Chris Elliott

TALBOT NOTES 2

D. Toledo

Affine Cubic Surfaces & Relative SL(2) Character Varieties

(Joint work with Bill Goldman)

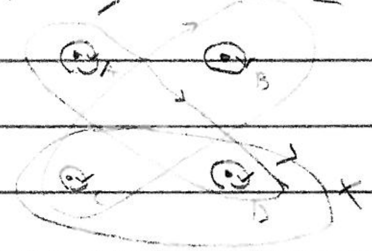
Fricke – Klein: (Vol II p. 270)

If you take representations

$$\{ \pi_1(S^2 \setminus 4 \text{ points}) \xrightarrow{\rho} SL(2,\mathbb{C}) \} / SL(2,\mathbb{C})$$

This is a codim 1 variety $V \subseteq \mathbb{C}^7$ given explicitly.

Consider loops



Put $a = Tr(\rho(A))$

$\vdots$

$z = Tr(\rho(Z))$

Then they satisfy

$$xyz + x^2 + y^2 + z^2 = (ab + cd)x + (bc + ad)y + (ac + bd)z + (4 - a^2 - b^2 - c^2 - d^2 - abcd)$$

& this cuts out V in $\mathbb{C}^7$.

Write this as $xyz + x^2 + y^2 + z^2 = px + qy + rz + s$

p, q, r quadratics in (a,b,c,d), s quartic.

Fixing p, q, r, s, get an affine cubic surface $V_{p,q,r,s}$.

Theorem:

The map $\mathbb{C}^4 \longrightarrow \mathbb{C}^4$ is surjective.

$$(a,b,c,d) \longmapsto (p,q,r,s)$$

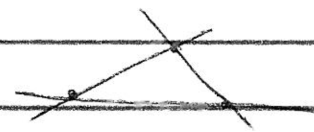
Thus all such cubic surfaces come from fixing traces as the punctures to looking at the moduli of representations.

__Cubic Surfaces__ (in $\mathbb{P}^3$: recover affine by fixing a hyperplane section).

1) The moduli space of cubic surfaces is 4-dimensional.

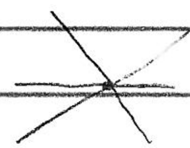
2) There are 27 __lines__ on a smooth cubic surface

3) There are 45 __tritangent planes__:



plane tangent to 3 points of the form.

or non-generically



__Eckardt points__

4) One can view the moduli space as

$$\mathcal{M} = (\mathbb{B}^4 \setminus \{\underset{\text{nodal surfaces}}{\text{hyperplanes}}\}) \big/ _{\text{discrete group}}$$

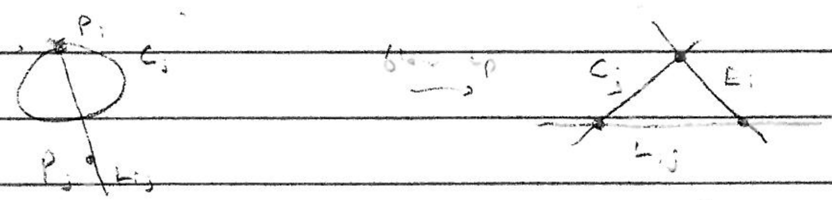
surfaces with Eckardt points come from Cartier divisors $\mathbb{B}^3 \subseteq \mathbb{B}^4$: more such $\mathbb{B}^3$'s you land in, the more Eckardt points you have, with the __Fermat cubic__ achieving the maximal number: 18. Thus every smooth cubic surface has generic tritangents, indeed, at least 27 of them.

__Fact__: Every smooth cubic surface arises from blowing up 6 points generically: no 3 colinear & not all 6 on a conic.

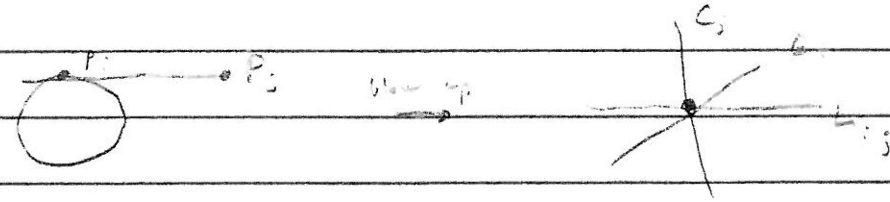
Given 5 points there is a conic through them

So look at the exceptional divisor E_j corresponding to p_j. This gives 6 lines that don't intersect. Furthermore there are lines through pairs: L_{ij} through p_i & p_j, & conics C_j through $\{p_i, i \neq j\}$. This gives $6 + 15 + 6 = 27$ lines.

Tritangents come from



or

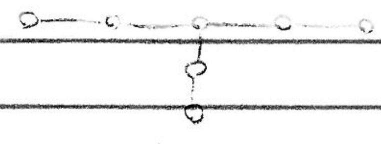


Now, assume the theorem on surjectivity. $V_{p,q,r,s}$ has a generic tritangent plane at ∞, & no singular points at ∞. Thus by this process every smooth cubic surface $\subset \mathbb{P}^3$ arises from a $V_{p,q,r,s}$.

What about singular surfaces? Which ones can arise? It turns out they're precisely those with the above conditions at infinity: non-singular generic tritangent.

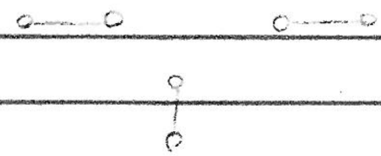
What singularities of the relative character varieties $V^{\circ}_{a,b,c,d}$ can occur?

In the study of symmetries of cubic surfaces, the Weyl group $W(E_6)$ appears. The singularities (isolated) of cubic surfaces were first classified in 1863 by Schläfli. In modern language, draw the affine Dynkin diagram of E_6:



The possible singularities correspond to the possible subdiagrams.

e.g.



would be $3A_2$ diagrams

- A_1, $2A_1$, $3A_1$, $4A_1$, A_2, A_3, D_4 can all be found in our $V_{p,q,r,s}$ cubic surfaces. Schläfli implies that in this restricted case this is all that can occur.

To prove the theorem, it suffices to show the map Φ is proper. Hence $\deg \Phi$ is defined. It's not zero, holomorphicity $\Rightarrow$ it is positive, so the map is surjective.

So, to prove properness, look at all differences & sums

$$(p-q) = (a-c)(b-d) \qquad (p+q) = (a+c)(b+d)$$
$$(q-r) = (b-a)(c-d) \qquad \vdots$$
$$(r-p) = (a-d)(b-c)$$

Suppose $|p|, |q|, |r| \leq C$. Then at least 3 of $\{a,b,c,d\}$ are within bounded distance of each other.

Case 1: $|a-d|$ bounded $\Rightarrow$ they're all bounded

Case 2: $|a-d|$ not bounded. Use the 4^{th} powers in S to get a contradiction of the above boundedness remark.

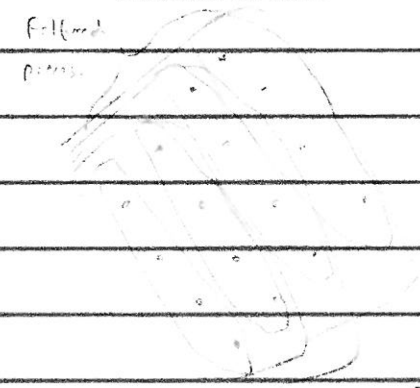
Alex

Hodge Theory

① A pure weight k Hodge structure is a finitely generated $\mathbb{Z}$ module $V_{\mathbb{Z}}$ with a filtration of $V_{\mathbb{C}} = V_{\mathbb{Z}} \otimes \mathbb{Z}$: $F^{\bullet}$ decreasing exhaustive s.t. $V_{\mathbb{C}} = F^{p} \oplus \overline{F^{k+1-p}}$ $\forall p$.

Example:

The usual Hodge diamond $H^{p,q}$ for a Kähler manifold.



Equivalently, set $H^{p,q} = F^{p} \cap \overline{F^{k-p}}$ $(p+q = k)$

So $F^{p} = \bigoplus_{p' \geq p} H^{p',q'}$ $\quad V_{\mathbb{C}} = \bigoplus_{p+q=k} H^{p,q}$

Definition.

A polarisation is a Hermitian inner product such that $(-1)^{p} \langle -, \bar{-} \rangle > 0$ on $H^{p,q}$.

Recall.

(X, h) is Kähler if $d\omega = 0$, where $\omega = \operatorname{Im}(h)$. Equivalently, we can write $h = \mathrm{Id} + O(z^2)$: osculating to euclidean metric to order 2.

Let L be the Lefschetz operator $- \wedge \omega$, with formal L^2-adjoint $L^* = \Lambda$.

(2) Kähler identities:

$$\partial^* := - * \bar{\partial} * = i[\Lambda, \bar{\partial}], \quad \bar{\partial}^* = -i[\Lambda, \partial].$$

Define

$\Delta_d = dd^* + d^* d$. The Kähler identities imply

$$\Delta_d = 2\Delta_{\bar{\partial}} = 2\Delta_\partial.$$

(c) Consequently, define harmonic (p,q) forms to be

$$\mathcal{H}^{p,q} = \ker(\Delta_d) = \ker(\Delta_\partial) = \ker(\Delta_{\bar{\partial}})$$
$$\subseteq H^{p,q}_{\bar{\partial}}(X)$$

equivalently, they are those forms that are both ∂ and ∂^*-closed

Theorem

$H^k(X;\mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}_{\bar{\partial}}$, & this is a weight k Hodge structure.

This comes by identifying $H^{p,q}_{\bar{\partial}} = \mathcal{H}^{p,q}$.

Proof: Bochner-Kodaira formula $\Rightarrow$ Δ is elliptic.

Alternatively, Deligne has a proof by showing the Hodge-to-de-Rham spectral sequence degenerates.

(b) Lefschetz formula

$[\Lambda, L] = \bigoplus_{p,q} (p+q-n)\,\pi^{p,q}$, proj onto (p,q) part.

This gives sl_2 action on the Hodge diamond, and consequentially Lefschetz decomposition etc.

Definition:

A variation of Hodge structure over manifold B is a local system $V_{\mathbb{Z}} \to B$ and a filtration of $V_{\mathbb{Z}} \otimes \mathbb{C}$ by holomorphic subbundles such that fibers are Hodge structure. Furthermore it should be equipped with a connection ∇ such that

$$\nabla F^i \subseteq F^{i-1} \otimes \Omega^1_B. \quad \text{(Griffiths transversality)}$$

There is a period map: if $D = \{\text{complete flags}\}$, get a diagram for fixed $b \in B$

$$\begin{tikzcd} \tilde{B} \arrow[r] \arrow[d] & D \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & D/\pi_1(B) \end{tikzcd}$$

The point being you can identify V_b with any $V_{b'}$, $b' \in B$. Draw

$$\begin{tikzcd} \tilde{V}_{\mathbb{C}} \arrow[r] \arrow[d] & \tilde{B} \arrow[r] \arrow[d] & D \arrow[d] \\ V_{\mathbb{C}} \arrow[r] & B \arrow[r] \arrow[ur, dashed] & D/\pi_1 \end{tikzcd}$$

to get $V_{\mathbb{C}} \to D$

Also impose Lefschetz conditions satisfied by "motivic" VHSs: those coming from smooth family of Kähler manifolds.

Define a putative VHS as $R^i f_* \mathbb{C}$ on X, where $f: Y \to X$ is the family.

Harmonic theory $\Rightarrow$ these are C^∞ vector bundles.

What about the underlying $\mathbb{Z}$-structure?

<u>Remark</u>:

The period map forgets it, but it gives an underlying <u>rigidity</u> to the system: adds canonicity.

<u>Example</u>:

Family of genus g curves, with H^1: get <u>Siegel upper half space</u>. Period matrix is a $2g+2$ matrix of periods that one can write. So have

$$\mathbb{H}^g = \{ (I, A) : A \text{ symmetric}, \operatorname{Im} A > 0 \}$$

Then $\mathbb{H}^g / Sp(g, \mathbb{Z})$ is the period domain.

We have a holomorphic map from moduli of curves to this, uniquely defined up to changing the symmetric basis.

$$\mathcal{M}_g \xrightarrow{\text{hol}} \mathbb{H}^g / Sp(g, \mathbb{Z})$$

Dario

<u>Mixed Hodge Structures</u>

A <u>mixed Hodge structure</u> (MHS) is a triple $(H, W_\bullet, F^\bullet)$, where
- H is a finite type $\mathbb{Z}$-module
- $W_\bullet$ is an increasing filtration on $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$, the <u>weight</u> filtration
- $F^\bullet$ is a decreasing filtration on $H_{\mathbb{C}} = H \otimes \mathbb{C}$

such that $Gr_F^p \, Gr_{\bar{F}}^q (Gr_n^W H_{\mathbb{C}}) = 0$ if $p + q \neq n$.

Observe subobjects & quotients of filtered objects inherit a filtration, so this makes sense.

$$F^\bullet(A/B) = F^\bullet(A)/B.$$

Example:

A split mixed Hodge structure: $H = \bigoplus H_k$ where H_k is a pure Hodge structure of weight k. This is equivalent to replacing $W_\bullet$ by a grading.

1) Why did we make this definition?
2) Can we make this more geometric? (e.g. Fano configuration)
3) Do MHSs form a category? What kind?
$\rightsquigarrow$ This leads to a Tannakian perspective

1) MHSs appear when we study cohomology of X a non-singular complex variety, maybe not compact. $H^\bullet(X;\mathbb{Z})$ carries a MHS.

Idea: Compactify X: $\exists\, \overline{X}$ a compact variety, $X \xrightarrow{j} \overline{X}$ such that $Y = \overline{X} \setminus X$ is a normal crossings divisor: say $Y \hookrightarrow \overline{X}$ looks analytic locally like $\{z_1 \dots z_k = 0\} \hookrightarrow \{(z_1, \dots, z_n)\} = \mathbb{C}^n$.

Fact: $H^\bullet(X;\mathbb{C}) \cong H^\bullet(\overline{X}, \Omega^\bullet_{\overline{X}}(Y))$

"logarithmic de Rham cx"

$\Omega^\bullet_{\overline{X}}(Y)$ is the locally free sheaf generated by $\Omega^\bullet_{\overline{X}}$ & $\dfrac{dz_i}{z_i}$.

gives → F
filtration on RHS → W

Observe this comes naturally with <u>two filtrations</u>

- $W_n(\Omega^\bullet_{\bar{X}}(Y)) = \{$ forms with $\leq n$ occurrences of the $dz/z \}$.
- $F^p(\Omega^\bullet_{\bar{X}}(Y)) = \{k$ forms with $k \geq p\}$

This gives spectral sequences

- ${}_F E_1^{p,q} = H^q(\bar{X}, \Omega^p_{\bar{X}}(Y)) \Rightarrow H^{p+q}(X; \mathbb{C})$
- ${}_W E_1^{p,q} \rightsquigarrow E_2^{p,q} = H^p(\bar{X}, R^q j_* \mathbb{C}) \Rightarrow H^{p+q}(X; \mathbb{C})$

"do some manouvers" to get Leray spectral sequence.

Theorem: (Deligne).

$(H^n(X, \mathbb{Z}), W[n]_\bullet, F^\bullet)$ is a MHS.

The Leray SS filtration is topological, so descends to a filtration on $H^n(X; \mathbb{Q})$.

Lecture 12

...he Skirvin

Factorisation Theorems & the Shafarevich Conjecture

Conjecture (Shafarevich)

Let X be a smooth complex projective variety. Then the universal cover $\tilde{X}$ is holomorphically convex.

Definition/Lemma

A complex analytic space S is holomorphically convex if either:

1) For any sequence (x_n) of points on S which doesn't have a limit point, then there's a function f unbounded on (x_n)

ii) $\exists$ a normal Stein space $Sh(S)$ and a proper map $S \longrightarrow Sh(S)$ with connected fibres.

Easy cases

1) If X is a curve, then we can apply the uniformisation theorem to see $\tilde{X} = \mathbb{P}^1, \mathbb{C}, \Delta$ with $Sh(\tilde{X}) = pt, \mathbb{C}, \Delta$.

2) If $\pi_1(X)$ is finite then $\tilde{X}$ is compact, so there's nothing to check.

Example

Hartogs' theorem implies the 2-ball $\setminus$ the origin is not holomorphically convex.

Kollar noticed a reformulation of (1): say

- $\exists$ a Stein space $Sh(\tilde{X})$
- $\exists$ a normal variety $Sh(X)$ such that $Sh(\tilde{X}) = \widetilde{Sh(X)}$ roughly and a proper map $X \to Sh(X)$ which connected fibres such that a subspace $Z \subseteq X$ is contracted to a point iff the image

Res: resolution

$$\mathrm{Im}\,(\pi_1(\mathrm{Res}(Z)) \to \pi_1(X)) \quad \text{is finite}$$

The map $X \to Sh(X)$ is known as the Shafarevich morphism.

Example:

We can construct the Shafarevich morphism if $\pi_1(X)$ is abelian. Let $\mathrm{Alb}(X)$ be the Albanese variety

$$\mathrm{Alb}(X) = H^0(X, \Omega^1_X)^* / H_1(X, \mathbb{Z}).$$

If we fix $x_0 \in X$ & a basis of 1-forms $\omega_1, \ldots, \omega_n$ then we get a map

$$X \longrightarrow \mathrm{Alb}(X)$$

$$x \longmapsto \left(\int_{x_0}^{x} \omega_1, \ldots, \int_{x_0}^{x} \omega_n \right)$$

Here, there exists Stein factorisation

$$X \xrightarrow{a_X} A(X) \to \mathrm{Alb}(X)$$

where a_X has connected fibres, $A(X)$ is normal and if $Z \subseteq X$, $a_X(Z) = pt \iff$

$$\mathrm{Im}(H_1(\mathrm{Res}(Z)) \to H_1(X))$$

is finite. So π_1 abelian $\Rightarrow \pi_1 \cong H_1$.

Kollar constructed a rational Shafarevich map, in general, which is slightly weaker, & not everywhere defined.

Hodge Theory is used to prove some powerful results.

Theorem:

If there exists a faithful representation $\pi_1(X) \to GL(n, \mathbb{C})$ then the Shafarevich morphism exists.

Theorem:

If $\pi_1(X)$ is nilpotent then the Shafarevich conj. holds.

Theorem:

If $\pi_1(X)$ is linear then the Shafarevich conj. holds.

The kind of facts one uses: given a representation $\rho : \pi_1(X) \to G$ with Zariski dense image in reductive G. Then the relative Malcev completion is a pro-algebraic group G_ρ which fits into a short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & U_\rho \arrow[r] & G_\rho \arrow[r] & G \arrow[r] & 1 \\ & & \pi_1(X) \arrow[u] \arrow[ur, "\rho"'] & & \end{tikzcd}$$

where U_ρ is the unipotent radical of G_ρ. G_ρ is universal among such things.

Lie(G_ρ) carries a mixed Hodge structure, provided ρ arises from a variation of Hodge structure. Furthermore, if $\pi_1(X)$ is linear then there exists a variation of Hodge structure such that $\pi_1(X)$ embeds into the corresponding Lie algebra of the relative Malcev completion.

Chen's π_1- de Rham theorem

Recall: The ordinary de Rham theorem says

$$\Omega^k_X \longrightarrow \mathrm{Hom}(C_k(X), \mathbb{C})$$
$$\omega \longmapsto \left(\sigma \mapsto \int_\sigma \omega\right)$$
induces an isomorphism between de Rham cohomology & singular cohomology.

Iterated integrals:

Given 1-forms $\omega_1, \ldots, \omega_k$ on X, then we have a map

PX path space

$$\int \omega_1 \cdots \omega_k : PX \longrightarrow \mathbb{C}, \text{ given by}$$
$$\gamma \longmapsto \int_\gamma \cdots \int \gamma^*(\omega_1) \cdots \gamma^*(\omega_k)$$
$$= \int_0^{t_1} \cdots \int_0^{t_k} f(t_k) \cdots f(t_1)\, dt_k \cdots dt_1$$

An iterated integral of length s is a linear combination of the constant function, and functions $\int \omega_1 \cdots \omega_k$ with $k \leq s$.

Fix a base point $x \in X$, & let $B_s(x)$ denote the iterated integrals of length s defined on loops based at x. Let $H^0(B_s(x))$ = elements of $B_s(x)$ depending only on the homotopy type of the loop. This gives us a map

$$H^0(B_s(x)) \to \mathrm{Hom}(\mathbb{Z}\pi_1(X,x), \mathbb{C})$$

Theorem (Chen's π_1 de Rham)

There is an isomorphism

$$H^0(B_s(x)) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{Z}\pi_1(X,x)/J^{s+1}, \mathbb{C})$$

where J is the augmentation ideal.

We can use this to place a mixed Hodge structure on the right hand side. Roughly:

Hodge filtration: # of dz's

weight filtration: $\omega_1 \cdots \omega_n$ has weight k = # of $\frac{dz}{z}$ terms

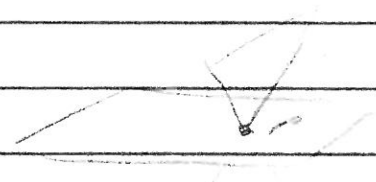
Hiro Tanaka

Singularities of $\mathrm{Rep}(\pi_1 X, G)$ and DGLAs

Lecture 13

We've been looking at the non-abelian Hodge theory from a bird's eye view. Like an eagle, we're going to come crashing down on $\mathrm{Rep}(\pi_1 X, G)$ (a look of singularities there, even before quotienting by conjugation

e.g.



two planes meeting transversally

Theorem (Goldman – Millson '88)

Let G be a real algebraic Lie group, $\Gamma = \pi_1(X, x)$ for X Kähler. If $\rho: \Gamma \to G$ has image contained in $K \subseteq G$ then $\mathrm{Rep}(\Gamma, G)$ has a quadratic singularity at ρ.

$K \subseteq G$ compact

Theorem (G – M '88)

Under the same assumptions, let ρ be the holonomy of a polarisable variation of Hodge structure. Then $\mathrm{Rep}(\Gamma, G)$ has a quadratic singularity at ρ.

Note: quadratic includes the empty set of quadratics, i.e. just a smooth point. If it's singular then the singularity is quadratic.

DGLAs:

Example:

Let P be a smooth manifold. Let $A^\bullet$ denote smooth de Rham forms. Then look at $A^\bullet(P;\mathbb{R}) \otimes \mathfrak{g}$.

Definition:

A DGLA is a triple $(L^\bullet, d, [\,,\,])$ satisfying

1) $[\,,\,] : L^i \otimes L^j \longrightarrow L^{i+j}$ is a (graded) skew-symmetric product: i.e. $[a,b] + (-1)^{|a||b|}[b,a] = 0$
2) $d : L^i \to L^{i+1}$, $d^2 = 0$, so $(L^\bullet, d)$ is a chain complex
3) d is a degree 1 derivation wrt $[\,,\,]$

Jacobi $\rightarrow$ 4) Fix $a \in L^i$. Then $[a, -]$ is a degree i derivation

Definition:

A degree i derivation is a map $\delta : L^j \longrightarrow L^{j+i}$ such that $\delta[a,b] = [\delta a, b] + (-1)^{i|a|}[a, \delta b]$

Examples:

0) Any Lie algebra $\mathfrak{g}$, placed in degree 0.
1) $A^\bullet$ a commutative dga, $\mathfrak{g}$ a Lie algebra. Then $A^\bullet \otimes \mathfrak{g}$ can be given a dgla structure by $[a \otimes v, a' \otimes v'] = aa' \otimes [v, v']$.
2) $L^\bullet$ a dgla, A a commutative (not necessarily unital) algebra. Then $L^\bullet \otimes A$ is a dgla as in 1).
3) Equivariant horizontal forms on a principal G-bundle $P \longrightarrow X$. We have the dgla $A^\bullet(P;\mathbb{R}) \otimes \mathfrak{g}$, and inside it we have $A^\bullet(X; \mathrm{ad}\, P)$: all forms ω s.t.
   - ω horizontal
   - ω equivariant: $R_g^* \omega = \mathrm{Ad}(g^{-1}) \cdot \omega$

Certainly this is a sub-graded Lie algebra, but not a dgla with differential d. Fix ω a connection on P. Define

$$d_\omega(\eta) = d\eta + [\omega, \eta]$$

This preserves being equivariant & horizontal, & if ω is flat, $d_\omega^2 = 0$.

Deformation functor associated to a dgla

Given $L^\bullet$ a dgla, we'll construct a functor

Local Artin Rings $\to$ Groupoids $\to$ Sets

Definition

An element $\eta \in L^1$ is called Maurer-Cartan if

$$d\eta + \frac{1}{2}[\eta, \eta] = 0$$

Exercise:

$\eta \in \Omega^1(X; \mathrm{ad}\, P)$ is Maurer Cartan iff $\omega + \eta$ is flat, for ω flat.

Exercise:

L^0 acts on the Maurer-Cartan elements of L^1 via

$$\eta \xrightarrow{\lambda} \exp(\mathrm{ad}\,\lambda)(\eta) + \frac{I - \exp(\mathrm{ad}\,\lambda)}{\mathrm{ad}\,\lambda} d\lambda$$

Definition:

An Artin ring is a commutative Noetherian algebra A of dimension 0.

Proposition

A a local k-Artin ring splits as $A = k \oplus \mathfrak{m}$ & $\mathfrak{m} = \sqrt{0}$ is nilpotent.

Definition:

The deformation functor associated to $L^\bullet$ sends

$$A \longmapsto DGM(L^\bullet, A) \quad \text{(Deligne-Goldman-Milson)}$$

where $\mathrm{ob}(DGM(L^\bullet, A)) = \{\text{Maurer-Cartan elements of } L^\bullet \otimes \mathfrak{m}\}$

and $\mathrm{Hom}(a, b) = \{\lambda \in L^0 \otimes \mathfrak{m} \text{ st } a \overset{\lambda}{\longmapsto} b\}$.

This is a functor $\mathrm{Artin}^{\mathrm{loc}} \to \mathrm{Gpd}$, & one maps to Sets by taking isomorphism classes: $A \longmapsto \mathrm{iso}\, DGM(L^\bullet, A)$.

One uses this functor to prove Goldman-Milson, by comparing to an infinitesimal deformation functor.

eter Dalakov

Applications of Goldman-Milson to NAHT

X will be a smooth projective variety, G a complex reductive group with Lie algebra $\mathfrak{g}$.

Examples of dglas

1) (P, ϕ), G-Higgs bundle. The controlling dgla is $L^\bullet_{Dol} = (A^\bullet(\mathrm{ad}P), \bar\partial + \mathrm{ad}\,\phi)$ with usual bracket.

e.g. on a curve this is the total complex of the double complex

$$\begin{array}{ccccc} & \longrightarrow & & \longrightarrow & \\ \bar\partial\uparrow & & \uparrow\bar\partial & & \\ A^{0,0}(\mathrm{ad}P) & \xrightarrow{\mathrm{ad}\,\phi} & A^{1,0}(\mathrm{ad}P) & \longrightarrow & \end{array}$$

2) (P, D) flat G-bundle. The controlling dgla is

$$L^\bullet_{dR} = (A^\bullet(\mathrm{ad}P), D).$$

e.g. on a curve this is the total complex of

$$\begin{array}{ccccc} A^{0,1}(\mathrm{ad}\,P) & \longrightarrow & A^{1,1}(\mathrm{ad}\,P) & \longrightarrow \\ {\scriptstyle d''}\uparrow & & \uparrow & \\ A^{0,0}(\mathrm{ad}\,P) & \xrightarrow{d'} & A^{1,0}(\mathrm{ad}\,P) & \longrightarrow \end{array}$$

d' & d'' $(1,0)$ & $(0,1)$ parts of D.

If $x \in X$, can look at $\varepsilon : A^0(\mathrm{ad}\,P) \longrightarrow \mathfrak{g}$, evaluation at x. This is a $\mathfrak{g}$-<u>augmentation</u>. A $\mathfrak{g}$-augmented dgla $(L^\cdot, d, \varepsilon)$ is also called a <u>deformation diagram</u>.

We want to know if our deformation functors are prorepresentable. Goldman–Millson proved some results. One may ask for prorepresentability of

$\mathrm{Iso}\,DGM(L^\cdot)$, $\mathrm{Iso}\,DGM(\ker \varepsilon)$ given $(L^\cdot, \varepsilon)$

or $\mathrm{Iso}\,DGM(L^\cdot, \varepsilon, -)$

where the latter is obtained by

$$\mathrm{Iso}\,DGM(L, \varepsilon, A) = DGM(L, A) \times \exp(\mathfrak{g} \otimes \mathfrak{m}) \big/ \exp(L^0 \otimes \mathfrak{m})$$

where $\exp(L^0 \otimes \mathfrak{m})$ acts on $\exp(\mathfrak{g} \otimes \mathfrak{m})$ through the augmentation ε.

We say an augmented dgla (L, ε) is <u>rigid</u> if ε embeds $H^0(L) \hookrightarrow \mathfrak{g}$.

<u>Theorem</u> (Goldman–Millson)

If (L, ε) is rigid & ε is surjective, $\varepsilon : L^0 \twoheadrightarrow \mathfrak{g}$, then $\mathrm{Iso}\,DGM(\ker \varepsilon)$ is prorepresented by a cone

Cone as in quadratic cone.

$C_{H^0(L)} \approx$ [illegible] $/ \exp(H^0)$ (or more precisely the germs at zero of this space.) Here

$$C_{H^0(L)} = \ker\left(H^1 \xrightarrow{f} H^2\right) \quad \text{for appropriate } f.$$

<u>Theorem</u>:

If we have a dgla with $d=0$ & $H^0(L)=0$, then $\mathrm{Iso}\,DGM(L)$ is prorepresented by $C_{H^0(L)}$.

<u>Caution</u>: In general, $\mathrm{Iso}\,DGM(L)$ is <u>not</u> prorepresentable, but it always admits a <u>hull</u>:

$$\ker \twoheadrightarrow \mathrm{Iso}\,(DGM(L))$$ giving an iso on tangent space.

Every <u>splitting</u> δ of (L, d) — a degree -1 map $L^\cdot \to L^\cdot$ such that $\delta^2 = 0$, $\delta d \delta = \delta$, $d \delta d = d$ — gives us such a hull $\ker^\delta$.

Think $\delta = G d^*$ in the abelian case.

There's an explicit construction of $\ker^\delta$: it is prorepresented by $\mathcal{K} = \ker\left(H^1 \longrightarrow L^2\right)$

$$\eta \mapsto [(\phi^*(\eta), \phi^*(\eta))]$$, brackets denoting projection onto harmonic part

where ϕ is a Kuranishi map.

<u>Fact</u>: If $L^\cdot$ is <u>formal</u>, $\mathcal{K}$ simply becomes the quadratic cone given by the cup product.

Exercises

1) Show that a polarised VHS gives a harmonic bundle, & show that the Higgs bundle is $\mathbb{C}^*$-fixed. Compare the period map & harmonic map.

2) Calculate the formula

$$ch_2(F) = c_1 \|F^-\|^2 - \|F^+\|^2$$

and $F^+ = \Lambda F + F^{0,2+2,0}$

to deduce Bogomolov-Gieseker for Hermitian Yang-Mills. Do in the Higgs case too.

3) Given a map $X \to Y$, Y a curve with some multiple fibres, make a map

$$X \longrightarrow Y[P/n] \quad (\text{root stack})$$

4) Describe the Hitchin moduli space corresponding to the character variety V_{abcd} described by Toledo, possibly assuming some conditions on a, b, c, d which correspond to finite order monodromy: Higgs bundles on a root stack over $\mathbb{P}^1$

5) Try to calculate the MHS on $H^*(V_{abcd})$.