David

Cohomology of Moduli Spaces

$\mathcal{M} = \mathcal{M}^{Higgs}(2, K, \omega_\Sigma)$. Σ is a Riemann surface of genus $g \geq 2$. So $\mathcal{M}$ is the moduli space of stable pairs (V, ϕ) with V a holomorphic vector bundle on Σ, with $\deg(V) = k$ odd, & fixed $\Lambda^2 V$.

$\mathcal{M}$ is a smooth variety with $\dim_{\mathbb{C}} \mathcal{M} = 6g - 6$. It's actually also hyperkähler. Consider complex structure I invariant under $(A, \phi) \longmapsto (A, e^{i\theta}\phi)$ circle action on the Higgs field.

$\mathcal{M}$ is naturally a quotient of $\mathcal{A} \times \Omega$, where $\mathcal{A}$ is the space of connections, & Ω is the space of Higgs fields.

Definition:

Define a moment map

$$\mu : (A, \phi) \longmapsto 2i \int_\Sigma \mathrm{Tr}(\phi \phi^*) = \|\phi\|^2_{L^2}$$

This μ is (up to a scalar) a moment map for the S^1-action $\phi \longmapsto e^{i\theta}\phi$ above.

Proof: Moment map means $d\mu = -2\, i(X)\omega_1$ where X is the generating vector field for the action. X here is

$$X(A, \phi) = (0, i\phi)$$

$$\Rightarrow \quad i(X)\omega_1(Y) = g(IX, Y) = -g(\phi, Y)$$

$$= -\tfrac{1}{2}\, d g(\phi, \phi)(Y)$$

since $g(\dot\phi + \varepsilon\gamma, \dot\phi + \varepsilon\gamma) = g(\dot\phi, \dot\phi) + 2\varepsilon g(\dot\phi, \gamma) + O(\varepsilon^2)$

$\square$

Corollaries:

1) The critical points of μ are precisely the fixed points for the circle action.
2) Each critical set Z for μ is non-degenerate, and its index λ_Z is the real rank of $N^- \subseteq$ normal bundle to fixed point set N; the things of negative rank.

μ will be a Morse function.

3) The Poincaré polynomial for M is now

$$P_t(M) = \sum_{\substack{\text{Fixed point} \\ \text{sets } Z}} t^{\lambda_Z} P_t(Z).$$

This is a theorem from Morse theory.

Proposition:

μ has the following properties

1) μ is proper
2) μ has critical values $0, \{(d-\frac{1}{2})\pi : 1 \leq d \leq g-1\}$
3) $\mu^{-1}(0)$ has index zero, it's non-degenerate critical & = the moduli space of rank 2 bundles of odd degree & fixed determinant.
4) $\mu^{-1}((d-\frac{1}{2})\pi)$ is non-degenerate critical of index $2g + 2d - 2$ & is isomorphic to a 2^{2g}-fold cover of $Sym^{2g-2d-1}\Sigma$.

More specifically, have

$$Jac(\Sigma) \longrightarrow Jac(\Sigma)$$

$$x \longmapsto 2x$$

and $S^{2g-2d-1}\Sigma \longrightarrow \mathrm{Jac}(\Sigma)$. The critical locus comes from pulling the cover back along this map.

For the non-zero type, we're seeking a gauge transformation g of form

$$g(\theta)^{-1}\,\Phi\,g(\theta) = e^{i\theta}\,\Phi$$

$$g(\theta)^{-1}\,d_A\,g(\theta) = d_A$$

$\Phi \neq 0$ means $g(\theta) \neq \mathrm{id}$ if $\theta \neq 2k\pi$.

Thus A reduces to a $U(1)$-connection.

So V splits as $V \cong L \oplus (L^* \otimes \Lambda^2(V))$ for L some line bundle. Since $g(\theta)$ is diagonal it has to act by

$$\begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}.$$

This says Φ has to be lower triangular

$$\begin{pmatrix} 0 & 0 \\ \phi & 0 \end{pmatrix} \text{ for } \phi \in \Gamma(\Sigma, L^{-2} \otimes K \otimes \Lambda^2 V)$$

This is where the Jacobians come in. Computations with self-duality equations give

$$d \leq g-1$$

We get

$\mu^{-1}((d-\frac{1}{2})\pi)$ = moduli of pairs $(V = L \oplus L^*(1))$ where L has degree d, ϕ has zeros a degree $(2g-2d-1)$ divisor.

pick a $\mathcal{U}$ & section $\hat{\Phi} \in \Gamma(U^{\cdot} \otimes K \otimes \Lambda^2(V))$. This gives the cover.

Given these explicit descriptions of the spaces, get

<u>Theorem</u> (Hitchin)

M is non-compact, connected, simply-connected, with Poincaré polynomial

$$P_t(M) = \frac{(1+t^3)^{2g}}{(1-t^2)(1-t^4)} - \frac{t^{4g-4}}{4(1-t^2)(1-t^4)}\Big[\big((1+t^2)^2(1+t)^{2g}$$
$$-(1+t)^2(1+t)^{2g} + 2^{2g-1}t^{4g-4}$$
$$\big((1+t)^{2g-2}-(1-t)^{(2g-2)}\big)\Big]$$

So for $g=2$

$$P_t(M) = 1 + t^2 + 4t^3 + 2t^4 + 34t^5 + 2t^6$$

<u>work</u> of Hausel & Rodriguez-Villegas. They use mixed <u>Hodge</u> <u>polynomials</u>

$$\mathcal{H}(M_n, x, y, t) = \sum h^{p,q;j} x^p y^q t^j.$$

$h^{p,q;j} = \dim_{\mathbb{C}}\big(Gr^F_p Gr^W_{p+q} H^j(M_n)\big)$

One can recover the Poincaré polynomial by putting $x = y = 1$.

<u>Conjecture</u> (Hausel, Rodriguez-Villegas)

$$H(M_n; q, t) = \frac{(t\sqrt{q})^{dn}}{H_n(\sqrt{q}, -\frac{1}{t\sqrt{q}})} \qquad x = y = \sqrt{q}$$

where H_n comes from symmetric functions.

This should recover & generalise Hitchin's result, which is for rank 2 only.

Sheel

Model Categories, Simplicial Sets & Simplicial Presheaves

Goal.

Understand homotopy theory for (higher) stacks.

First, consider the abstract situation where we have a category $\mathcal{C}$ & notion Σ of "weak equivalences".
e.g. $\mathcal{C} = \mathrm{Top}$, $\Sigma = \{ \text{maps } X \xrightarrow{f} Y \text{ s.t. } \pi_* X \xrightarrow[\sim]{f_*} \pi_* Y \}$.
& produce $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[\Sigma^{-1}]$

Motto: • Model categories are a tool for constructing / analysing category $\mathrm{Ho}(\mathcal{C})$.
• Simplicial model categories are similar for the $(\infty, 1)$-category $\mathrm{Ho}(\mathcal{C})$

Definition:

A model category is a category $\mathcal{M}$ with 3 distinguished classes of morphisms

$\cdot \xrightarrow{\sim} \cdot$ weak equivalences, $\cdot \hookrightarrow \cdot$ cofibrations, $\cdot \twoheadrightarrow \cdot$ fibrations

e.g. $\mathcal{M} = \mathrm{Top}$, weak equivalences = usual iso on π_*
Fibrations = Serre Fibrations
Cofibrations = ?

Satisfying. MC1: $\mathcal{M}$ is closed & coclosed
MC2: If 2 of $\{f, g, fg\}$ are in a class so is the third.

MC3: each class is closed under retracts

MC4: (lifting). If we have a diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the lift exists if i or p is a weak equivalence (call it an <u>acyclic</u> or <u>trivial</u> (co) fibration.

<u>e.g</u>: For spaces this is like homotopy lifting. the definition of Serre Fibration

MC5: (Factorisation) any morphism $X \to Y$ can be factored (non-uniquely) as a cofibration followed by an acyclic fibration or acyclic fibration followed by a cofibration.

Immediately there's some things we can do. Call the initial / terminal objects $\emptyset$ & $*$.

<u>Definition:</u>

An object X is <u>fibrant</u> if unique $X \to *$ is a fibration, or <u>cofibrant</u> if unique $\emptyset \to X$ is a cofibration.

<u>Definition:</u>

A <u>cofibrant replacement</u> for X is a factoring

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[dr, hook] & X \\ & QX \arrow[u, "\sim"'] \end{tikzcd}$$

(so Qx is cofibrant)

A **Fibrant replacement** is a factoring

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\sim"'] & * \\ & RX \arrow[u] \end{tikzcd}$$

(so RX is fibrant

They exist & are non-unique.

Exercise:

Weak equivalences & cofibrations determine fibrations & vice versa. i.e. $f: X \to Y$ is a fibration iff

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Z \arrow[d, "\wr"] \\ Y \arrow[r] \arrow[ur, dashed] & W \end{tikzcd}$$

$\exists$ lift in all

Homotopy Category

In a model category $\exists$ a notion of homotopy (in fact two) for maps $f, g: X \to Y$, via "cylinder objects" generalising $X \times I$ or "path objects" generalising $Map([0,1], Y)$.

Claim: For objects that are fibrant and cofibrant, these notions of homotopy are the same, & give an equivalence relation.

Definition:

The homotopy category $Ho(\mathcal{M})$ has

$$ob\ Ho(\mathcal{M}) = ob\ \mathcal{M}$$

& $$Hom_{Ho(\mathcal{M})}(X, Y) = [RQX, RQY]$$

homotopy classes of maps between fibrant cofibrant replacements.

It turns out this is independent of all choices.

Derived Functors

If $F : \mathcal{M} \to \mathcal{A}$, $\mathcal{M}$ a model category, s.t.

$\left\{ \begin{array}{l} \text{weak equivalences} \\ \text{between cofibrant objects} \end{array} \right\} \longmapsto \{\text{isomorphisms}\}$ (left exact)

then F factors through

$LF : Ho(\mathcal{M}) \longrightarrow \mathcal{A}$. LF is a <u>left derived functor</u>.

If $\mathcal{A}$ is a model category too, then there's a notion of a <u>total left derived functor</u>:

$$LF : Ho(\mathcal{M}) \longrightarrow Ho(\mathcal{A})$$

If F sends weak equivalences between cofibrants to weak equivalences.

<u>Example</u>:

Chain complexes of R-modules in degree ≥ 0, $Ch_R^{\geq 0}$, has various model structures e.g

<u>Projective model structure</u>

i) weak equivalences are isos on H_*

ii) cofibrations are injective maps with projective cokernel. (so cofibrant objects are projective).

Then in fact

$$Hom_{Ho(Ch_R^{\geq 0})}(A[n], B[m]) = Ext^{n-m}(A, B).$$

Homotopy Limits (& colimits)

Limits (& colimits) of diagrams are not homotopy invariants.

We want to fix this with model categories.

To do this, let I be a finite category, e.g $(\bullet \to \bullet \leftarrow \bullet)$. Form a diagram category

$\mathcal{M}^I := \text{Functors}(I, \mathcal{M})$.

So objects are diagrams
$$\begin{tikzcd} & X \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}.$$

There's a diagonal functor
$$\Delta : \mathcal{M} \longrightarrow \mathcal{M}^I$$
$$\begin{tikzcd} & & X \arrow[d, equal] \\ X \arrow[r, mapsto] & X \arrow[r, equal] & X \end{tikzcd}$$

If Δ has a right adjoint, we call it the limit.

Exercise: This agrees with the universal property definition.

To produce the homotopy limit functor:

Claim:

Sometimes (e.g. if I is finite), there's an induced model structure on $\mathcal{M}^I$. Weak equivalences are pointwise weak equivalences
$$\begin{tikzcd} & X \arrow[d] \arrow[r, "\sim"] & X' \arrow[d] \\ Z \arrow[r] \arrow[dr, "\sim"'] & X \arrow[r, "\sim"] & X' \\ & Z' \arrow[ur] & \end{tikzcd}$$

Cofibrations are also pointwise (though fibrations won't be generally).

In this situation, $\lim_I : \mathcal{M}^I \to \mathcal{M}$ satisfies the conditions to have a total right derived functor
$$\operatorname{holim}_I : Ho(\mathcal{M}^I) \longrightarrow Ho(\mathcal{M}).$$

Simplicial Sets

Moral: These are combinatorial models for spaces.

Definition:

The simplex category Δ has

objects: finite sets $[n] = \underset{0}{\cdot} \to \underset{1}{\cdot} \to \cdots \to \underset{n}{\cdot}$

morphisms: order preserving morphisms of sets.

There are some distinguished morphisms.

Face maps $d_i : [n] \to [n-1]$

$(0 \to \cdots \to n) \longmapsto (0 \to \cdots \to i-1 \to i+1 \to \cdots \to n)$

degeneracy maps $s_i : [n] \to [n+1]$

$(0 \to \cdots \to n) \longmapsto (0 \to \cdots \to i \to i \to \cdots \to n)$

The simplicial category is uniquely characterised by these and some relations called simplicial identities

e.g. $d_i d_j = d_{j-1} d_i$ for $i < j$

Definition:

A simplicial set is a functor $F : \Delta^{op} \to Set$

The n-simplices F_n are just $F([n])$.

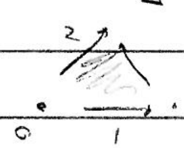
e.g.: X a topological space, $S(X)$ singular chains:

$S(X)_n = Hom(|\Delta^n|, X)$, $|\Delta^n|$ the topological n-simplex

There are standard n-simplices

$$\Delta^n := Hom_{\Delta}(-, [n])$$

think of e.g.



Definition:

The k^{th} horn Λ^n_k is generated by all faces of Δ^n

except the k^{th}.

e.g.:

$\Lambda^2_2 =$

$\Lambda^2_1 =$

Definition:

A map $X \longrightarrow Y$ of simplicial sets is a Kan fibration if $\forall$ diagrams & $\forall k, n$

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

the lift exists.

This should be like a Serre fibration:

$$\begin{tikzcd} |\Lambda^n_k| \arrow[r] \arrow[d] & Z_{top} \arrow[d] \\ |\Delta^n| \arrow[r] \arrow[ur, dashed] & W_{top} \end{tikzcd}$$

A Kan complex (fibrant object) is a Kan fibration $Y \longrightarrow *$, i.e.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & Y \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

lift always exists.

Geometric Realisation:

This is a functor $|\cdot| : sSet \longrightarrow Top$, defined as

$$|F| = \coprod F_n \times |\Delta^n| / \sim$$

where $\sim$ are gluing relations, $(a_i, d_i^{|\Delta^n|}(t)) \sim (d_i a, t)$

Now we can produce the model structure. A map $X \longrightarrow Y$ of sSets is a weak equivalence if $|X| \longrightarrow |Y|$ is a weak equivalence. Fibrations are Kan fibrations.

Remarks:

- There's a characteration of acyclic cofibrations: they're anodyne extensions. Cofibrations are just inclusions.
- $\exists$ a simplicial notion of homotopy: just

$$\begin{tikzcd} X \times \Delta^0 \arrow[d, "0"] \arrow[dr, "f"] & \\ X \times \Delta^1 \arrow[r, "h"] & Y \\ X \times \Delta^0 \arrow[u, "1"'] \arrow[ur, "g"'] & \end{tikzcd}$$

This is not an equiv rel - in general, but is for Kan complexes.

So we've produced adjoint functors

$$|\cdot| : sSet \rightleftarrows Top : S.$$

They induce equivalences of homotopy categories.

Example:

$\mathcal{C}$ a category (e.g. a group). The nerve of $\mathcal{C}$
$N\mathcal{C} \in sSet$, with n-simplices

$$\text{Functors}\ (\underset{0}{\bullet} \to \cdots \to \underset{n}{\bullet},\ \mathcal{C})$$

so 0-simplices are objects $\bullet$

1-simplices are morphisms $\bullet \to \bullet$

k-simplices are composable chains $\bullet \to \cdots \to \bullet$

Simplicial Model Categories

These are model categories enriched in sSet.

There's mapping objects $Map(X, Y) \in sSet$

we should ask $Map(\cdot,\cdot)$ to play nicely with (co)fibrations. $\exists$ nice constructions of holims & hocolims

Example: sSet with $Map(X, Y)_n = Hom(X \times \Delta^n, Y)$.

Simplicial Presheaves

Motivation:

Make sense of homotopy theory for (higher) sheafs.

e.g.: $H^n(X; \mathcal{F})$ "=" "$[X, K(\mathcal{F}, n)]$"

Let $(\mathcal{C}, \mathbb{T})$ be a site (with a Grothendieck topology $\mathbb{T}$).

e.g. $\mathcal{C} = \{$open sets of a space$\}$

$\mathbb{T} = \{$ collections of covers $\{i_\alpha : U_\alpha \to U\}$ $\}$

Have presheaves: functors $\mathcal{C} \to Set$ or $Grpd$

We have descent if for any cover $\{U_\alpha \to U\}$ then

$$F(U) \xrightarrow{\sim} \lim\left(\prod F(U_\alpha) \rightrightarrows \prod F(U_\alpha \cap U_\beta)\right)$$

Definition:

Simplicial presheaves are functors $\mathcal{C} \to sSet$

Simplicial sheaves are those satisfying descent.

For a cover $\{U_\alpha \to U\}$ require

$$F(U) \xrightarrow{\sim} \operatorname{Holim}\left(\prod(F(U_\alpha)) \rightrightarrows \prod F(U_\alpha \cap U_\beta) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\right)$$

So have $sSh(\mathcal{C}) \xhookrightarrow{i} sPSh(\mathcal{C})$. i has a left adjoint, called <u>sheafification</u>.

<u>Global model structure</u> comes from thinking of simplicial presheaves as diagram categories (<u>pointwise</u> w.e.'s & cofibrations)

<u>Local model structure</u> (Jardine): Morally <u>weak equivalences</u> are local equivalences on stalks. Really, have $F \to G$, $\pi_* F \to \pi_* G$ presheaves. Sheafify to get $\tilde{\pi}_* F \xrightarrow{\sim} \tilde{\pi}_* G$. <u>Cofibrations</u> are pointwise.

<u>Theorem</u>: (Hollander, Dugger-Hollander-Isaksen)
Fibrant objects are exactly sheaves.

Morey

Perverse Sheaves:

The focus will be the 2010 paper of de-Cataldo, Hausel & Migliorini.

Let C be a non-singular complex projective curve. Have moduli spaces $\mathcal{M}_{Dol}$ & $\mathcal{M}_B$: both q uasi-projective & non-singular. Have Hitchin map

$$\mathcal{M}_{Dol} \longrightarrow \mathbb{C}^d$$

whose fibres are abelian varieties. $\mathcal{M}_B$ is affine, & $\mathcal{M}_{Dol}$ is not.

NAHT says $\exists\, \phi : \mathcal{M}_{Dol} \longrightarrow \mathcal{M}_B$ (not algebraic) which gives an isomorphism ϕ^* on singular cohomology:

$$\underbrace{H^*(\mathcal{M}_{Dol};\mathbb{Q})}_{\text{pure MHS}} \xrightarrow{\phi^*} \underbrace{H^*(\mathcal{M}_B;\mathbb{Q})}_{\text{not pure MHS.}}$$

Call the weight filtration W_{Dol} & W_B

Theorem (dCHM)

In cases $G = GL(2)$, $PGL(2)$, $SL(2)$, ϕ^* identifies W_B with the perverse Leray filtration on $H^*(\mathcal{M}_{Dol};\mathbb{Q})$.

Whether this works in higher dimensions is unknown.

Leray Filtration

If $f: X \to Y$ is a fibre bundle, Y an n-dim cell complex let $Y_* = \{Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n\}$ be the filtration on skeleta inducing $X_* = f^{-1}(Y_*)$.

The <u>Leray filtration</u> is

$$L = L_i H^j(X, \mathbb{Z}) = \ker\left(H^j(X; \mathbb{Z}) \to H^j(X_{j-i-1}; \mathbb{Z})\right)$$

the filtration used for the Leray spectral sequence

$$E_2^{p,q} = H^p(Y, R^q f_* \mathbb{Z}) \Rightarrow Gr_L^p H^{p+q}(X; \mathbb{Z}).$$

<u>Definition</u>

A <u>stratification</u> of a space X is a finite set of subspaces S such that
1) $X =$ disjoint union of $s \in S$
2) each $s \in S$ is a manifold
3) $\overline{s}$ is the union of strata.

If you have a <u>sheaf</u> $\mathcal{F}$ on stratified X, call it <u>constructible</u> wrt S if for all $s \in S$, $\mathcal{F}|_s$ is locally constant.

A complex $\mathcal{F}^\bullet$ of sheaves is constructible if each cohomology sheaf $\mathcal{H}^i(\mathcal{F}^\bullet)$ is constructible.

<u>Definition</u>:

Let X be a stratified space with strata S. A <u>perversity</u> p is a map $S \xrightarrow{p} \mathbb{Z}$.

Denote the inclusion $\iota_s : s \hookrightarrow X$ thus.

<u>Definition</u>

The <u>perverse t-structure</u> on the bounded derived category is

$${}^pD^{\leq p} = \{ \mathcal{F}^\bullet : \iota_s^{-1} \mathcal{F} \in D^b(s)^{\leq p(s)} \ \forall s \in S \}$$

$$^pD^{\geq 0} = \{ \mathcal{F} : \iota_s^! \mathcal{F} \in D^b(s)^{\geq p(s)} \quad \forall s \in S \}$$

A perverse sheaf on X wrt perversity p is a constructible sheaf in the heart of the t-structure: $^pD^{\geq 0} \cap {^pD^{\leq 0}}$.

Definition:

Let $\mathcal{P}_X$ be the abelian category of perverse sheaves.

Definition:

For $H^*(Y, \mathcal{K})$, $\mathcal{K} \in D(Y)$ a complex of constructible sheaves, there is a perverse spectral sequence of a filtered complex $(R\Gamma(Y, \mathcal{K}), P)$.

P is the perverse filtration on $H^*(Y, \mathcal{K})$:

$$P^p H^*(Y, \mathcal{K}) = \mathrm{Im}(H^*(Y, \tau^{\leq -p}\mathcal{K}) \to H^*(Y, \mathcal{K}))$$

where $\tau^{\leq -p}$ is the left adjoint of $^pD^{\leq -p} \hookrightarrow D$: like a perverse truncation functor.

The spectral sequence is then

$$E_2^{p,q} = H^{p+q}(Y, {^p\mathcal{H}^p}(\mathcal{K})) \Rightarrow R\Gamma(Y, \mathcal{K})$$

Example:

Let $f : X \to Y$ be a smooth proper map of varieties, $C \in D(X)$, & $\mathcal{K} = Rf_* C$, then we recover the usual Leray spectral sequence.

Intersection Cohomology

On Y a complex algebraic variety there is something called intersection cohomology $IH^*(Y;\mathbb{Q})$ such that

1) Duality holds:

$$IH^{n+j}(Y) \times IH^{n-j}(Y) \xrightarrow{\sim} \mathbb{Q} \quad \text{perfect pairing}$$

2) There is a constructible complex of sheaves of $\mathbb{Q}$-vector spaces IC_Y s.t.

$$IH^j(Y;\mathbb{Q}) = H^{j+n}(Y, IC_Y)$$

Example:

If Y is smooth, $IH^*(Y,\mathbb{Q}) \cong H^*(Y,\mathbb{Q})$.

$$IC_Y = \mathbb{Q}_Y[n].$$

Theorem: (Beilinson, Bernstein, Deligne, Gabber). (Decomposition)

Let $f: X \to Y$ be a proper map. Then

$$f_* IC_X \cong \bigoplus_{b\in B} IC_{Z_b}(L_b)[d_b].$$

B is a uniquely determined set of triples $\{(Z_b, L_b, d_b)\}$, $Z_b \subseteq Y$ closed irreducible

$L_b \neq 0$ simple local system on Z_b°

$d_b \in \mathbb{Z}$

Outline of proof of dCHM

① Use the Hitchin map $h: M_{Dol} \to \mathbb{A}^d$ to give a refined decomposition theorem. (Use Ngô's support theorem) to get

$$h_{u*}\mathbb{Q}_{M_u}[2d] \cong \bigoplus_{i=-d}^{d} IC_u R^{i+d}[-i]$$

where $u \subseteq \mathbb{A}^d$

② Define a perversity p so that some element $x \in P_{j-b-1} H^j(M_{Dol};\mathbb{Q})$ iff it vanishes when restricted to the fibre $h^{-1}(U)$. $U \subseteq \mathbb{A}^d$ here is chosen by taking general linear subspace $\Lambda^b \subseteq \mathbb{A}^d$

③ Use polynomials from earlier (Poincaré polynomials etc) to complete the proof: relabelling gives equalities of the filtrations.

Lecture 17

Carlos Simpson

Higher Non-abelian Hodge Theory

Cohomology with Coefficients

Classically, if X is a space & L is a local system, on the one hand you can think of its monodromy rep $\pi_1(X, x) \to GL(n)$, or on the other hand as a locally constant sheaf. In this situation, you can look at $H^i(X; L)$.

In algebraic geometry, this comes up in the Leray spectral sequence: if $X \to Y$ is smooth & proper,

$$E_2^{j,k} = H^j(Y; R^k f_* \mathbb{C}) \Rightarrow H^{j+k}(X; \mathbb{C})$$

L will be a VHS. L" (In fact this is a spectral sequence of mixed Hodge structures).

L can vary in a family. Then we expect to get some kind of family of $H^i(X; L)$. How do we make sense of this - dimensions can jump!

X connected smooth projective

In the de Rham point of view, look at $\mathcal{M}_{dR}(X, n)$ the moduli stack of (E, ∇) rank n flat connections: a finite type Artin stack. One can define $R_{dR}(X, x, n) = \{(E, \nabla, \beta) : \beta : E_x \xrightarrow{\sim} \mathbb{C}^n\}$ fine moduli of framed flat bundles. This is a scheme, & $GL(n, \mathbb{C})$ acts by change of framing to give $\mathcal{M}_{dR}$ the quotient stack.

$$R_{dR} /\!\!/ _{GL_n} = \mathcal{M}_{dR}$$

We have a <u>universal family</u> $(\mathcal{E}, \nabla)$ on $\begin{array}{c} X \times \mathcal{M}_{dR} \\ \downarrow p \\ \mathcal{M}_{dR} \end{array}$.

$\mathcal{E}$ vector bundle on $X \times \mathcal{M}_{dR}$

$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_{X \times \mathcal{M}_{dR} / \mathcal{M}_{dR}}$ <u>relative connection</u>.

For a single L, corresponding to (E, ∇) on X,

$$H^i_B(X^{top}; L) \cong \mathbb{H}^i(X; E \xrightarrow{d_\nabla} E \otimes \Omega^1_X \xrightarrow{d_\nabla} \cdots \longrightarrow E \otimes \Omega^d_X)$$

In families then, we can look at the hyper derived direct image

$\mathbb{R}p_*(\mathcal{E} \otimes \Omega^\bullet_{rel}) =$

$$\mathbb{R}^\bullet p_* (\mathcal{E} \xrightarrow{d_\nabla = \nabla} \mathcal{E} \otimes \Omega^1_{X \times \mathcal{M}_{dR}/\mathcal{M}_{dR}} \xrightarrow{d_\nabla} \cdots \xrightarrow{d_\nabla} \mathcal{E} \otimes \mathcal{E}^d_{X \times \mathcal{M}_{dR}/\mathcal{M}_{dR}})$$

a complex of sheaves on $\mathcal{M}_{dR}$ (i.e. a $GL(n)$-equivariant complex on $\mathcal{R}_{dR}$).

$\mathbb{R}p_*$ is a complex of quasi-coherent $\mathcal{O}_{\mathcal{M}_{dR}}$-modules.

argument from Mumford's "abelian varieties"

Take $X = \bigcup U_i$ an affine Zariski-open cover.

Form the Čech complex

$$\check{C}(\mathcal{U} \times \mathcal{M}_{dR}; \mathcal{E} \otimes \Omega^\bullet_{rel}, d_\nabla)$$

<u>Exercise</u>: write this double complex down.

Let $\mathcal{U}_\bullet = \{U_{i_1} \cap \cdots \cap U_{i_n}\} \xrightarrow{i} X$. Take

$i_* \check{C}(\mathcal{U} \times \mathcal{M}_{dR}, \mathcal{E} \otimes \Omega^\bullet_{rel}, d_\nabla)$.

The vertical differentials are d_∇, <u>not</u> $\mathcal{O}_X$-linear, but rather $\mathcal{O}_{\mathcal{M}}$-linear.

<u>Fact</u>: $\mathbb{R}p_*(\mathcal{E} \otimes \Omega^\bullet_{rel}) \cong p_*(i_* \check{C}(\mathcal{U} \times \mathcal{M}_{dR}; \mathcal{E} \otimes \Omega^\bullet_{rel}))$ on $\mathcal{M}_{dR}$

So it's quasi-coherent, & $\mathcal{O}_{\mathcal{M}_{dR}}$-flat.

Its i^{th} hypercohomology $\mathbb{R}^i p_*(-)$ is <u>coherent</u>: by Grauert's theorem & the spectral sequence.

In this situation, Mumford explains how to get a perfect complex.

<u>Definition</u>

A <u>perfect complex</u> is a bounded complex of finite rank vector bundles.

If Z is a scheme with a map $Z \xrightarrow{\phi} \mathcal{M}_{dR}$, then

$$\phi^* \mathbb{R}p_* (\mathcal{E} \otimes \Omega^{\bullet}_{rel})$$

is quasi-isomorphic to a perfect complex on Z, though not generally canonically. Put $Z = \mathcal{R}_{dR}$: this won't necessarily be $Gl(n)$ equivariant on-the-nose.

<u>Aim</u>: Understand this better.

Mumford's argument

Call the perfect complex we've produced $\mathcal{C}^\bullet$

$$\mathcal{C}^0 \to \mathcal{C}^1 \to \cdots \to \mathcal{C}^n \to 0$$

$H^m(\mathcal{C}^\bullet)$ is coherent & each term is flat.

Locally on Z choose $\mathcal{O}_Z^a \to \mathcal{C}^m$ such that $V \twoheadrightarrow H^m$ exists for V a vector bundle.

Look at

$$\cdots \to \mathcal{C}^{n-2} \to \ker(V \oplus \mathcal{C}^{n-1} \to \mathcal{C}^n) \to 0$$. Call it $\mathcal{B}$.

Then $\mathcal{B} \to V[-m] \to 0$ is quasi-isomorphic to $\mathcal{C}$.

But $\mathcal{B}$ is a complex of shorter length than $\mathcal{C}$, so $\mathcal{B}$ quis to a perfect by induction, so produce $\mathcal{C}$ quis to a perfect.

Now we have a "perfect complex" $\mathcal{H}^\bullet$ on $\mathcal{M}_{dR}$, up to some homotopy, with the property that if $\rho \in \mathcal{M}_{dR}$, $\mathcal{H}^\bullet_\rho$ = complex calculating $H^\bullet(X, L_\rho)$.

Over some open set $\mathcal{U} \subseteq \mathcal{M}_{dR}$, the dimensions will all be constant, & $\mathcal{H}^i(\mathcal{H}^\bullet)$ will be vector bundles : the family of $H^i(X; L)$.

First case of this work of Green-Lazarsfeld on cohomology jump Loci. They did this for rank 1 local systems. A goal is to understand the NAHT of these kinds of objects.

Lecture 18

Peter Dalakov

Mixed Hodge Structures

We'll talk about some results of R. Hain

Motivation:

Let X be a smooth proper variety $/\mathbb{C}$, then we can look at its moduli of deformations of X by

i) Find a topological invariant, e.g. $H^k(X;\mathbb{Z})$. finitely generated abelian group

ii) Put a Hodge filtration on it, & see how the filtration varies with the complex structure. For this we look at periods.

Question: How can we do the same thing for pointed varieties (X, x_0)?

Natural candidate for an invariant is $\pi_1(X, x_0)$, but it's not abelian. Try its group ring $\mathbb{Z}\pi_1(X, x_0)$, which is abelian but not finitely generated. Then try truncating $\mathbb{Z}\pi_1(X, x_0)/J^{k+1}$, J augmentation ideal.

What is the Hodge filtration analogue? Chen considered iterated integrals $\int_\gamma \omega_1 \omega_2 \cdots \omega_r$, where γ is a cycle i.e.

$$\int \cdots \int_{0 \le t_1 \le \cdots \le t_r \le 1} \gamma^*\omega_1(t_1) \cdots \gamma^*\omega_r(t_r)\, dt_1 \cdots dt_r$$

Chen's π_1 Theorem:

Let $H^0(I_x)$ be the set of iterated integrals depending only on the homotopy class. Then

1) $H^0(I_X)_s \cong \mathrm{Hom}(\mathbb{Z}\pi_1(X, x_0)/J^{s+1}, \mathbb{Z})$ elements with at most s differentials

2) $H^0(I_X) \cong \mathcal{O}(\mathcal{U})$
as Hopf algebras. $\mathcal{O}(\mathcal{U})$ is the coordinate ring of the unipotent completion of π_1.

Using this, Hain proved the following

Theorem 1:

Let X be smooth complex algebraic (or a complement of a simple normal crossings divisor in a compact Kähler manifold). Pick $x_0 \in X$. Let V be an admissible VHS on X, and let $S = \mathrm{Aut}(V_0, \langle , \rangle)$. Suppose $\rho: \pi_1(X, x_0) \to S$ is zariski-dense, & let $\tilde{\rho}: \pi_1(X, x_0) \longrightarrow \mathcal{G} = \mathcal{G}(X, x_0)$ be the "completion relative to ρ".

Then $\mathcal{O}(\mathcal{G})$ has a canonical MHS with non-negative weights, such that the antipode, multiplication & comultiplication are morphisms of MHSs, as is the inclusion $\mathcal{O}(S) \hookrightarrow \mathcal{O}(\mathcal{G})$.

Let's explain some of these ideas.

Recall:

A Hopf algebra H over k is k-algebra with $m: H \otimes H \to H$, with comultiplication $\Delta: H \to H \otimes H$, counit $H \to k$, and antipode $H \to H$, with properties that will follow this example:

Example:

$\mathcal{O}(G)$, the ring of functions on a group G,

with comultiplication = the diagonal $\Delta f(g \otimes h) = f(gh)$

counit = evaluation at e

antipode = inverting the argument.

Fact:

If G is an algebraic group, you can recover $\mathfrak{g} = \mathrm{Lie}(G)$ from $\mathcal{O}(G)$.

Relative Malcev Completion

Data: an abstract group π

a linear algebraic group S/k char $k=0$

$\rho: \pi \to S$ a representation with Zariski-dense image.

Look at diagrams of the form

$$\begin{tikzcd} 1 \arrow[r] & U \arrow[r] & E \arrow[r] & S \arrow[r] & 1 \\ & & \pi \arrow[u, "\tilde{\rho}"] \arrow[ur, "\rho"'] & & \end{tikzcd}$$

where U is unipotent, E linear algebraic.

The relative Malcev completion of π relative to ρ is the limit $\mathcal{G} = \varprojlim E$ over such diagrams, i.e. the universal such E.

(In case $S = \mathrm{id}$ for simplicity) There is an explicit construction of such completions, by completing the group ring & taking group-like elements: consider $\hat{k\pi}$ with comultiplication Δ induced from comultiplication on $k\pi$.

$$\mathcal{G} = \{ x \in \hat{k\pi} : \Delta(x) = x \otimes x,\ \varepsilon(x) = 1 \}$$