Chris Elliott

TALBOT NOTES 3

Saul Glasman — Crazy Things You Can Put Mixed Hodge Structures on — Lecture 19

Definition:

The homotopy Lie algebra $\mathfrak{g}_\cdot(X)$ of a space X

- If X is not nilpotent $\mathfrak{g}(X) = 0$
- If X is nilpotent (i.e. $\pi_1(X)$ is nilpotent & acts unipotently on higher homotopy) Then

$\mathfrak{g}_1(X) = \mathfrak{g}_0(\pi_1(X))$: Malcev Lie algebra of $\pi_1(X)$

$\mathfrak{g}_k(X) = \pi_k(X; \mathbb{Q})$ for $k \geq 2$

The bracket is the Whitehead bracket: $S^k \times S^\ell$ has cell decomposition with 1 0-cell, 1 k-cell, 1 ℓ-cell & 1 $k+\ell$-cell with attaching map $S^{k+\ell-1} \to S^k \vee S^\ell$.

So if $\alpha \in \pi_k(X)$, $\beta \in \pi_\ell(X)$, $[\alpha, \beta] \in \pi_{k+\ell-1}(X)$ by this map.

Theorem:

The homotopy Lie algebra (of a smooth complex variety) has a mixed Hodge structure.

In fact Hain has shown that almost everything in the universe has a MHS.

Theorem

- $H^\bullet(\Omega X)$ has a MHS, for ΩX based loop space
- Consider $X \xrightarrow{f} Y$. Take a cofibrant replacement

$$\begin{tikzcd} & E_f \arrow[d] & \\ X \arrow[r, hook] & X' \arrow[r, two heads, "f'"] & Y \end{tikzcd}$$

& take homotopy fibre.

If 1) E_f is connected, & 2) Y is nilpotent

Then $H^\bullet(E_f)$ has a MHS

- If X as before, $\tilde{X} \to X$ connected normal cover with Galois group acting unipotently. Then $H^\bullet(\tilde{X})$ has a MHS.

Main Idea:

If we have a nice model for the rational de Rham complex $A^\bullet(X;\mathbb{Q})$ of X, then under favourable conditions one can construct algebraically nice models for $A^\bullet(\Omega X;\mathbb{Q})$ or $A^\bullet(E_f;\mathbb{Q})$ etc.

If one can produce an MHS on $A^\bullet(X;\mathbb{Q})$, & our constructions preserve MHSs then we can pass them over.

Definition:

many details supressed

If X is a space, we define a ~~n~~ de Rham mixed Hodge complex for X to be a "mixed Hodge complex" $C^\bullet$, with a quasi isomorphism to $A^\bullet(X;\mathbb{Q})$.

Construction

If $C^\bullet$ is such a MHC, & $M^\bullet, N^\bullet$ are "$C^\bullet$-modules", then one can perform a Bar construction to form a MHC $B(M^\bullet, C^\bullet, N^\bullet)$

Theorem (Hain) (de Rham $\to$ Homotopy Hodge Theorem)

Let (X,x) be any pointed space, whose

Betti numbers are finite, and which has a de Rham MHC. Then

a) The homotopy Lie algebra of X has a MHS in which the filtered pieces are ideals.

b) The completed group ring $\widehat{\mathbb{Z}\pi_1(X,x)}$ has a MHS.

<u>Theorem</u> (Deligne)

Any smooth variety has a de Rham MHC.

<u>Moral</u>: Not only do all these things have mixed Hodge structures, but anything we can build from them by nice constructions does too.

$$\begin{array}{c} E \\ \downarrow \\ \mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2 \end{array}$$

For instance, the homotopy fibre of $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^2$ has a MHS we can compute by these methods.

<u>Functoriality</u>:

This is a delicate matter: these constructions are often not functors. But, our homotopy fibre construction has attached long exact sequence on homotopy

$$\cdots \to \pi_{n+1}(Y) \xrightarrow{\partial} \pi_n(E_f) \xrightarrow{i_*} \pi_n(X) \xrightarrow{f_*} \pi_n(Y) \to \cdots$$

∂, i.e. f can be made morphisms of MHSs.

Jacob Lurie

Higher Nonabelian Cohomology I

One can look at the abelian cohomology, which is a representable functor

$$H^n(X; A) = [X, K(A,n)]$$

where $\pi_i(K(A,n)) = \begin{cases} A & i = n \\ 0 & \text{otherwise} \end{cases}$

If G is a general group, we can describe

$$H^1(X; G) = [X, BG]$$
$$= \{\text{isomorphism classes of principal } G\text{-bundles}\}$$
$$= \mathrm{Hom}(\pi_1(X), G) / \text{conjugation}$$ assuming X is connected.

More generally, for any space Y, you can write

$H(X; Y) = [X, Y]$: recovering above examples if $Y = K(A,n)$ or BG.

Definition: (Higher stacks)

A prestack is a functor $\mathcal{X} : \mathbb{C}\text{-algebras} \longrightarrow s\mathrm{Set}$. So it's a simplicial presheaf on affine schemes. Roughly we'll expect to land in Kan complexes $\approx$ topological spaces.

Example:

If X is a scheme, we can define a prestack via

$$X(R) = \mathrm{Hom}(\mathrm{Spec}\, R, X)$$

Example:

If G is an algebraic group, define a prestack BG via

$$BG(R) = \text{Nerve of groupoid of } G\text{-torsors on } \mathrm{Spec}\, R$$

Another formula for this would be

$$X(R) = B(G(R))$$

Example

Let V be a commutative linear algebraic group (a vector space. Then define

$$K(V,n)(R) = K(\underset{\overset{\shortparallel}{V(R)}}{V \otimes R}, n)$$

Remark:

Let X be a scheme, thought of as a prestack. Then

- $[X, R]$ is the set of isomorphism classes of algebraic G-torsors on X.
- $[X, K(V,n)] = H^n(X; \mathcal{O}_X \otimes V)$.

Here we're using the model category structure on simplicial presheaves.

Definition:

We say that a prestack X is a stack if X satisfies descent for the flat topology. i.e. given cover

$$\cdots \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; U \times_{\mathrm{Spec}\,R} U \rightrightarrows U \longrightarrow \mathrm{Spec}\,R$$

we require

abusing notation, writing $U = \mathrm{Spec}\,U$.

$$X(R) \rightarrow \mathrm{holim}\left(X(U) \rightrightarrows X(U \times_{\mathrm{Spec}\,R} U) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \cdots\right)$$

to be a homotopy equivalence (& also for hypercoverings).

e.g.: If $\mathrm{Spec}\,R = U \cup V$, this says

$$\begin{array}{ccc} X(\mathrm{Spec}\,R) & \rightarrow & X(V) \\ \downarrow & & \downarrow \\ X(U) & \longrightarrow & X(U \cap V) \end{array}$$

must be a homotopy pullback.

Our world of prestacks contains objects one thinks of in two ways: schemes & cohomology theories, & evaluating becomes just looking at maps between them.

Let X be a pointed prestack, i.e. either a functor $X : \mathbb{C}\text{-algebras} \longrightarrow \text{pointed sSets}$, or a prestack with a functor $* \longrightarrow X$.

Definition:

The homotopy groups of X can be defined as follows:

$$\pi_i^{\text{naive}}(X) : \mathbb{C}\text{-alg} \to \begin{cases} \text{abelian groups} & i \geq 2 \\ \text{groups} & i = 1 \\ \text{sets} & i = 0 \end{cases}$$

by

$$\pi_i^{\text{naive}}(X)(R) = \pi_i(X(R)).$$

Then

$$\pi_i(X) = \text{sheafification of } \pi_i^{\text{naive}}(X).$$

Example

$\pi_0^{\text{naive}}(BG)$ is the functor of isomorphism classes of G-torsors $[X, BG]$. Then $\pi_0(BG) = *$ point.

Definition:

A pointed prestack X is a higher gerbe if

1) X is a stack
2) $\pi_0(X) = *$
3) $\pi_1(X)$ is represented by an affine unipotent group scheme
4) $\pi_i(X)$ are represented by affine commutative group schemes (i.e. $R \longmapsto \mathrm{Hom}_{\mathbb{C}}(V, R)$ for V a vector space).

Roughly, these are sometimes also called very presentable n-stacks.

<u>Remark</u>:

The collection of these stacks is organised into a model category. Further, you have <u>Postnikov towers</u>: If X is any stack you can associate to it $\tau_{\leq n} X$ killing higher homotopy. For X a higher gerbe have

$$\begin{tikzcd} & \vdots \arrow[d] & \\ & \tau_{\leq 3} X \arrow[d] & K(\pi_3(X), 3) \arrow[l] \\ X \arrow[ur] \arrow[r] \arrow[dr] & \tau_{\leq 2} X \arrow[d] & K(\pi_2(X), 2) \arrow[l] \text{ homotopy fibre} \\ & \tau_{\leq 1} X = B(\pi_1(X)) & \end{tikzcd}$$

where $K(\pi_2 X, 2)(R) = K(\pi_2 X(R), 2)$ etc.

For these functors X can be recovered as the limit of its Postnikov tower.

<u>Question</u>: What is a quasi-coherent sheaf on a prestack X?

<u>Case 1</u>: X a scheme.

<u>Definition</u>:

A <u>quasi-coherent sheaf</u> on X is a functor assigning to each R-point $\eta \in X(R)$: $(\eta : \operatorname{Spec} R \to X)$ an R-module M_η $(\eta^* \mathcal{F})$. This is functorial in η, i.e. in maps between rings

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "\eta"] & X \\ \operatorname{Spec} R' \arrow[u] \arrow[ur, "\eta'"'] & \end{tikzcd} \qquad M_{\eta'} \cong R' \otimes_R M_\eta.$$

In an abstract way, define the category

$$QCoh(X)^{\heartsuit} = \underset{\substack{\longleftarrow \\ R \in X(R)}}{\text{holim}} (R\text{-modules})$$

This makes sense for X any prestack.

Examples:

- $QCoh(BG)^{\heartsuit}$ = category of algebraic representations of G
- $QCoh(K(V,2))^{\heartsuit} = \mathbb{C}$-vector spaces

Analogies:

Spaces	Prestacks
Covering spaces	Quasicoherent Sheaves
$G \curvearrowright$ Set	BG linear representations of G

dg-Categories (over $\mathbb{C}$).

Definition:

A dg-category $\mathcal{C}$ over $\mathbb{C}$ has

1) objects X_i
2) For every X_1, X_2, a chain complex $Hom(X_1, X_2)$.
3) Multiplication $Hom(X_1, X_2)_{\bullet} \otimes Hom(X_2, X_3)_{\bullet} \to Hom(X_1, X_3)_{\bullet}$ a map of chain complexes, i.e.

$$d(\alpha \circ \beta) = d(\alpha) \circ \beta + (-1)^{|\alpha|} \alpha \circ d(\beta).$$

4) Associativity, in the way we'd expect.

Example:

Let R be a $\mathbb{C}$-algebra. Chain (R) is the category of chain complexes of R-modules, naturally a dg-category.

This category has a model category structure, so consider $\mathrm{Chain}(R)^{\mathrm{cof}}$: cofibrant objects : a sub-dg-category.

dg-categories form a model category <u>themselves</u>, the homotopy category has

$$\mathrm{Hom}_{h\mathcal{C}}(X,Y) = \mathrm{Ho}(\mathrm{Hom}(X,Y)_\bullet).$$

So say $F: \mathcal{C} \to \mathcal{D}$ is a weak equivalence if it's an equivalence, & $\mathrm{Hom}(X,Y)_\bullet \to \mathrm{Hom}(FX,FY)_\bullet$ is a quasi-isomorphism $\forall X, Y$. The fibrations & cofibrations are somewhat technical.

<u>Definition</u>:

$$Q\mathrm{Coh}(X) = \underset{\eta \in X(R)}{\overleftarrow{\mathrm{holim}}} \left(\mathrm{Chain}(R)^{\mathrm{cof}}\right) \quad \text{for } X \text{ prestack.}$$

This is a dg-category, closely related to $Q\mathrm{Coh}(X)^{\heartsuit}$.

<u>Theorem</u> (A version of higher Tannaka duality)

If X is a prestack & Y is a higher gerbe, you get an inclusion

$$\begin{aligned} \mathrm{Map}(X,Y) &\longrightarrow \mathrm{Fun}^{\otimes}(Q\mathrm{Coh}(Y), Q\mathrm{Coh}(X)) \\ f &\longmapsto f^* \end{aligned}$$

If you look at functors preserving the symmetric monoidal structure up to coherent homotopy, preserving homotopy colimits, & preserves being concentrated in degree ≥ 0, then you can recover f from f^*.

Lecture 21

Jacob Lurie

Higher Nonabelian Cohomology II

We'll try to give analogues of H_{dR} & H_{Dol} in our language of higher stacks.

Question: What does it mean to have a flat connection on a vector bundle?

Well, our connections give a notion of parallel transport between fibres $\mathcal{E}_x$, $\mathcal{E}_y$, on a path p from x to y. Flatness means this doesn't depend on the element in the homotopy class of paths.

In algebraic geometry, for X a variety, say $x, y \in X(R)$ are infinitesimally close if x, y have the same image in $X(R/Nil(R))$.

Definition (Grothendieck).

A quasicoherent sheaf with connection on a smooth variety $/\mathbb{C}$ is a quasi-coherent sheaf $\mathcal{F}$, with the data of an isomorphism $x^*\mathcal{F}, y^*\mathcal{F}$ whenever x & y are infinitesimally close, with transitivity.

Claim 1: The following are equivalent.

1) A quasicoherent sheaf with connection
2) A quasicoherent sheaf $\mathcal{F}$ with connection in the traditional sense: $\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_X$, such that $\nabla^2 = 0$ (Flatness) (& usual Leibnitz rule).

3) A sheaf on X with action of the sheaf D_X of algebraic differential operators, which is quasi-coherent as an $\mathcal{O}_X$-module.

Note:

If $X = \mathbb{A}^1 = \operatorname{Spec} \mathbb{C}[t]$, then D_X has global sections $\mathbb{C}[t, \frac{\partial}{\partial t}]$; $[t, \frac{\partial}{\partial t}] = 1$.

Claim 2:

The analogous statement to 1 holds where quasi-coherent sheaves are replaced by algebraic vector bundles, & the D_X-modules are required to be $\mathcal{O}_X$-coherent.

If X is projective, this is further equivalent to giving a smooth vector bundle on X^{an} with flat connection in the usual sense.

Definition:

X_{dR} : $\mathbb{C}$-algebras $\longrightarrow$ Sets $\subseteq$ sSets is the functor

$X_{dR}(R) = X(R/Nil(R))$

$= X(R)/$infinitesimal closeness (using X smooth variety)

This is a prestack.

Consequence: $QCoh(X_{dR})^{\heartsuit}$ = quasicoherent sheaves with connection.

Example: $[X_{dR}, BGL_n]$ = flat vector bundles of rank n on X

$[X_{dR}, BG]$ = flat G-bundles on X.

$[X_{dR}, K(\mathbb{A}^1, n)] = H^n_{dR}(X)$. This is why we call it X_{dR}.

<u>Question</u>: How can we get Dolbeault cohomology?

Think of $X_{dR}(R)$ as a groupoid, with objects $X(R)$, & isomorphisms when points are infinitesimally close. We can take its nerve: a sSet.

$$\begin{array}{c} \widehat{X \times X \times X}\ (R) \\ \downarrow\downarrow\downarrow \\ X \hat{\times} X\ (R) \quad \subseteq \quad (X \times X)(R) = X(R) \times X(R) \\ \downarrow\ \downarrow \\ X(R) \end{array}$$

Consider

where $\hat{X}^n(R) = \{ \eta_1, \dots, \eta_n \in X(R) \text{ all infinitesimally close to each other} \}$

This is precisely the nerve.

So define a homotopy equivalent version of $X_{dR}(R)$: the simplicial set above.

This sits inside $X(R) \leftleftarrows X^2(R) \leftleftarrows X^3(R) \cdots$.

A fancy way of describing this is

$$X^{n+1}(R) = \mathrm{Hom}(\mathrm{Spec}(R) \times \{0,1,\dots,n\}, X)$$

via functor $n \longmapsto \{0,1,\dots,n\}$ from finite ordered sets to schemes: so a cosimplicial scheme. Thus $X^{n+1}(R)$ is a simplicial scheme.

We want to generalise this.

Our starting point will be a cosimplicial scheme $T^\bullet$, where T^n = some scheme Spec A_n, where A_n is $(n+1)$-dimensional as a vector space over $\mathbb{C}$.

$A_0 = \mathbb{C}$: A_1 = some 2d $\mathbb{C}$-algebra.

Maps $A_0 \rightleftarrows A_1 \rightleftarrows A_2 \cdots$ simplicial $\mathbb{C}$-algebra

The map $d_0, d_1 : A_1 \longrightarrow A_0$ allow us to write

$$A_1 = \mathbb{C} \oplus \underbrace{\ker(d_0)}_{I \text{ ideal}}$$

$$d_1 : A_1 \longrightarrow \mathbb{C}, \text{ restrict to } I \xrightarrow{\lambda} \mathbb{C}.$$

<u>Conclusion:</u>

Any time we have the above data we get a 1-d $\mathbb{C}$-vector space I, & a map $I \xrightarrow{\lambda} \mathbb{C}$.

In fact this essentially uniquely determines $A_\bullet$:
This is dictated by multiplication on I: choose a generator $x \in I$. So $x^2 = \mu x$.

$\Rightarrow \lambda(x)^2 = \mu\lambda(x)$ suggesting $\mu = \lambda(x)$.

This defines A_1 as an algebra.

Continuing, $A_2 = A_1 \times_{\mathbb{C}} A_1$

$A_n = A_1 \times_{\mathbb{C}} \cdots \times_{\mathbb{C}} A_1$ n-times.

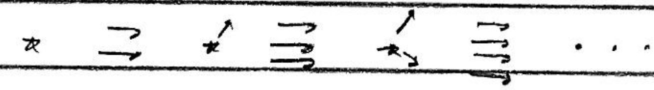
There are fundamentally two cases, $\lambda = 0$ & $\lambda \neq 0$.

<u>Example:</u>

$\lambda = 0$. wlog $I = \mathbb{C}$. This produces

$$\mathbb{C} \leftleftarrows \mathbb{C}[\varepsilon]/(\varepsilon^2) \leftleftarrows \mathbb{C}[\varepsilon]/(\varepsilon^3) \cdots$$

simplicial ring, & hence cosimplicial scheme which we'll draw as



Look at $\mathrm{Hom}(T^{\bullet}, X)$ now. This looks like

$$X \leftleftarrows TX \leftleftarrows TX \times_X TX \leftleftarrows \cdots$$

points | points with tangent vector | points with two tangent vectors | ...

One can complete this to $(\widehat{TX})^n$ at each level.
This is a functor $\mathbb{C}$-algebras $\to$ Simplicial sets, i.e.
a <u>prestack</u> X_{Dol}.

$$X_{Dol}(R) = \coprod_{\eta \in X(R)} B(\mathrm{Nil}(R) \otimes_R T_{X,\eta})$$

<u>Question</u>: What is a quasicoherent sheaf on X_{Dol}?
<u>Answer</u>: It's a rule which assigns to each $\eta \in X(R)$ an R-module acted on by $\mathrm{Nil}(R) \otimes_R T_{X,\eta}$.
i.e. Quasicoherent sheaves on X with an action of $\hat{T}_X$.

<u>Example</u>

$X = \mathbb{A}^1$, $T_X \cong X \times \mathbb{A}^1$. An action of $\mathbb{A}^1$ on a vector space V means an algebraic map $\mathbb{A}^1 \to \mathrm{Aut}(V)$.
This is necessarily given by $t \mapsto e^{tE}$: $E \in \mathrm{End}(V)$.
<u>Note</u>: this is algebraic $\Leftrightarrow$ E is nilpotent, so the power series is eventually zero.

What if we formally complete T_X? Then we only need to be defined in a formal neighbourhood of 0.
So any E will do.

More generally, if $X = V$, $T_X = X \times V$

$\hat{T}_X = X \times \hat{V}$

So a representation of $\hat{V}$ is the same data as an action of $\mathrm{Sym}^* V$: need on E for each generator, all commuting.

For more general X, locally trivialise the tangent bundle, giving

Upshot: Quasi-Coherent sheaves on X_{Dol}

= Quasi-Coherent Sheaves on X with an action of $\mathrm{Sym}^* T_X$

= Quasi-Coherent Sheaves on T^*_X

= Higgs Sheaves $\mathcal{F} \xrightarrow{\theta} \mathcal{F} \otimes \Omega^1_X$ such that $\theta^2 = 0$.

Relatedly, $[X_{Dol}, G] =$ G-Higgs bundles

$[X_{Dol}, K(\mathbb{A}^1, n)] = H^n_{Dol}(X)$

More generally

Definition:

For any stack Y,

$$H_{Dol}(X; Y) = [X_{Dol}, Y]$$

$$H_{dR}(X; Y) = [X_{dR}, Y]$$

Question:

How are these related?

Certainly there was a family of constructions with parameter λ, $\lambda = 0$ giving Dolbeault, $\lambda = 1$ giving de Rham.

Assume Y is a higher gerbe. Then we can compare using Tannaka duality:

$$H_{Dol}(X;Y) \qquad\qquad H_{dR}(X;Y)$$
$$\text{\rotatebox{90}{$\cong$}} \qquad\qquad \text{\rotatebox{90}{$\cong$}}$$
$$\pi_0 \mathrm{Fun}^{\otimes}(QCoh(Y), \text{complexes of Higgs sheaves}) \qquad \pi_0 \mathrm{Fun}^{\otimes}(QCoh(Y), \text{complexes of algebraic D-modules})$$

How are these related?

$$\text{Harmonic Bundles} \ni (\mathcal{E}, D = \partial + \bar\partial + \theta + \bar\theta, h)$$
$$D'' = \bar\partial + \theta$$

$$\swarrow \qquad \searrow$$

(*) $=$ polystable Higgs bundles with vanishing c_i $\ni \mathcal{E}, \mathcal{F}$ $\qquad$ $\mathcal{V}, \mathcal{W} \in$ semisimple vector bundles with flat connection $=$ (**)

$$\cap\!| \qquad\qquad \cap\!|$$

complexes of Higgs sheaves $= QCoh(X_{Dol})$ $\qquad$ $QCoh(X_{dR}) =$ complexes of D-modules

where $Map(\mathcal{E}, \mathcal{F})_\bullet = (\mathcal{A}^*(Hom(\mathcal{E}, F)), \bar\partial + \theta)$

$Map(\mathcal{V}, \mathcal{W})_\bullet = (\mathcal{A}^*(\underline{Hom}(\mathcal{V}, \mathcal{W})), D)$ give dg-category structures.

<u>Idea</u>: Try to define, for $\mathcal{E}, \mathcal{F}$ harmonic bundles, a chain complex $Map^{harm}(\mathcal{E}, \mathcal{F})$ mapping to the above, q.is to both.

Try $Map^{harm}(\mathcal{E}, \mathcal{F}) =$ harmonic forms $\subseteq \mathcal{A}^*(Hom(\mathcal{E}, \mathcal{F}))$ with trivial differential.

However, we lose the dg-category structure, as multiplication of forms doesn't preserve harmonicity.

Instead, look at $\ker(\partial + \bar\theta = D' : \mathcal{A}^* Hom(\mathcal{E}, \mathcal{F}) \to \mathcal{A}^* Hom(\mathcal{E}, \mathcal{F}))$ with differential $D = D''$. So this complex maps to both.

<u>All</u> these dgas are <u>equivalent</u>. This can be extrapolated to larger chunks of the dg-categories <u>formally</u>.

Indeed, it gives

$$\begin{array}{ccc} \text{Higgs sheaves built from } ⊛ & \xleftrightarrow{\sim} & \text{D-modules built from } ⊛⊛ \\ \cap\!\mid & & \cap\!\mid \\ \text{Complexes of Higgs sheaves} & & \text{Complexes of D-modules} \end{array}$$

This gets you a long way toward

$H_{Dol}(X; Y)$ $\qquad\qquad$ $H_{dR}(X; Y)$

On RHS: $\otimes$-functors factor through these subcategories, & using duals on $QCoh(Y)^{\heartsuit} \subseteq QCoh(Y)$, as Y a higher Gerbe

On LHS: This doesn't quite work

But we at least get an injection RHS $\hookrightarrow$ LHS.

With assumptions on Y you can say more, e.g. if $\pi_1(Y)$ is unipotent it's bijective.

Anthony

Schematic Homotopy Types

Lecture 22

Goal: Given a variety X, smooth quasiprojective $/\mathbb{C}$, define a stack $(X \otimes \mathbb{C})^{Sch}$, & a $\mathbb{C}^*$ action on it. This recovers the Hodge structure.

This is contained in the paper Affine Stacks of Toën.

The schematisation Problem

First stated in "Pursuing stacks" by Grothendieck, but we'll discuss Toën's vision of it.

Given an abelian group object in $Ho(k) =$ $Ho($Simplicial presheaves on the site of schemas $/k)$, define a classifying stack BA, itself an abelian group stack. Iterating gives Eilenberg - Maclane stacks

$$K(A,i) = B\,K(A,i-1) \quad , \quad K(A,0) = A.$$

Grothendieck: The stacks $K(\mathbb{G}_a, i)$ are fundamental examples of schematic homotopy types $/k$. These ought to be stable under holims in $Ho(k)$.

Definition:

Given a subcategory of $Ho(k)$, call it a schematic homotopy type category if it contains $K(\mathbb{G}_a, i)$, & is stable under Holim.

Consider the functor $Ho(k) \xrightarrow{R\Gamma} Ho(sSet)$. We can consider this restricted to $\mathcal{C}$ a schematic homotopy type category.

Schematization Problem:

Find such a $\mathcal{C} \subseteq Ho(\mathbb{Q})$ such that $R\Gamma|_{\mathcal{C}}$ ~~has~~ a left adjoint $\otimes\mathbb{Q}: Ho(sSet) \to \mathcal{C}$ which is fully faithful when restricted to rational connected finite type homotopy types.

Affine Stacks:

Work over a field k. Let Alg_k be the category of commutative k-algebras, Sch_k the category of schemes over k so $Spec : (Alg_k)^{op} \longrightarrow Sch_k$.

Want a homotopy category version.

Replace Alg_k by Alg_k^{Δ} : model category of cosimplicial algebras

Sch_k by $SPr(k)$: simplicial presheaves $/Aff_k^{Flat}$.

N normalisation

The equivalences on Alg_k^{Δ} are $A \longrightarrow B$ such that $H^i(NA) \to H^i(NB)$ are isomorphisms

fibrations are surjections level wise

Cofibrations determined by lifting property.

Definition:

$Spec : (Alg_k^{\Delta})^{op} \to SPr(k)$ is defined to be

$Spec(A)(Spec(B)) = \underline{Hom}(A, B)$, where $\underline{Hom}$ is simplicial hom.

$\underline{Hom}(A,B)_n = Hom(A_n, B)$.

Fact:

Spec is right Quillen, so has a right derived functor

$\mathbb{R}Spec : Ho(Alg_k^{\Delta}) \longrightarrow Ho(k)$

Proposition (Toën)

$\mathbb{R}\mathrm{Spec}$ is fully faithful.

Definition:

A stack is *affine* if it is in the essential image of $\mathbb{R}\mathrm{Spec}$: $F \sim \mathbb{R}\mathrm{Spec}(A)$.

Examples:

$K(\mathbb{G}_a, i) \sim \mathrm{Spec}(S(i))$

where $S(i)$ is the free cosimplicial algebra on S^i_k, S^i_k the cosimplicial k-module corresponding to the complex with k in degree i, 0 elsewhere.

In fact all affine stacks are holimits of such stacks. Toën has proved we can characterise affine stacks: F is affine iff it is subaffine & $\mathcal{O}$-local. i.e.

- $F = h_X$, X simplicial affine scheme
- F is local wrt $\mathcal{O}$-equivalences,

Schematic Homotopy Types

Given a stack F we can talk about its *affination*

$$F \to (F \otimes k)^{uni}$$

it is the universal stack with such a morphism.

This provides a solution to the Schematisation problem:

$(F \otimes k)^{uni} = \mathbb{R}\mathrm{Spec}\,\mathcal{O}(F)$ where $\mathcal{O}$ is right adjoint to Spec.

What is the schematisation $(X \otimes k)^{sch}$?

Idea: Replace $\mathbb{O}$-equivalences by P-equivalences: morphisms inducing isomorphism on cohomology with coefficients in all $K(A, V, n)$, A an affine group scheme, V a linear finite-dimensional representation of A. $K(A,V,n) = K(V,n) \times_{\cdot} A$.

The schematic homotopy types are pointed connected stacks F that satisfy

- $\mathbb{R}\Omega F$ is affine

& • F is P-local

So $(X \otimes k)^{sch}$ is the the universal such stack among $X \longrightarrow (X \otimes k)^{sch}$.

Ting. Based on the paper "Schematic homotopy types & non-abelian Hodge Theory" of Katzarkov-Pantev & Toën

Remark: The Hodge filtration on $H^n(X; \mathbb{C})$ is equivalent to the $\mathbb{C}^\times$-action on $H^n(X; \mathbb{C})$;

if $y \in \mathcal{H}^{p,q}$, $\lambda \in \mathbb{C}^\times$, $\lambda(y) = \lambda^p y$.

If X is smooth projective over $\mathbb{C}$, let $(X \otimes \mathbb{C})^{sch}$ be its schematic homotopy type.

Properties:

- $H^*((X \otimes \mathbb{C})^{sch}; \mathbb{G}_a) \cong H^*(X; \mathbb{C})$
- $\pi_i((X \otimes \mathbb{C})^{sch}, *) \cong \pi_i(X, *) \otimes \mathbb{C}$. if $i > 1$, X simply connected.

- $\pi_1((X\otimes\mathbb{C})^{sch}, *) \cong \pi_1(X,x)^{alg}$ pro-algebraic completion

The main result of KPT is to define a $\mathbb{C}^*$-action $\mathbb{C}^* \circlearrowright (X\otimes\mathbb{C})^{sch}$ that will recover the $\mathbb{C}^*$-action on cohomology, hence the Hodge structure.

__Morgan__ in '78 defined a MHS on $\pi_i(X,*)\otimes\mathbb{C}$ for $i>1$, X simply connected.
__Simpson__ in '98 defined a $\mathbb{C}^*$-action on $\pi_1(X,*)^{red}$, the proreductive completion.

__Theorem:__
The $\mathbb{C}^*$ action on $F=(X\otimes\mathbb{C})^{sch}$ exists, such that

1) The induced action on $H^i(F,\mathbb{G}_a)\cong H^i(X;\mathbb{C})$ is the above action.
2) The induced action on $\pi_1(F,x)^{red}$: the maximal reductive quotient of the proreductive completion, $\cong \pi_1(X,*)^{red}$, agrees with the above.
3) The induced action on $\pi_i(F,x)$ $i>1$ also agrees with the above.

There's also an analogue of the mixed structure: a __weight tower__:

$$F \longrightarrow \cdots \longrightarrow F_i \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 = *$$

of pointed stacks with homotopy fibres W_i.

The associated long exact sequence on homotopy groups

$$\begin{tikzcd}[column sep=small]
\pi_*(F) \arrow[r] & \cdots \arrow[r] & \pi_*(F_i) \arrow[r] \arrow[dr] & \cdots \arrow[r] & \pi_*(F_1) \arrow[r] \arrow[dr] & \pi_*(F_0) \\
& & \pi_*(W_{i-1}) \arrow[u] & & \pi_*(W_0) \arrow[u] &
\end{tikzcd}$$

Forms an exact couple
$$\begin{tikzcd} A \arrow[rr] & & A \arrow[dl] \\ & B \arrow[ul] & \end{tikzcd}$$
by taking direct sums, whose associated spectral sequence can be used to calculate $\pi_*(F)$. This gives new constraints to the homotopy types of smooth projective varieties.

Construction:

Start with the category L_{Dol} of Higgs bundles over X. (polystable with vanishing c_i).

Objects : (V, D'') V C^∞ vector bundle

$$D'' : V \longrightarrow V \otimes A^1$$

with $D''(a \cdot s) = \bar{\partial} a \cdot s + a D''(s)$.

T_{Dol} will be the category of Ind-objects in L_{Dol}: inductive limits of objects. One can think of it as a completion.

Similarly, have L_{dR} : Flat bundles over X.

Objects : (V, ∇) as usual.

T_{dR} will be the category of Ind-objects in L_{dR}.

Pick $x \in X$. Let $G_x = \pi_1(X, x)^{red}$

Non-abelian Hodge correspondence tells us these categories are equivalent. $L_{Dol} \xrightarrow{\sim} L_{dR}$, $T_{Dol} \xrightarrow{\sim} T_{dR}$.

Facts : • There's a $\mathbb{C}^\times$ - action on L_{Dol}

$$(V, \bar\partial + \theta) \xmapsto{\lambda} (V, \bar\partial + \lambda\theta).$$

• Let $\omega_x : L_{Dol} \to Vect$ be the functor taking fibre at x

$v_x : T_{Dol} \to Vect$ similarly.

The latter has a right adjoint ρ:

$\rho(\mathbb{1}) = \mathcal{O}(G_x)$ using NAHT.

• $\mathbb{1}$ is an algebra, hence so is $\rho(\mathbb{1})$.

• ω_x is $\mathbb{C}^\times$ - invariant, hence so is ρ.

$\Rightarrow \rho(\mathbb{1})$ is fixed by $\mathbb{C}^\times$.

• If $(V, D'') \in L_{Dol}$, one can associate to it

$$A^0(V) \xrightarrow{D''} A^1(V) \to \cdots$$

$\rho(\mathbb{1})$ is an inductive system of such, so it also has a Dolbeault complex $A^{\cdot}_{Dol}(\rho(\mathbb{1}), D'')$.

Now we can start to describe the $\mathbb{C}^\times$ - action.

$\mathbb{C}^\times \circlearrowright A^{\cdot}_{Dol}(\rho(\mathbb{1}), D'')$ by

$$\rho(\mathbb{1}) \xmapsto[\sim]{U_\lambda} \lambda\rho(\mathbb{1}), \quad \text{as } \rho(\mathbb{1}) \ \mathbb{C}^\times\text{-fixed}.$$

inducing U_λ on the Dolbeault complex.

Also $[\lambda]$: mult by λ^p on p,q forms is another $\mathbb{C}^\times$ action.

So our action is $\mathcal{U}_\lambda^{-1} \circ [\lambda] : A_{Dol} \to A_{Dol}$.

$G_x \circlearrowright A^{\cdot}_{Dol}(\rho(\mathbb{1}), D'')$ also, compatibly, i.e. quotient still has a $\mathbb{C}^\times$ - action. Let

$A = C^{\cdot}_{Dol}(X, \mathcal{O}(G_x)) = D(A^{\cdot}_{Dol}(\rho(\mathbb{1}), D''))$: denormalisation using Dold - Kan.

Claim:

$$\mathbb{R}\mathrm{Spec}\, A / G_x \cong (X \otimes \mathbb{C})^{sch}$$

This gives a $\mathbb{C}^\times$-action on the RHS from that on the left.

Remark:

This also works identically for $A_{dR}(\mathcal{O}(G_x), \nabla)$.

The model you get is called $(X \otimes \mathbb{C})^{diff}$.

Carlos Simpson

Lecture 23

Local Systems on Non-compact Varieties

Focus on the paper "a weight two phenomenon": look at rank 1 local systems on the easiest non-compact variety:

$$U = \mathbb{P}^1 \setminus \{0, \infty\} = \mathbb{G}_m.$$

Review: twistor space picture.

Say X is a compact curve. Let $\mathcal{M}$ be the Hitchin hyperKähler moduli spaces for some group G.

$$(\mathcal{M}, I) = \mathcal{M}_{Dol}$$

$$(\mathcal{M}, J) = \mathcal{M}_{dR}.$$

Let $\mathbb{P}^1 = \{ xI + yJ + zK : (x^2+y^2+z^2) = 1 \}$

Then $Tw = M \times \mathbb{P}^1$ has a natural complex structure

$$\downarrow$$
$$\mathbb{P}^1$$

such that on the fibre over λ, it's the complex structure given by λ. The horizontal $\mathbb{P}^1$'s - preferred sections are holomorphic.

We have antipodal involution on $\mathbb{P}^1$, which extends to an antilinear involution $\sigma : Tw \to Tw$. Then

$$\mathcal{M} = \{\text{preferred sections}\} \subseteq \Gamma(\mathbb{P}^1, Tw)^{\sigma} \quad : \sigma\text{-invariant}.$$

In fact it's a union of connected components.

This says that if we deform a preferred section σ-invariantly it stays preferred. Or equivalently

$$(T_{\rho}\, \Gamma(\mathbb{P}^1, Tw))^{\sigma} \xrightarrow{\sim} T_{\rho}\mathcal{M} \qquad \rho \in M \leftrightarrow (\mathbb{P}^1 \to Tw)$$

More precisely, this should be an isomorphism when we evaluate at any point λ.

Why is this? Well the normal bundle satisfies

$$N_{\lambda(\mathbb{P}^1)/T_w} = \rho^* T(T_w/\mathbb{P}^1) = \mathcal{O}_{\mathbb{P}^1}(1)^a \qquad a = \dim_{\mathbb{C}}(M)$$

the <u>twistor property</u>:

$$T_p(\Gamma(\mathbb{P}^1, T_w)) \cong \Gamma(\mathbb{P}^1, \rho^* T(T_w/\mathbb{P}^1)) \quad \text{compatibly with } \sigma$$
$$\cong \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^a)$$

so we have

$$\sigma \circlearrowright \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^a) = \mathbb{C}^{2n} \longrightarrow \mathbb{C}^a = (\mathcal{O}_{\mathbb{P}^1}(1)^a)_\lambda = T_p(M)$$

$$\begin{tikzcd} \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^a) \arrow[u, hook] \arrow[ur, "\sim"'] \end{tikzcd} \text{ iso for any } \lambda$$

$$\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)^a)^\sigma = \mathbb{R}^{2n}$$

<u>Conclusion</u>:

The normal bundles to the preferred sections being semistable of weight / slope 1 corresponds to the existance of the quaternionic structure.

More precisely, if we have $H_{\mathbb{R}} \subseteq H_{\mathbb{C}} = H^{0,1} \oplus H^{1,0}$ a weight 1 real Hodge structure <u>then</u> it has a natural quaternionic structure. We can also form the <u>Rees bundle</u> by taking the Rees constructions for $F^\cdot$ & $\bar{F}^\cdot$ at 0 and ∞ respectively. This is really

$$(H^{1,0} \otimes \mathcal{O}_{\mathbb{P}^1}(0)) \oplus (H^{0,1} \otimes \mathcal{O}_{\mathbb{P}^1}(\infty))$$
$$\cong \mathcal{O}_{\mathbb{P}^1}(1)^{2g}$$

<u>Weight 2 case</u>

Now lets go back to $U = \mathbb{P}^1 \setminus \{0, \infty\} = X \setminus D$.

$\pi_1(U) = \mathbb{Z}$: So a rank 1 local system is just a map $\mathbb{Z} \xrightarrow{\rho} \mathbb{C}^*$ given by monodromy $f = \rho(1)$.

<u>A rank 1 bundle with connection</u> (logarithmic at D) is a pair (L, ∇), L a line bundle on X, ∇ a connection

$$\nabla : L \longrightarrow L \otimes \Omega^1_X (\log D).$$

Assume $L = \mathcal{O}_{\mathbb{P}^1}$, so $\nabla = d + \beta \frac{dz}{z}$. Then the monodromy is $f = e^{2\pi i \beta}$

<u>A rank 1 Higgs bundle</u> is a pair (E, θ), again assume $E = \mathcal{O}_{\mathbb{P}^1}$, $\theta = \alpha \frac{dz}{z}$ since

$$\Gamma(\mathrm{End}(E) \otimes \Omega^1_X(\log D)) = \Gamma(\Omega^1_X (\log D))$$

<u>How do these correspond?</u> we need to choose a harmonic metric h_a: $|e|_h = z^a$ where e is a unit section, z is the coordinate, so we need polynomial growth: <u>tameness</u>. a will determine a growth rate of sections of L or E.

Near $z = 0$ we get the <u>parabolic filtration</u>

$$E_u = \{ \text{sections } v \text{ of } E \text{ such that } |v|_h \leq |z|^{-u-\varepsilon} \ \forall \varepsilon > 0 \}$$

Giving a <u>parabolic bundle</u>.

If you do this, producing a parabolic weight, α residue

then this will correspond to a connection (b, β): related by a formula

Motivically, what does this mean?

$$H^1(U, \mathbb{Z}) = \mathbb{Z}(1) : \underline{\text{weight 2 motive}}$$

due to the 2 punctures. Create a bundle $\mathcal{R}$ over $\mathbb{A}^1$: the $\underline{\lambda\text{-line}}$ of "possible residues of λ-connections": $\mathcal{O}_{\mathbb{A}^1} = \mathbb{C} \times \mathbb{A}^1$. But now

$$\lambda\text{-connections on } U \Leftrightarrow \mu\text{-connections on } \bar{U}$$

for $\mu = \lambda^{-1}$. This is the Deligne gluing.

In total, these glue together as an $\mathcal{O}_{\mathbb{P}^1}(2)$. We'll have an involution σ on this, & the residues of harmonic bundles will be the σ-invariant sections, like in the compact case.

But $\Gamma(\mathcal{O}_{\mathbb{P}^1}(2)) = \mathbb{C}^3$, so $\Gamma(\mathcal{O}_{\mathbb{P}^1}(2))^\sigma = \mathbb{R}^3$.

The evaluation maps down to $\mathcal{R}_0, \mathcal{R}_1, \mathcal{R}_\lambda \cong \mathbb{C}$.

Botong Wang

The Higher Dimensional Non-compact Case

We'll work locally, so let

$X = \Delta^*$ punctured disc

$\bar{X} = \Delta$, $E = \mathcal{O}_X$, $\theta = a \frac{dz}{z}$, $a \in \mathbb{C}$.

Example:

$|1|_h = |z|^\alpha$ $\alpha \in \mathbb{R}$. In this setting,

So $\partial_h = \partial_0 + \alpha \frac{dz}{z}$, $\bar{\theta} = \bar{a} \frac{d\bar{z}}{\bar{z}}$

$$D = \bar{\partial}_h + \theta + \bar{\theta} = \bar{\partial} + \partial_0 + \bar{a}\frac{d\bar{z}}{\bar{z}} + (a+\alpha)\frac{dz}{z}$$

This makes it easy to compute the residue.

Example:

$E = \mathcal{O}_X^2$ now, $h = h^1 \oplus h^2$, with holo unit sections w_1, w_2.

Say $\theta(w_1) = \frac{1}{2} w_2 \frac{dz}{z}$, $\theta(w_2) = 0$,

So $\operatorname{res}(\theta) = \begin{pmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}$. A harmonic metric will be

$$|w_1|_h = |\log|z||^{\frac{1}{2}}, \quad |w_2|_h = |\log|z||^{-\frac{1}{2}}.$$

Now, let $X \hookrightarrow \bar{X}$, X quasi projective, $\bar{X} \setminus X = D$ a simple normal crossings divisor. One can show there are correspondences

$$\left\{\begin{array}{c}\text{tame harmonic} \\ \text{bundles on } X\end{array}\right\}$$

$$\left\{\begin{array}{c}\text{polystable filtered} \\ \text{regular flat bundles} \\ \text{with } c_i = 0\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{tame harmonic} \\ \text{bundles on } X\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{polystable filtered regular} \\ \text{Higgs bundles} \\ \text{with } c_i = 0\end{array}\right\}$$

This was proved by Simpson, Mochizuki, Biquard. The directions <u>from</u> harmonic are easier. We'll focus on Higgs $\longrightarrow$ Harmonic.

<u>Theorem</u> (Mochizuki) (Higgs $\longleftrightarrow$ Harmonic)

Let (E_*, θ) be a <u>regular filtered Higgs bundle</u> on $(\bar{X}, D)$ (relative version). Put $E := E_*|_X$: no longer filtered as the filtration all happens at the divisor. It is polystable with $c_i = 0$ iff $\exists$ a pluriharmonic metric h for (E, θ) on X, which is adapted to the filtration. Such a metric is unique up to obvious ambiguity.

We'll define the objects in this statement.

<u>Definition</u>:

E_*, a <u>filtered sheaf</u> on $(\bar{X}, D)$, is a family of sheaves parameterised by $\mathbb{R}^\ell$, (where $D = D_1 \cup \dots \cup D_\ell$), such that

① $a \leq b \Rightarrow E_a \subseteq E_b$, ($\leq$ in all coordinates)

② $E_{a'} = E_a \otimes \mathcal{O}_{\bar{X}}(-\Lambda_i D_i)$, where $a' = a - (\Lambda_1, \dots, \Lambda_\ell)$ $\quad \Lambda_i \in \mathbb{Z}$

③ $E_a = E_{a+\varepsilon}$ for ε small enough.

So understanding $E_\alpha \subseteq \dots \subseteq E_{\alpha+1}$ suffices, as $E_{\alpha+1} = E_\alpha \otimes \mathcal{O}(D)$.

$E_{\alpha+1} / E_\alpha = E_{\alpha+1}|_D$, so really this is a filtration on $E|_D$.

Chern Characters of E_*.

How do we define c_i? We need to take into account the whole family.

Example:

Suppose we have jump .

Then we should have

$$c_1(E_*) = \tfrac{1}{2} c_1(E_0) + \tfrac{1}{2} c_1(E_{1/2})$$

One generalises this by linear algebra.

Stability: If $F_* \subseteq E_*$, stability as usual means $\text{Slope}(F_*) < \text{Slope}(E_*)$ as usual: using the above to define degree.

Sketch Proof of $\Rightarrow$ in the Theorem

We want to construct a pluriharmonic metric.

Strategy:

Step 1: Take appropriate initial metric

Step 2: Deform it along the heat equation

Step 3: Take limit: show it's Hermite - Einstein

In the compact case $\underline{1}$ is trivial, $\underline{2 \,\&\, 3}$ are hard, but were solved by Simpson

Theorem:

Let (Y, ω) be a Kähler manifold satisfying some conditions. (e.g. Finite volume). Let $(E, \bar\partial_E, \theta, h_0)$ be a Higgs bundle with a metric s.t. $F(h)$ is bounded wrt h_0 & ω.

$F(h) = \partial\bar\partial + \bar\partial\partial = \theta\bar\theta + \bar\theta\theta$

If it is analytically stable, then $\exists$ a Hermite-Einstein metric h satisfying

- h & h_0 are mutually bounded (so they give the same filtration).

This is fine in the non-compact case, so we must solve 1): Find a good initial metric.

Need to specify something on the smooth part of Δ & its singular part. They look something like

$$\Delta^* \times \Delta^{n-1}$$

$$\& \quad (\Delta^*)^k \times \Delta^{n-k} \quad \text{locally.}$$

How can one globalise local constructions: gluing local pieces? Recall the following

Definition:

Suppose (E_*, θ) is a filtered Higgs bundle. Fix a jump α. Then one can define $res_\alpha(\theta)$ acting on $res_\alpha(E)$.

Since D_i are projective, eigenvalues of $res_\alpha(\theta)$ are constant: $res(\theta)$ is a section of $End(res(E)|_{D_i})$.

Take its characteristic polynomial:

$$t^r + a_1 t^{r-1} + \dots + a_r.$$

$$a_i \in H^0(D_i, \mathcal{O}_X^{\otimes i}|_{D_i}).$$

Thus the situation is very similar to the curve case.

Good Case:

res(Θ) diagonalisable. Mochizuki calls this case "graded semisimple". Here we can pick a good initial metric by working locally.

What about the bad case? This is hard, so we need to reduce to the good case. We "disturb" the filtration.

e.g: res(Θ) = $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \phi \circlearrowright V$.

Filter by $V_0 = \ker \phi$, $V_1 = V$.

If there is bad behaviour at jump α, split into two jumps $E^{\varepsilon}_{\alpha-\varepsilon}$, $E^{\varepsilon}_{\alpha+\varepsilon}$ corresponding to this splitting by weight filtration

Suppose (E_*, Θ) is in the bad case.
Disturb to $(E^{\varepsilon}_*, \Theta)$ to make it good.
Then we use:

Theorem: (to take a limit $\varepsilon \to 0$)

Let $(E_m, \bar{\partial}_m, \Theta_m, h_m)$ be a sequence of tame harmonic bundles on X. Assume the sections $\{\det(t - \Theta_m)\} \in \Gamma(\mathrm{Sym}^{\cdot}\, \Omega^1_X(\log D)[t])$ are convergent. Then $\exists$ a subsequence which converges to a tame harmonic bundle $(E_\infty, \bar{\partial}_\infty, \Theta_\infty, h_\infty)$ on X. (weakly in L^p_2, locally on $X \setminus D$ for $p >> 0$).

Apply this theorem to the harmonic metrics for each $(E_n^\varepsilon, \theta)$, & take a limit. Note the condition is satisfied because θ is fixed.

Note: This theorem is essentially properness of the Hitchin map.

Idea: bound eigenvalues of θ

$\Rightarrow$ bound $|\theta|$

$\Rightarrow$ bound $|F(h)| = |\theta\bar{\theta} + \bar{\theta}\theta|$

Then use Uhlenbeck compactness & an idea of Atiyah-Bott.

Sam Gunningham

Local Systems on $\mathbb{P}^1 \setminus$ a Finite Set of Points & Middle Convolution.

We'll understand local systems on $\mathbb{P}^1 \setminus S = \mathcal{U}$, $S = \{q_1, \dots, q_n\}$.

One can think of these

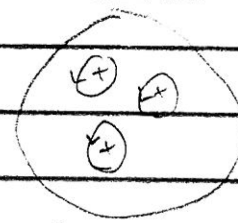
Betti: representations $\pi_1(\mathcal{U}) \to GL(r, \mathbb{C})$,

i.e a collection $M_1, \dots, M_n$, where $M_1 M_2 \cdots M_n = I$ of matrices in $GL(r, \mathbb{C})$, up to simultaneous conjugation.

de Rham: Meromorphic connections with logarithmic singularities at the q_i.

Associated to such an object we get local data.

$\underline{f}$ = conjugacy classes of local monodromies around the puncture, or conjugacy classes of residue matrices.



We'll fix this local data $\underline{f}$, & consider the set $M(\underline{f}) = \{$ local systems on $\mathcal{U}$ with local data $\underline{f} \}$.

Questions:

1) When is $M(\underline{f})$ empty? This is called the Deligne - Simpson problem.
2) How can we construct these local systems when they exist?

Important Special Case : $M(\underline{f}) = pt$.

Such a local system is called rigid : i.e. it is determined by its local data.

In the rigid case, Katz gave an answer to 1) & 2).

He showed every rigid irreducible local system on $\mathcal{U}$ could be built from 1-dimensional local systems explicitly.

Historical Motivation:

The classical perspective is thinking of the local systems as coming from systems of ODEs.

An important class of equations on $\mathcal{U}$ is the hypergeometric equations

$$z(1-z) f'' - (c - (a+b+1)z) f' - ab f.$$

where $a, b, c \in \mathbb{C}$. Singularities are at $0, 1, \infty$.

They have 2-d spaces of solutions built from the hypergeometric function ${}_2F_1(a,b,c;z)$.

The corresponding local system of solutions has eigenvalues of local monodromy

$$1, e^{-2\pi i c} \quad \text{around } 0$$
$$1, e^{2\pi i(c-a-b)} \quad \text{around } 1$$
$$e^{2\pi i a}, e^{2\pi i b} \quad \text{around } \infty$$

Theorem (Riemann, in other words).

The local system "${}_2F_1(a,b,c)$" is rigid & irreducible, and every (rigid) irreducible rank 2 local system on $\mathbb{P}^1 \setminus \{0,1,\infty\}$ is of the form $\mathcal{L} \otimes {}_2F_1(a,b,c)$ where $\mathcal{L}$ is a rank 1 local system.

irreducible will imply rigid here

Note in rank 1 matrix = conjugacy class of matrices, so the problems are easy.

Key Idea:

There's a particular formula for ${}_2F_1$:

$$ {}_2F_1(a,b,c;z) = \int t^{a-c}(1-t)^{c-b-1}(z-t)^{-a}\,dt $$

$$ = \left(z^{a-c}(1-z)^{c-b-1}\right) * z^{-a} $$

So think of the rank 2 local system as a convolution of rank 1 local systems

Middle Convolution will be an assignment

$$ \{\text{local system on } U\} \times \{\text{1-d local system on } \mathbb{G}_m\} \xrightarrow{*_m} \{\text{local system on } U\} $$

"Kummer local system".

Properties:

1) Local data for $\mathcal{L} *_m \mathcal{K}^\mu$ only depends on the local data $\underline{t}$ of $\mathcal{L}$ & μ: $\underline{t} \longmapsto \kappa(\underline{t},\mu)$.

2) If $\mathcal{L}$ is irreducible, then the operation is invertible, with inverse $*_m \mathcal{K}^{\mu^{-1}}$.

Theorem (Katz)

If $\mathcal{L}$ is a rigid irreducible local system on U of rank ≥ 2, then by doing combinations of

1) $\otimes$ with rank 1 local system

2) $*_m$ with Kummer local system,

then we can reduce the rank.

Katz's Algorithm

Suppose we start with a local datum. Then apply the above theorem until either:

1) We get to a 1-dimensional local system on $\mathcal{U}$. In this case we can invert the operations.

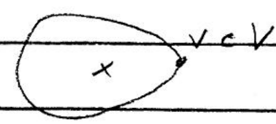
2) We get to local data $\underline{f}$ that "doesn't make sense". Then you deduce that the original moduli space was empty.

<u>Remark</u>:

Similar methods have been applied in various non-rigid situations too.

<u>Remark</u>:

The word <u>middle</u> here is the same middle as middle perversity. Given a local system can consider $H^*(\mathcal{U}, \mathcal{L})$. Say $\mathcal{U} = \mathbb{C}^*$. So one can look at 1-chain's say, for monodromy $M \in \mathrm{End}(V)$



& singular chain complex

$$V(\bullet) \xleftarrow{M-I} V(\bigcirc)$$

Cohomology comes from fixed vectors for the monodromy.

The <u>middle/intersection cohomology</u> of $\mathcal{L}$ is the cohomology of $j_{!*}\mathcal{L} = R^0 j_* \mathcal{L}$.

Unpacking this, have distinguished triangle

$$R^0 j_* \mathcal{L} \longrightarrow R j_* \mathcal{L} \longrightarrow R^1 j_* \mathcal{L}[-1] \xrightarrow{+1}$$

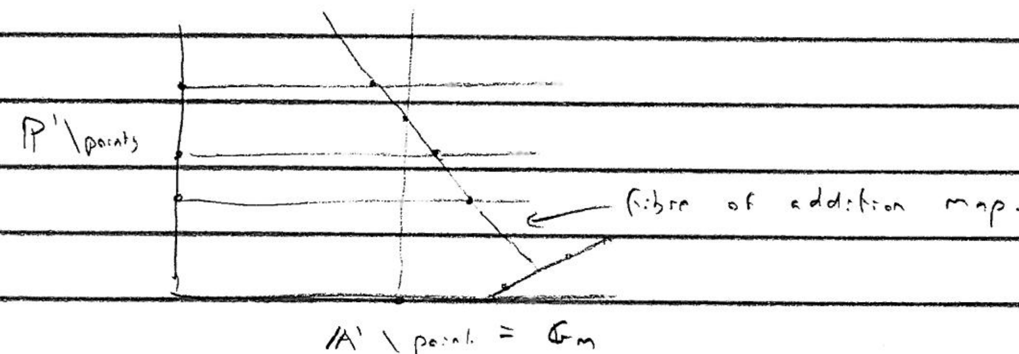
computing middle cohomology; computing ordinary cohomology; computing local cohomology. $\bigoplus_s H^1(D_s^*, \mathcal{L}|_{D_s^*})$

From this we produce a long exact sequence

$$0 \to MH^0 \xrightarrow{\sim} H^0 \to 0$$

$$0 \to MH^1 \to H^1 \to \bigoplus_s H^1(D_s^*, \mathcal{L}) \to MH^2 \to 0$$

Ordinary convolution of sheaves on $\mathbb{A}^1$ is



$a: \mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$, $\mathcal{F} * \mathcal{G} = Ra_*(\mathcal{F} \boxtimes \mathcal{G})$:
integrate along fibres of the addition map.

Definition:

What we wrote before is just $Rp_{2*}(\mathcal{F}_x \otimes \mathcal{G}_{t-x})$.

Consider diagram

$$\begin{array}{ccccc} U \times \mathbb{G}_m & \xrightarrow{\sim} & V & \hookrightarrow & \mathbb{P}^1 \times \mathbb{A}^1 \\ (x,t) & \longmapsto & (x, t-x) & & \downarrow p_2 \\ & & & & \mathbb{A}^1 \end{array}$$

So if $\mathcal{L}$ is on U, $\mathcal{K}$ on $\mathbb{G}_m$, then we define

$$\mathcal{L} *_m \mathcal{K} = R^1 p_{2*} j_{!*} (\mathcal{L}_x \otimes \mathcal{K}_{t-x}).$$