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- Introduction
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Local Systems are Representations of π_1

X algebraic varieties / $\mathbb{C}$

topological space associated to X (assume connected)

X^{top}

Ex: $X \subseteq \mathbb{P}^n$ $\quad X^{top} \subseteq \mathbb{CP}^n$ same ~~space~~ subset
but w/ analytic topology

we'll basically be talking about local systems on X^{top} (may drop top superscript)

local system $:=$ representations $\rho : \pi_1(X^{top}, x) \to GL(n,\mathbb{C})$

What is the idea behind local systems?

This basic theme is common in math

Galois Theory $:=$ Local systems of finite sets over Spec(k); k = field (Grothendieck's interpret)

Riemann surfaces: local systems of sets over $V \subseteq \mathbb{C}$

Liouville, etc $\to$ systems of O.D.E's
Riemann - Hilbert Correspondence

Riemann-Hilbert - what monodromy systems can you get

Picard-Lefschetz theory - start of Hodge theory

$$\begin{array}{c} X \\ \downarrow \\ Y \end{array} \quad \text{you get a local system } L \text{ on } Y \qquad L_y = H^i(X_y, \mathbb{C})$$

It turns out that these things have a lot of interesting structure.

<u>Topological world</u>   complex analysis   <u>Algebraic World</u>

a local system $\longleftarrow$ an algebraic map

(just a variation on $X \longmapsto X^{top}$)

if we have a local system

$L \longmapsto$ (complex analytic vector bundle w/ connection) (E, ∇) $L = E^{\nabla}$ GAGA $\Rightarrow$ algebraic

$E^a = L \otimes_{\mathbb{C}} \mathcal{O}$ $\nabla =$

considered as a sheaf

Question: Explain more how we go from alg map to local system?

$$\begin{array}{c} X \\ f\big\downarrow \\ Y \end{array}$$

f smooth, projective

$$L = R^i f_* \mathbb{C}$$

if not smooth/projective get something a bit more complicated

constructible sheaves

Weil, Grothendieck, Deligne $\rightsquigarrow$ do "local systems"
étale topology, number theory, geometry / $\mathbb{F}_p$
This motivated a lot of stuff : Deligne motivated
by Hodge theory

The study of situations like
$$\begin{array}{c} X \\ \downarrow f \\ Y \end{array}$$

led to Griffith's theory of variations of Hodge theory
(VHS = variation Hodge stru.)
we get extra structure (E, ∇) vector bundle w/
connection associated to one of our topological ~~systems~~
local systems. $L = Rf^i_* \mathbb{C}$

Hodge filtration : $H^i(X, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X_y)$

$F^p = \sum_{p' \geq p} H^{p,q} \subseteq E$ holomorphic subbundle

$H^{p,q} \subseteq E$ are just C^∞ subbundles
+ flat hermitian form $\langle , \rangle$ ...
+ more

New in the 1980's : Local systems can
vary

$$\mathrm{rep}(\pi_1)/_{\sim} = \{\rho : \pi_1(X^{top}, x) \to Gl(n,\mathbb{C})\} = \underset{\substack{\text{subset} \\ \text{defined by} \\ \text{relations}}}{\odot} \subseteq GL(n,\mathbb{C})^{\overset{\#\text{ of generators}}{a}} \Big/ \mathrm{conj}$$

Donaldson's new proof of a theorem of Narasimham-Seshardi (spelling?)

1960's : {Unitary Reps} = {Stable bundles}

~~Hitch~~

Topologists using the ~~character variety~~ $\mathrm{Rep}(\pi_1)/\sim = \chi(\pi_1)$

to study Smith conjecture

Masur-Wiels: deformations of $\mathbb{Q}_p$-local systems

generalized Shau : rank 1 case

~~How does that interact~~ struc

How does a ~~the~~ structure of a varying local system interact w/ the algebraic structure of the variety?

Hitchin: generalize N-S ~~conjecture~~ theorem to complex groups
(Corlette & Donaldson)

$$\left\{\begin{matrix}\text{Rep} \\ \rho : \pi_1(X,x) \to Gl(n,\mathbb{C})\end{matrix}\right\} \longmapsto \left[\begin{array}{l}\text{Higgs bundles} \\ (E,\phi) \\ E \text{ a real line bundle} \\ \phi : E \to E \otimes \Omega^1_X \\ \phi \wedge \phi : E \to E \otimes \Omega^2_X \text{ is } = 0 \\ \text{semistable} \\ c_i = 0 \text{ in } H^{2i}(X, \mathbb{Q})\end{array}\right.$$

N.S. case:

$$\rho : \pi_1 \to U(n) \longmapsto (E, \phi = 0)$$

How does this ~~[illegible]~~ become non-Abelian Hodge Theory?

Nonabelian interpretation of this correspondence

$\{\rho : \pi_1 \to Gl(n, \mathbb{C})\} = H^1(X^{top}; Gl(n, \mathbb{C}))$

non-abelian

local system $\leftrightarrow$ cocycle ($g_{ij} \in GL(n, \mathbb{C})$ for $U_i \cap U_j \neq \emptyset$ we assume $U_i \cap U_j$ constractible

$g_{jk} \cdot g_{ij} = g_{ik}$ if $U_i \cap U_j \cap U_k = \emptyset$)

assume $U_i \cap U_j$ is either empty or ~~non~~ contractible so our cocycles ae w/ the non-empty intersections

Observation: an (E, ϕ) can be viewed as a nonabelian Dolbeault cohomology

Recall $H^1_{Dol}(X) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X)$

$= H^1(A^{\bullet\bullet}(X), \bar{\partial})$

~~$A^{0,1}$ $A^{0,0}$~~

(E, ϕ)

E a vector bundle

$\in H^1(X, GL(n, \mathbb{C}))$

ϕ: a relation $H^\circ(\underbrace{\mathrm{End}(E) \otimes \Omega'_X})$

twisted form $\mathfrak{g}$
of the Lie algebra
$\mathfrak{gl}(n, \otimes \Omega'_X)$

<u>Q</u>: Smith conjecture
if you rotate S^1 around the knot its the unknot ?

Flat to Harmonic Bundles

Corlette - Flat G-bundles w/ canonical metrics

<u>Motivating Picture</u> (Kempf + Ness:)

G semi-simple alg group / $\mathbb{C}$.

V cplx vector spac w/ hermitian form h

$G \curvearrowright V$ linearly

Define an orbit of G to be stable if it is closed and of maximal dim amongst closed orbits

<u>observation</u> Gv is stable iff Gv preserves a shortest vector

(h comes in only at the last moment?)

<u>GIT</u> $K \subset G$ K a stabalizer of h

K compact

$\exists$ moment map for the K-action

$\mu : V \to K^*$

<u>Thm</u>: $\mu(v) = 0$ iff v is the shortest vector in Gv

Carlette's set up: M a compact Riemann manifold

G as before

$K \subset G$ max

$$P \supset SL(n, \mathbb{C}) \qquad K = SU(n)$$

$\downarrow$ principal G-bundle

M w/ fixed reduction of the structure group to K

$\mathcal{C}^{\infty}$ = smooth $SL(n, \mathbb{C})$-connections on P

$$\Omega^1(P) \otimes \mathfrak{g} \underset{\text{affine}}{\cong} \Omega^1(M, ad(P)) = \mathfrak{g} \underset{G}{\times} P$$

$\exists\, \mathcal{L}^2$ metric on $\mathcal{C}^{\infty}$ from integrating $g^* \otimes h$

g^* on $T^*(M)$

h on $ad(P)$

$\mathcal{G}$ = group of gauge transformations of P

$$Aut(P) \cong \Gamma(G \times_{Ad} P)$$

analog of $G \curvearrowright V$

$\mathcal{U}^{K}$ = K-trans gauge transformations preserving the hamil metrics on P (or K-structure on P)

Carlette: $\exists$ moment map for the action of $\mathcal{U} \curvearrowright \mathcal{C}$

$$\Phi : \mathcal{C} \to \Gamma^*$$

lie algebra of $\mathcal{U}$ } dual? in what sense

~~= skew alg~~ skew adjoint sections of $ad(P)$

(linearization of sections that preserve the K structure)

slight digression : If D is a connection on P
~~[illegible]~~ can split $D = D^+ + \Theta$
~~[illegible]~~

preserves K structure (the skew adjoint part)

self adjoint part of D

means adjoint of D^+

$\Phi_D := D^{+,*}\Theta$ Θ is a 1-form so Φ_D will be a 0-form

Thm : $d\|\Theta_D\|^2_{L^2}(\xi) = 2i\langle \Phi_D, \xi \rangle$

this is restricted to the G-orbit of D

i.e. fix a G orbit

Thm (Corlette) D is stable, flat then $\exists!$ K-orbit on G-orbit of D on which $\Phi_D = 0$

Equivalently : $\exists!$ hermitian metric on P for which Φ_D vanishes

Corlette : D stable $\Leftrightarrow$ holonomy of D is not entirely contained in ~~[illegible]~~ a parabolic subgroup of ~~[illegible]~~ G.

(in easy cases just irreducible representation)

In the second formulation: D flat on P
K-structures on P

$\hat{M} \xrightarrow{P} M$ universal cover

$\Gamma(G/K \times_G P) = Q$

$p^*Q \leftarrow$ trivial bundle because it preserves a flat connection comming from D
$\downarrow$
$\hat{M}$

$f :=$ section $\hat{M} \longrightarrow G/K$ (fibred structure ~~on G/K from holonomy~~ on Q)
Riemannian

f descends to define a harmonic metric if
$f : \hat{M} \longrightarrow G/K$ is $\pi_1 M$ equivariant
Furthermore, $\Phi_D = 0$ actually $\Rightarrow$ Φ is harmonic as Riem-manifolds

Method of developing correspondence
solve $A_t \leftarrow$ family of connections

$$\frac{dA_t}{dt} = -D_A \Phi_A$$

using other formulations

$$\boxed{\frac{dg}{dt} g^{-1} = \Phi_g \quad \text{; } g \text{ a gauge trans}}$$

the derivative moved back to the Lie algebra

A note on $\left\{ \begin{array}{l} \frac{dA_t}{dt} = -D_A \Phi_A \\ \frac{dg}{dt} g^{-1} = \Phi_g \end{array} \right\}$

~~What the exact~~

Eells - Sampson -

$f: M \longrightarrow N$ riemannian

$\downarrow$

negative curvature

~~heat flow~~

f + heat equation + negative curvature

G/K always has ~~neg curvature~~ $\underbrace{\text{non-positive curvature}}$ - sectional

— energy minimizing = same as harmonic

energy = $\mathcal{L}^2$ norm of ~~Df~~

Bochner Formula (where does this go?)

$\mu: M \to \mathbb{R}$ $\Delta \frac{1}{2} |\nabla \mu|^2 + Ric(\nabla\mu, \nabla\mu)$

$=$

$f: N \longrightarrow M$ $\sum_{\alpha\beta\partial\sigma} R \, \xi^{\alpha\beta} \xi^{\partial\sigma} > 0$

$\downarrow$

strongly negative curvature

Heat flow in Gauge group $\frac{dg}{dt} g^{-1} = \Phi g$

Q1 converging to element in the orbit?

$\frac{dA_t}{dt} = -D_A \Phi_A$ as $t \to \infty$ what does it converge to

Q: How functorial ~~f~~ is this K-reduction / Harmonic structure?
Φ, D are induced $\rightsquigarrow$ so should follow?

Harmonic is a triple $(E, \Phi, -)$
bundle
flat connection
metric

Spelling
Bochner formula tells you harmonic $\Rightarrow$ pulri harmonic
$\Rightarrow$
when you restrict complex subvariety

Higgs to Harmonic

Yang-Mills equations come from physics

$$\begin{array}{c} P \\ \downarrow \\ M^4, g \end{array}$$ principle bundle

~~SU(n) 2 2V associated~~

Connections A $\qquad$ $M \subset SU(n)$

curvature $F_A = dA + A \wedge A$

YM Functional - can write in 2 ways

$$\|F_A^2\|_{L^2} = \int_{M^4} |F_A|^2 dvol$$

$$= \underbrace{-\int_M \mathrm{Tr}\, F_A \wedge * F_A}_{\text{more convient}}$$

$A \longrightarrow A + \delta A$

$\delta F_A = d\delta A + \delta A \wedge A + A \wedge \delta A$

$\quad = d_A \delta A$

$\delta_1 F_A \wedge * F_A = \delta F \wedge * F + F \wedge * \delta F$

$\quad = 2 d_A \delta A \wedge * F$

$$\delta(YM) = 2\int \mathrm{Tr}\, d_A \delta A \wedge * F$$

$$= 2\int \mathrm{Tr}\, \delta A \wedge d_A * F_A$$

<u>YM</u>: $d_A * F_A = 0$

$* : \Lambda^2 \longrightarrow \Lambda^2$

$*^2 = 1$ eigenvalues ± 1

$F = F^+ + F^-$

$F = *F \quad d_A * F_A = d_A F_A = 0$ <u>Self Dual</u>

$F = -*F \quad \partial_A * F_A = -d_A F_{A^2} = 0$ <u>Anti Self Dual</u>

X a complex surface. Suppose $\begin{matrix} P \\ \downarrow \\ X \end{matrix}$ SU(2)-bundle hermitian

E the associated bundle to P

$\begin{matrix} E = \mathbb{C}^2 \times P \\ \downarrow \\ X \end{matrix}$, h = hermitian metric

<u>Fact</u> If E is a smooth vector bundle w/ hermitian form. $\begin{matrix} E \\ \downarrow \\ X \end{matrix} \leftarrow$ hol manifld

$$\bar\partial_E : \Omega^{1,0}(E) \longrightarrow \Omega^{0,1}(E)$$

furthermore: $\bar\partial_E(fx) = (\bar\partial f)v + f \cdot \bar\partial_E v$ & $\bar\partial \circ \bar\partial = 0$

Then $\exists$ holomorphic structure on E s.t.
∂_E is the $\bar{\partial}$ operator on $\Omega(E)$

Any hermitian, hol, v.B admits a unique hermitian connection D such that $D'' = \bar{\partial}_E$

$$\begin{array}{c}(E, h, \bar{\partial}_E)\\ \downarrow \\ X\end{array}$$

- point: there is a 1-1 correspondence

- the curvature of such a connection is of (1,1)-type
- (E,h) smooth herm v.B. and D is hermit connection s.t. F_D is a (1,1)-type then $\exists !$ hol structure of F making D the hermition connection of $(E, h, \bar{\partial}_E)$

Hermitian-Einstein Connections / Motives

$S(U(2))$ - connection $\tilde{D}$

$$\begin{array}{c}P\\ \downarrow \\ X\end{array}$$

P principal $S(U(2))$-bundle
$X \leftarrow$ compact Kahler surface

$E :=$ ass Her bundle
$h :=$ herm metric on E

$D =$ connection induced on E by $\tilde{D}$

Prop: If $\tilde{D}$ is anti self dual then
1] F_D is of (1,1) type
2] $F_D \wedge \omega = 0$

$\}$ is Hermitian-Einstein condition

(Def of h being Hermitian-Einstein)

Def: E hol VB h = herm metric
$\downarrow$
X Kähler surface

$$F_h \wedge \omega = \mu \cdot \omega^2 \cdot \frac{rk E}{2} \cdot Id$$

$$\left(F_h \wedge * \omega = c \cdot \omega \cdot Id \right)$$

ex: X Kähler $TX = E$; h hermitian, hol, struct
then H-E is the condition
$Ric = \mu \cdot g$

$$\left(\mu = \frac{1}{Vol(X)} \frac{\deg E}{rk E}, \quad \deg(E) = \int_X c(E) \wedge \omega \right)$$

Main Thm
Any stable holomorphic V.B / X
where X is compact Kähler admits a unique
H-E connection

Stable means "slope stable"
$L \subseteq E \quad \deg L < \frac{1}{2} \deg E$

SD = Self Dual = $F_A = \sum_i F_{ij}\, dx^i \wedge dx^j$

We can look for solutions to the self duality problem

- Also look for solutions which are invariant (variables dont depend on 2 coordinates)

$d + A =$

$$A = \sum A_i + dx^i = A_1 dx^1 + A_2 dx^2 + \phi_1 dx^3 + \phi dx^4$$

$F_{12} = F_{34} \quad F_{13} = F_{42} \quad F_{14} = F_{23}$

$F_{ij} = [\partial_i + A_i$

rewrite these curvature equations in terms of Higgs fields $\phi_1 \phi_2$

~~$D_i = \partial_i + A_i$~~

$$\phi = \phi_1 + i\phi_2$$

$F = \frac{1}{2}[\phi, \phi^*]$

Self duality equations

(1) $F_A = -[\Phi, \Phi^*]$

$A = A_1 dx^1 + A_2 dx^2$

(2) $D_A'' \Phi = 0$

Φ is a 1-form

If you have a solution to the S.D. equations ~~(A,Φ)~~

$(A, \Phi) \rightsquigarrow$ holomorphic structure on E

$\Phi \in \Omega^{1,0}(\mathrm{End}(E))$

Ex: $\Phi = 0$. Flat connection

Ex: $K = K^{1/2} \oplus K^{1/2}$ $\quad V = K^{1/2} \oplus K^{-1/2}$

tautological Higgs

$$\begin{array}{cc} & K^{1/2} \\ K^{1/2} & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{array} = \Phi$$

an element of $\mathrm{Hom}(K, K)$

$F = -2\underbrace{h\, dz\, d\bar{z}}_{\omega}$

in pervious theorem $D^+ + \bar{\Phi}$

$D = D_A + \phi + \bar{\Phi}^*$

$D^2 = F_A + (\cancel{\phi^2} + \phi\phi^* + \phi^*\phi + \cancel{\phi^*\phi^*})$

$= 0$ if self duality equations are satisfied

$\boxed{\text{Every Flat connection can be produced from a solution to the self duality equations}}$

~~V a~~ $\in \Omega^{1,0}(End(V))$

$(V, \bar{\Phi})$ is there an A which satisfies self duality?

Yes

Start w/ Higgs $(V, \bar{\Phi})$ $\rightsquigarrow$ hermitian h
$\rightsquigarrow$ A connection
$\rightsquigarrow$ run through self dual
$\rightsquigarrow$ which is equivalent to harmonic

$p : Y \longrightarrow X$ a local epi of presheaves

$B\mathcal{G}(p)$

$\mathcal{G}(p) \xrightarrow{\quad k \quad}$ Cem Spectra

A_2 - cat

$X \to Y$

$\pi_0 X \longrightarrow \pi_0 Y$

$\pi_n(X, x) \longrightarrow \pi_n(Y, y)$

is of groups

$$\boxed{\mathrm{Map}^h(X, Y) = \mathrm{Hom}(X^c, Y^f)}$$

$$\begin{tikzcd}
 & X \arrow[r, "f"] \arrow[d, Rightarrow] & Y \arrow[d, Rightarrow] \arrow[dr] & \\
X^c \arrow[ur] \arrow[dr] & & & Y^f \\
 & X \arrow[r, "f"] & Y \arrow[ur] &
\end{tikzcd}$$

$$\mathrm{Map}^h(X, Y)$$

$\mathrm{Hom}_{Ho(\mathcal{C})}(X, Y)$

$$A \amalg A \xrightarrow{i_0 + i_1} A \wedge I \xrightarrow{1} A$$

$$X \longrightarrow X \times X \longrightarrow$$

$$X \xrightarrow{\sim} X^I \overset{p}{\twoheadrightarrow} X \times X$$

II Homotopy Theory of Simplicial Sets / Homotopy Theory of Topological Spaces

A. A simplicial set is a functor $\Delta^{op} \to Set$

X_n is called the n-simplices of X

A simplicial map $f: X \to Y$ is a natural transformation

(1) Ex: Suppose that a category $\mathcal{C}$ is small in the sense that morphisms $Mor(\mathcal{C})$ and objects $Ob(\mathcal{C})$ are sets.

If $\mathcal{C}$ is a small category there is a simplicial set $B\mathcal{C}$.

$$B\mathcal{C}_n = \underbrace{hom(n, \mathcal{C})}_{\text{functors } [n] \to \mathcal{C}}$$

(simplicial structure comes from precomposition)

$B\mathcal{C}$ = classifying space or nerve

(Ex 2) Suppose I is a small cat
$X : I \to \mathrm{Set}$ is a set valued funct

The category of elements (or slice cat or translation category)

$$*/X = E_I(X)$$

associated to X has as objects all pairs (i, x) w/ $x \in X(i)$ ~~or equivalently~~

(just functions $* \xrightarrow{x} X(i)$)

$$\begin{tikzcd} * \arrow[r, "x"] \arrow[dr, "y"'] & X(i) \arrow[d, "\alpha_*"] \\ & X(j) \end{tikzcd} \quad \Big\} \begin{array}{l} \text{morphism} \\ \alpha : (i,x) \to (j,y) \end{array}$$

$B(E_I X)$ is called the homotopy colimit for the functor X ~~holim~~

$$\underrightarrow{\mathrm{holim}}_I X = B(E_I X)$$

0-simplices colim ?

B. Axioms

(technically closed model cat)

MC1: Finite limits and colimits exist &

less important

MC2: Weak equiv satisfy 2-out-of-3

MC3: If f is a retract of g

probably irrelevant

Ex: In Mod_R X is a retract of Y iff $\exists$ a module Z such that $Y \underset{iso}{\cong} X \oplus Z$

Ex: If $f, g \in Mor(\mathcal{C})$ f is a retract of g if the object of $Mor(\mathcal{C})$ represented by f is a retract of the object of $Mor(\mathcal{C})$ represented by g

in pictures

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Y \arrow[r, "r"] \arrow[d, "g"] & X \arrow[d, "f"] \\ X' \arrow[r, "i'"] & Y' \arrow[r, "r'"] & X' \end{tikzcd}$$

ri and $r'i'$ are identity maps

important

MC4:
$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed, "h"] & D \end{tikzcd}$$
liftings

MC5: f can be factored

Simplicial Presheaves

I. Model Categories and Simplicial Sets.

A. Def: A model cat is a category w/ 3 distinguished classes of morphisms

(1) Why have a model structure?

we want to localize a category $\mathcal{C}$ at its weak equivalences $\rightsquigarrow$ we have a class of maps we want to behave like isomorphisms. Problem: If we just invert weak equivalences we have no idea what $\mathcal{C}$ will be (it might not even be a category). The model structure allows us to say what $Ho(\mathcal{C})$ is

$Ho(\mathcal{C})$ has same objects as $\mathcal{C}$

$$Hom_{Ho\mathcal{C}}(X, Y) = Hom_{\mathcal{C}}(RQX, RQY)/\sim$$

where $\sim$ is homotopy equivalence.

Ex 2 Expanded

Special Case:

$$BI = \underrightarrow{holim}_I *$$

$\exists$ functor $E_I X \longrightarrow I$

$$(i, x) \longmapsto i$$

this induces a ~~fun~~ simplicial set map

$$B(E_I X) = holim_I X \longrightarrow BI$$

$$(holim_I X)_n = B(E_I X)_n = \coprod_{i_0 \to \cdots \to i_n} X(i_0)$$