get Higgs bundles $\phi \wedge \phi = 0$

more on this

Ex: λ-connections
$\Lambda = D_{X \times \mathbb{A}^1 / S \times \mathbb{A}^1}$

deformation of as gr between these two examples

M_B = Betti Moduli Spaces

Q: ~~Ca~~ Are ~~we~~ we talking about Mosp of sheaves or schemes

Γ a finite gen grp

$\Gamma = \langle \gamma_1, \ldots, \gamma_k \mid w \rangle$

$R(\Gamma, n) = \{ \mathrm{Hom}(\Gamma, Gl(n, \mathbb{C})\}$ $\circlearrowleft Gl(n, \mathbb{C})$
by finite conjugation

orbits ~~measure~~ are isom classes of representations

Need $R(\Gamma,n) / Gl(n,\mathbb{C})$

First: $R(\Gamma,n) \subset Gl(n,\mathbb{C}) \longrightarrow Gl(n,\mathbb{C})$

ψ

$(m_1,\ldots,m_k)$ ψ

$r(m_1,\ldots,m_k)=1 \quad \forall r \in W$

So $R(\Gamma,n)$ is a subring

Thm: $\exists$ a map $R(\Gamma,n) \longrightarrow M(\Gamma,n)$

between quotient by $Gl(n,\mathbb{C})$

Pf: $A = K[R(\Gamma,n)] \qquad B = A^{Gl(n,\mathbb{C})}$

$M(\Gamma,n) = \operatorname{Spec} B$

Riemann-Hilbert Correspondence

Sym structure on $M_B(X,G)$ X coarse

Γ = fin. gen. group
G = reductive

B = bilinear form invariant under G action

$M_B(\Gamma, G)$

$\phi \in \mathrm{Hom}(\Gamma, G)$

$$\phi_t(x \cdot y) = \phi_t(x)\phi_t(y)$$

$$\mu(xy) = \mu(x) + \mathrm{Ad}\,\mu(x)\mu(y)$$

$\mu : \Gamma \to \mathfrak{g}$

$$T\phi : M_B(\Gamma, n) = \underbrace{H^1(\Gamma, \mathfrak{g}_{ad})}_{\text{group cohomology?}}$$

[The tangent space is the group cohomology?]

$$H^i(\Gamma, \mathfrak{g}) + H^{2-i}(\Gamma, \mathfrak{g}) \to H^2(\Gamma, \mathbb{R}) = \mathbb{R}$$

$$\upsilon : H^1(\Gamma, \mathfrak{g}) \times H^1(\Gamma, \mathfrak{g}) \to \mathbb{R}$$

υ symplectic structure on M_B

R-H-correspondence

$$M_B(X/S, n) \cong M_{DR}(X/S, n)$$

identification of analytic spaces

locally free rank n

rank n { rep $\pi_1(X)$ in $Gl(n, \mathbb{C})$ } / local systems

flat bundles

$\nabla : E \to E \otimes \Omega^1_X$

$$L \longmapsto L \otimes_{\mathbb{C}} \mathcal{O}_X$$

Perverse Sheaves in 5 minutes or less:

A complex of sheaves K is a collection of sheaves $\{K^i\}_{i\in\mathbb{Z}}$ and maps $d^i : K^i \to K^{i+1}$ subjects $d^2=0$. The i^{th} cohomology sheaf $\mathcal{H}^i(K)$ is $\ker d^i / \operatorname{im} d^{i-1}$

Perverse sheaves live on spaces w/ singularities
- limit ourselves to v.b on complex alg varieties
- to further ignore pathological examples
  D_X = the category of bounded constructible complexes of Sheaves which sits in the derived cat and is closed under ~~pushouts/pullbacks~~, pushforward/pullback/ cup and cap products

$K \in D_X$ only finitely many $\mathcal{H}^i(K) \neq 0$ and for every i the set $\operatorname{supp} \mathcal{H}^i(K)$ the closure of the set of pts at which the stalk is non-zero is an alg subvariety

$*$ We have the <u>perverse filtration</u>

$${}^{p}S_i^{p} H^j(Y, K) = \operatorname{Im}\{H^j(Y, {}^{p}\tau_{\le i} K) \hookrightarrow H^j(Y, K)\}$$

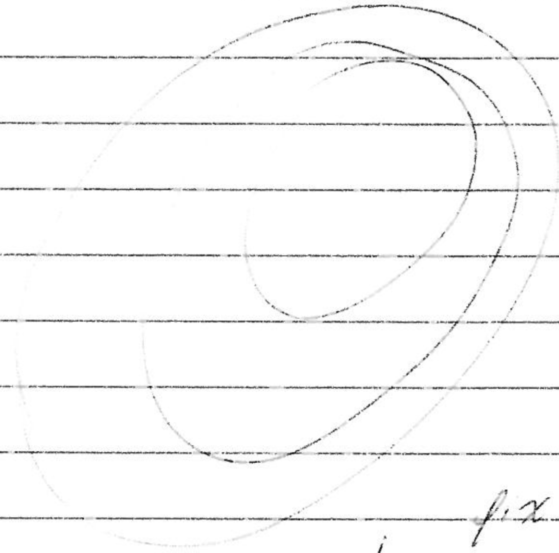
Let $f: X \to Y$ be a map of algebraic varieties and let $C \in D_X$

<u>Def</u>: The perverse Leray filtration ${}^{p}\mathcal{L}_i$ on $H^j(X, C)$ is the perverse filtration on $H^j(Y, f_* C) = H^j(X, C)$

Thm: Let $f: X \to Y$ be a map of algebraic varieties w/ Y affine of dim n. Then $\exists$ a filtration Y_* by closed subvarieties Y_i of dim i s.t. if we take $X_* = f^{-1} Y_*$

$$\mathcal{L}^{p}_i H^j(X, \mathbb{Z}) := {}^{p}S_i H^j(Y, f_* \mathbb{Z}_X)$$

$$= \operatorname{Ker}\{H^j(X, \mathbb{Z}) \to H^j(X_{n+j-i-1}, \mathbb{Z})\}$$



no codim 1
stratum

fix a stratification
by using a S'action

~~the~~ the generalization of transversality
is support

$M_D \xrightarrow{f} Y$

$Rf_* \mathbb{C}$ Complex of sheaves

i^{th} graded piece of $\mathbb{R}f_*\mathbb{C}$
is H^i ⇒ ~~complex~~
⇒ is a sheaf

another t-structure ${}^pH^i$ perverse thing
complex of sheave

inter

$$E_2^{p,q} = \mathbb{H}^p({}^pH^q \mathbb{R}f_* \mathbb{C})$$

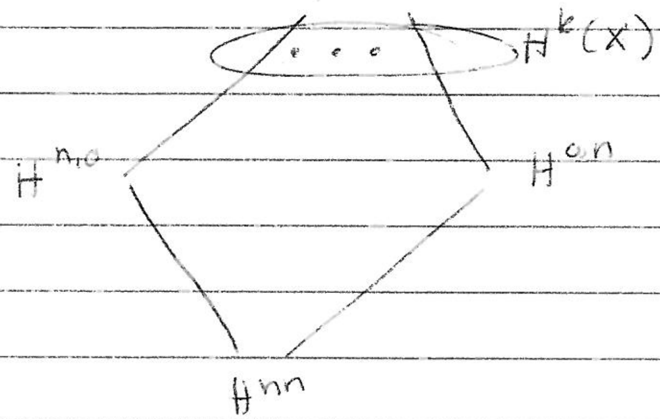
$$\mathbb{H}^p(Y, {}^pH^q Rf_* \mathbb{C})$$

Hodge Theory:

① A pure weight k Hodge structure consists of a fin. gen. $\mathbb{Z}$-mod $V_{\mathbb{Z}}$

• filtration $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{C}$: $F^i \supset F^{i+1} \supset \dots \supset 0$

st $V_{\mathbb{C}} = F^p \oplus F^{k+1-p}$ $\forall p$



cohomology of Kähler manifold

Equiv $H^{p,q} = F^p \cap \overline{F^{k-p}}$ $q+p=k$

$V_{\mathbb{C}} = \bigoplus H^{p,q}$ $F^p = \bigoplus_{p' \geq p} H^{p',q}$

Also: Pol

inner form

(X,h) is Kähler if $d\omega = 0$

equiv: $h = I_d + O(2)$

Let $L = (- \wedge \omega)$; $\Lambda = L^{op}$

operators 5

② Kähler identities

$$\partial^{op} = i[\Lambda, \bar\partial]$$

$$\bar\partial^{op} = -i[\Lambda, \partial]$$

2 key consequences

Mixed Hodge Structure (MHS) is a tripple
(H, W, F)

(1) H is a finite type $\mathbb{Z}$-mod

(2) W is an increasing filtration on $H \otimes_{\mathbb{Z}} \mathbb{Q}$ (~~Hodge~~ weight filtration)

(3) F is a decreasing filtration on $H_{\mathbb{C}} = H \otimes \mathbb{C}$ s.t. $Gr_F^p G_F^q (Gr_n^W(H_{\mathbb{C}})) = 0$ if $p+q=n$

Observation: Subobjects or quotients of filtered objects are filtered

If A is a ~~filtration~~ filtered $F^k(A/B) = \frac{F^k(A)}{B \cap F^k}$

Ex: Split MHS: $H = \bigoplus H_k$

↖ pure Hodge structure of weight k

$\Leftrightarrow$ W is a grading of $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$

Q: Why this def?
Q: Can we make this more geometric?
(Reese construction)
Q: Categorical Observations?
[Tannaka duality]

A: MHS appear in nature — in cohomology of $\mathbb{C}$ varieties X
non-compact → key pt
non-singular

$H^\circ(X;\mathbb{Z}) = ?$

Idea: Compactify $X \rightsquigarrow \exists \bar{X}$ compact variety containing X s.t.
$Y = \bar{X}/X$ is a normal crossing divisor

$Y \hookrightarrow \bar{X}$ locally $\{z_1 \cdots z_k = 0\} \hookrightarrow \{(z_1, \dots, z_k)\}$
"$\mathbb{C}^k$"

Fact $H^\cdot(X, \mathbb{C}) \cong \mathbb{H}^\cdot(\bar{X}, \Omega^\cdot_{\bar{X}}\langle Y\rangle)$
logarithmic dR cohomology
$\Omega^\cdot_{\bar{X}}\langle Y\rangle$ locally free sheaf generated by ~~automorphic~~ forms on X
$\Omega_X \wedge dz_i/z_i$

$\Omega^\bullet_{\bar{X}}\langle Y\rangle$ naturally has two filtrations

$$W_n(\Omega^\bullet_{\bar{X}}\langle Y\rangle) = \left(\begin{array}{l}\text{forms w/} \le n \\ \text{occurrences of } dz_i/z_i\end{array}\right)$$

$$F^p(\Omega_{\bar{X}}\langle Y\rangle) = \{k\text{-forms } \omega \ \ k \ge p\}$$

Hyper cohomology w/ coeff w/ 2 filtrations $\rightsquigarrow$ 2 spectral sequences

s.s.

$${}_F E_1^{p,q} = H^q(\bar{X}, \Omega^p_{\bar{X}}\langle Y\rangle) \Rightarrow H^{p+q}(X;\mathbb{C})$$

$${}_W E_2^{p,q} = H^p(\bar{X}, R^q j_* \mathbb{C}) \Rightarrow H^{p+q}(X;\mathbb{C})$$

$j : X \hookrightarrow \bar{X}$

<u>Thm</u>: $(H^n(X,\mathbb{Z}), W[n], F^\bullet)$ is a MHS
(Deligne)

a filtration on $H^n(X,\mathbb{Q})$
(this filtration is topological
comes from L.S.S)

independent of choice of $\bar{X}$ and functorial