Lecture 7

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Moduli of Higgs Bundles

The Hitchin Fibration:

Consider $Higgs_\Sigma(r, d, \omega_\Sigma)$ where Σ is a smooth projective complex curve, r & d rank & degree.
The Hitchin Fibration is a map to a f.d vector space $H : Higgs_\Sigma(r, d, \omega_\Sigma) \to B_{\omega_\Sigma}$

Historically: $H : T^*\mathcal{M} \to B$, where $\mathcal{M}$ is the moduli space of stable vector bundle of rank r & degree d.

Definition:

A vector bundle E is stable (μ-stable or slope stable). If for all $F \subseteq E$, $F \neq 0$, sub-bundles

$$\frac{\deg(F)}{rk(F)} < \frac{\deg(E)}{rk(E)}.$$

It is semistable if we also allow equality.

A Higgs bundle (E, ϕ) is stable / semistable if the above inequality holds for all $F \subseteq E$ a sub-Higgs bundle; i.e. for all subbundles F preserved by ϕ

Definition (equivalence on Higgs bundles)

Say $(E, \phi) \sim (E', \phi')$ if there are ϕ, ϕ' invariant filtrations on E, E' by semistable pairs of constant slope with isomorphic associated gradeds.

Fact: There is an algebraic coarse moduli scheme parameterising these equivalence classes.

To understand the Hitchin map, let's consider a toy case: $\Sigma \rightsquigarrow \{pt\}$, so a vector bundle is just a vector space E, & consider $\phi : E \longrightarrow E \otimes K$ for K a 1-dimensional vector space.

Define: $H(E, \phi) = \det(I y - \phi)$ characteristic poly

$$= y^r + a_1 y^{r-1} + \dots + a_r \qquad a_i \in K^{\otimes i}$$

So think $\{(E, \phi)\} \xrightarrow{H} \bigoplus_{i=1}^{r} K^{\otimes i}$

Definition

Let Σ be a smooth proj curve, K a line bundle.

Define

$$H : \mathrm{Higgs}_\Sigma(r, d, K) \to B = \bigoplus_{i=1}^{r} H^0(K^{\otimes i})$$

$$(E, \phi) \longmapsto \det(I y - \phi)$$

This should be a completely integrable Hamiltonian system, so the fibres should be tori...

Spectral Curves

Let $a \in B$. Define $p_a(y) = y^r + a_1 y^{r-1} + \dots + a_r$

$\mathbb{K}$ total space of K

$$p_a : \mathbb{K} \longrightarrow \mathbb{K}^{\otimes r} \text{ say.}$$

$\mathbb{K}^{\otimes r}$ = total space of $K^{\otimes r}$

Define the spectral curve C_a to be

$$p_a^{-1}(\text{0-section of } \mathbb{K}^{\otimes r})$$

roughly the "eigenvalues".

Properties:

- C_a is generically smooth (in a)
- $\pi_* \mathcal{O}_{C_a} = \mathrm{Sym}(K^{-1})/\mathcal{J}$

where $\pi : \mathbb{K} \longrightarrow \Sigma$ projection,

$\mathcal{J}$ generated by the image of the map

$$K^{-r} \longrightarrow \mathrm{Sym}(K^{-1})$$
$$u \longmapsto (a_1 + \ldots + a_r) \otimes u$$

- C_a is an r-fold ramified cover of Σ

Proposition (Hitchin)

Let $a \in B$. There's a bijection between

$$\left\{ \begin{array}{c} \text{rk } 1 \text{ torsion-free sheaves} \\ \text{on } C_a \end{array} \right\} \Big/ \cong \; \longleftrightarrow \; \underbrace{\left\{ \begin{array}{c} \text{Higgs pairs of rank } r \\ (E, \phi) \text{ with } \mathrm{char}(\phi) = p_a(x) \end{array} \right\} \Big/ \cong}_{H^{-1}(a)}$$

assuming C_a is integral.

Basically, given L a rk 1 torsion-free sheaf on C_a, look at $\pi_* L$. This is a rk r vector bundle E, with a $\pi_* \mathcal{O}_{C_a}$-algebra structure (in fact this construction is bijective). Such $\pi_* \mathcal{O}_{C_a}$ structures are in bijective correspondence with $\phi : E \longrightarrow E \otimes K$ s.t. $\mathrm{char}(\phi) = p_a(x)$.

Indeed, $\pi_* \mathcal{O}_{C_a} = \mathrm{Sym}(K^{-1})/\mathcal{J}$, so such algebra structures are homomorphisms

$$K^{-1} \otimes E \longrightarrow E$$

giving

$$E \longrightarrow E \otimes K$$

This helps us prove generic fibres of H are Jacobians for the spectral curve.

Harold

One can compute

$$\dim \mathcal{M}_{Higgs} = 2r^2(g-1) + 2$$

$$\dim Jac(C_a) = \dim \bigoplus_{i=1}^{n} H^0(K^{\otimes i}) = r^2(g-1) + 1$$

The Hitchin map is a maximal collection of Poisson-commuting holomorphic functions; this - combined with the fact that the fibres are abelian varieties - means it is an algebraically completely integrable Hamiltonian system.

Theorem

This remains true for all reductive complex G.

In general have by Chevalley

$$k[\mathfrak{h}]^W \cong k[\mathfrak{g}]^G \hookrightarrow k[\mathfrak{g}]$$

where $\mathfrak{h}$ is a Cartan subalgebra, W is the Weyl group

gives $\mathfrak{g} \longrightarrow \mathfrak{h}/W$ a polynomial map.

So this defines a map on bundles

$$(P \times_G \mathfrak{g}) \otimes k \longrightarrow (P \times \mathfrak{h}/W) \otimes k$$

$$\& \quad H^0(X, \operatorname{ad} P \otimes k) \longrightarrow H^0(X, (P \times \mathfrak{h}/W) \otimes k)$$

$$\text{"} \bigoplus_{i=1}^{m} H^0 K^{d_i} \text{"} \leftarrow \text{"exponents"}$$

Example:

Take $G = Sp(2m, \mathbb{C})$. So a G-Higgs bundle is a triple $(E, \langle , \rangle, \Phi)$,

where $\langle , \rangle$ is a nondegenerate [illegible] form, & we have

$$\langle \Phi s, t \rangle = - \langle s, \Phi t \rangle .$$

generally Φ has distinct eigenvalues and eigenvectors

$\{\lambda_i\}, \{v_i\} : \omega$

$$\langle \Phi v_i, v_j \rangle = - \langle v_i, \Phi v_j \rangle$$

$$\Rightarrow \lambda_i \langle v_i, v_j \rangle = - \lambda_j \langle v_i, v_j \rangle$$

The Hitchin fibration now goes

$$Sp(2m, \mathbb{C}) - \mathcal{M}_{Higgs} \longrightarrow \bigoplus_{i=1}^{m} H^0(K^{\otimes 2i})$$

<u>Note</u>: the form gives an isomorphism $E \xrightarrow{\sim} E^*$, changing the sign of the eigenvalues.

The spectral curve $\tilde{C}$ has an involution σ: multiplying by -1, so the fibre of H is the sub-abelian variety of the Jacobian consisting of line bundles L such that $\sigma^* L \cong L^*$.

Hendrik

Construction of Moduli Spaces

Lecture 8

We'll construct M_B, M_{dR}, M_{Dol}, M_{Hod}, using roughly the same ideas as coarse moduli schemes which we can produce complex analytic equivalences between.

Definitions:

X projective scheme over $\mathbb{C}$, E a coherent sheaf thereon. E is μ-(semi) stable if

$$\text{slope}(F) < \text{slope}(E) \quad (\text{or } \leq)$$

for all coherent subsheaves $F \subseteq E$. $\text{Slope}(E) = \frac{\deg(E)}{rk(E)}$.

E is p-semistable if instead we require

$$\frac{p(F,n)}{rk(F)} \leq \frac{p(E,n)}{rk(E)} \quad \text{for } n >> 0$$

where p is the Hilbert polynomial.

E is pure of dimension d if $\dim(\text{supp}(F)) = d = \dim(\text{supp}(E))$ for all $F \leq E$ coherent subsheaves.

Now, given a functor $Y^{\#}: Sch \longrightarrow Set$, then a scheme Y universally corepresents if we have a map $Y^{\#} \to Y$ such that $\forall$ schemes $V \longrightarrow Y$, the fibre product

$$\begin{array}{ccc} V \times_Y Y^{\#} & \longrightarrow & Y^{\#} \\ \downarrow & & \downarrow \\ V & \longrightarrow & Y \end{array}$$

is corepresented by V.

corepresenting
$\Downarrow$
universally
corepresenting
but not conversely

This is the idea of coarse moduli schemes.

If an algebraic group G acts on X, one can take the set-theoretic quotient $X^\#/G^\#$. Say a map of schemes $X \longrightarrow Y$ is a universal categorical quotient if Y universally corepresents $X^\#/G^\#$.

Now, take X/S projective. Let $M^\#(\mathcal{O}_X, P)$ be the functor

$$\left\{ \begin{matrix} S' \\ \downarrow \\ S \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{Semistable sheaves on } X' \\ \text{with Hilbert polynomial } P \\ \text{\& pure dimension } d \end{matrix} \right\}$$

Here
$$\begin{matrix} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{matrix}$$
is the base change.

To construct a scheme that universally corepresents, we use the following starting point.

<u>Theorem</u> (Grothendieck).

$\exists$ a scheme $\mathrm{Hilb}(W, P)$ where $W \in \mathrm{Coh}(X)$, P a polynomial, such that if $\sigma : S' \to S$, the S'-points of $\mathrm{Hilb}(W,P)$ are quotients

$$\sigma^* W \to \mathcal{F} \longrightarrow 0$$

on X', ~~with~~ $\mathcal{F}/S'$ flat & Hilbert polynomial P.

Now, take $N \in \mathbb{Z}^+$, $W = \mathcal{O}_X(-N)$, $V = \mathbb{C}^{P(N)}$. So $\mathrm{Hilb}(V \otimes W, P)$ exists. It turns out one can choose

$$Q_2 \subseteq \mathrm{Hilb}(V \otimes W, P)$$

which corepresents

Betti Moduli M_B:

Let Γ be a finitely generated group $\langle \gamma_1, \dots, \gamma_k \mid w \rangle$, we'd like to form the moduli space of its representations.

Form the set $\{ \Gamma \longrightarrow GL(n, \mathbb{C}) \}$. $GL(n, \mathbb{C})$ acts on this space by conjugation, & the orbits are isomorphism classes of representations. So we need

$$\{ \Gamma \to GL(n,\mathbb{C})\} / GL(n,\mathbb{C}) =: R(\Gamma, n) / GL(n, \mathbb{C}).$$

<u>First</u>: Claim that $R(\Gamma, n)$ is a subscheme of $\underbrace{GL(n,\mathbb{C}) \times \dots \times GL(n, \mathbb{C})}_{k}$. So $(m_1, \dots, m_k) \in R(\Gamma, n)$

if $v(m_1, \dots, m_k) = 1 \quad \forall v \in w$.

<u>Theorem</u>:

There exists a map $R(\Gamma, n) \longrightarrow M(\Gamma, n)$ which universally corepresents the quotient by $GL(n, \mathbb{C})$

To prove this, if $A = k(R(\Gamma, n))$ is the ring of functions, set $B = A^{GL_n(\mathbb{C})}$, & put $M(\Gamma, n) = \operatorname{Spec} B$.

If X is smooth projective over $\mathbb{C}$, fix $x \in X$, & set $\Gamma = \pi_1(X^{an}, x)$.

<u>Definition</u>:

The <u>Betti moduli space</u> is

$$\mathcal{M}_B(X, n) = M(\Gamma, n).$$

Some more work is required to produce a relative version.

Pietro

Symplectic structure on $M_B(X, n)$

Assume X is a curve now. We can compute the tangent space to $M(\Gamma, G)$, where Γ is finitely generated, G a reductive algebraic group. Fix B a G-invariant bilinear form on $\mathfrak{g}$.

$\phi \in \mathrm{Hom}(\Gamma, G)$. Take a family (ϕ_t) where $\phi_0 = \phi$ satisfying some conditions.

$$\phi_t(x) = \exp(tu(x) + t^2)\phi(x) \qquad u: \Gamma \to \mathfrak{g}$$

So $\quad \phi_t(xy) = \phi_t(x)\,\phi_t(y)$

$$u(xy) = u(x) + \mathrm{Ad}\,\phi(x)\,u(y)$$

Can look at the composite $\quad \Gamma \xrightarrow{\phi} G \xrightarrow{\mathrm{Ad}} \mathfrak{g}$

Now look at group cohomology, with $C^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\phi})$.
The above $\Rightarrow u \in Z^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\phi})$.

If ϕ_t is contained in a G-orbit then $u \in B^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\phi})$ is a coboundary, and so

$$T_\phi(M_B(\Gamma, G)) = H^1(\Gamma, \mathfrak{g}_{\mathrm{Ad}\phi}).$$

If $\Gamma = \pi_1(X^{an}, x)$ now, we have a product

$$\cup : H^i(\Gamma, \mathfrak{g}) \times H^{2-i}(\Gamma, \mathfrak{g}) \to H^2(\Gamma, \mathbb{R}) \cong \mathbb{R}$$

In particular, apply this for $i = 1$

$$\cup : H^1(\Gamma, \mathfrak{g}) \times H^1(\Gamma, \mathfrak{g}) \to \mathbb{R}.$$

We'd like to show this is a symplectic structure on M_B.

To see it is closed, take $\tilde{X}$ the universal cover & look at the principal G-bundle on X.

$$\tilde{X} \times G / \Gamma = P$$

given by a representation ϕ. We can take the adjoint $ad\, P_\phi$ with flat connection d_ϕ to see

$$H^1(\Gamma, \mathfrak{g}_{Ad\phi}) = H^1(X, ad\, P_\phi).$$

Now, let ω^B be the form described.
Let $\mathfrak{A}$ be the affine space of connections on P, modelled on $\Lambda^1(X, ad\, P)$. That is,
$T_A \mathfrak{A} = \Lambda^1(X, ad\, P) \quad \forall A \in \mathfrak{A}.$
Invoking our map $B: \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$

produces $B_*: ad\, P \times ad\, P \longrightarrow \mathbb{R}$

If $\eta, \Theta \in \mathfrak{A}$, we can produce

?

$$\int_X B_*(\eta \wedge \Theta) \in \mathbb{R} \quad \Rightarrow \text{closedness}$$

restricted to $Z^1(X, ad\, P) \subseteq \Lambda^1$ the form may be degenerate, but if we assume $(\omega(\eta, -)) \in B^1(X, ad\, P)$ & descend to cohomology it is again non-degenerate, & closedness is preserved.

Riemann-Hilbert Correspondence

There is an identification of complex analytic spaces

$$\mathcal{M}_B^{an}(X/S, n) \cong \mathcal{M}_{dR}^{an}(X/S, n)$$

$$L \longmapsto (L \otimes_{\mathbb{C}} \mathcal{O}_X, 1 \otimes d)$$

$$E^\nabla \longleftarrow (E, \nabla)$$

$\{ s \in E,\ \nabla s = 0 \}$

Lecture 9

Pavel

Homeomorphisms between Moduli Spaces

Definition:

(X, g) Riemannian manifold is hyperkähler if there are $I, J, K \in End(TX)$ covariant constant, satisfying

$$I^2 = J^2 = K^2 = IJK = -1$$

In particular, I, J, K are integrable, so (X, g, I) etc. are kähler.

This allows us to introduce quaternionic structure on TX, so $\dim_{\mathbb{R}} X = 4n$ say.

The holonomy is contained in $GL(n, \mathbb{H}) \cap O(4n) = Sp(n)$

$$= \underbrace{U(2n)}_{\text{holonomy of a kähler}} \cap \underbrace{Sp(2n, \mathbb{C})}_{\text{holonomy of } X \text{ with holo symplectic fo}}.$$

Thus a hyperkähler admits ω_I of type $(2,0)$ such that $(\underbrace{X, I}_{\text{kähler}}, \omega)$ is holomorphic symplectic.

Conversely, any such (X, I, ω) is hyperkähler. Indeed, ω^{2n} is a holomorphic trivialisation of the canonical bundle. So by Yau's theorem $\exists$ a Ricci-flat metric g, & by Bochner formula ω is covariant constant wrt the Levi-Civita connection ∇_g.

Examples:

1) $\mathbb{H}^n$

2) Recall $\mathbb{CP}^n$ & its complex manifolds are Kähler, however $\mathbb{HP}^n$ is not hyperkähler.

3) $n=1$. Always $Sp(n) \leq SU(2n)$ so hyperkähler $\Rightarrow$ Calabi-Yau

However $Sp(1) = SU(2)$, so in this dimension

Calabi-Yau's are hyperkähler, e.g. T^4 & K3 surfaces.

4) Many examples come from gauge theory:

$\mathbb{H}^4 / \text{some group}$

5) Consider $\Gamma \leq SU(2)$ finite. Resolve the singularity $\widetilde{\mathbb{C}^2/\Gamma} \to \mathbb{C}^2/\Gamma$. Then this is hyperkähler (ADE resolutions).

Like in the Kähler case, get Kähler forms

$\omega_1 = g(Ix, y)$, ω_2, ω_3 similarly, of type $(1,1)$.

$\omega_2 + i\omega_3$ is a $(2,0)$ form for I : holo symplectic forms

Twistor Space:

Consider $I_{\underline{u}}$ for $\underline{u} \in \mathbb{R}^3$. $I_{\underline{u}} = Iu_1 + Ju_2 + Ku_3$.

Then $I_{\underline{u}}^2 = -1 \Leftrightarrow \underline{u} \in S^2$. We'll use this to encode X in a complex manifold.

Define the twistor space to be the C^∞ manifold $Z = X \times S^2$.

$S^2 \cong \mathbb{C}\mathbb{P}^1$: call its complex structure $I_{\mathbb{C}\mathbb{P}^1}$.

Given $(x, u) \in Z$, put

$$I_{(x,u)} = (I_u, I_{\mathbb{C}\mathbb{P}^1})$$

Claim: This almost complex structure is integrable

Proof: Use Newlander - Nirenberg.

Theorem.

On Z we have:

1): $p: Z \to \mathbb{CP}^1$ a holomorphic fibre bundle

2) an antilinear involution σ of Z covering the antipodal map on S^2.

3) There is a section $\underline{\omega} \in \Gamma(\Lambda^2 T^*_{Z/\mathbb{CP}^1} \otimes p^* \mathcal{O}(2))$ such that $\underline{\omega}$ is fibrewise symplectic in I_ω, and real wrt σ.

4) We have real sections (twistor lines) with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes_{\mathbb{C}} p^*\mathcal{O}(1)$: holo sections in I preserved by σ.

σ is just the involution of S^2

$\underline{\omega}$ is given by

$\underline{\omega}(\xi) = \omega_2 + i\omega_3 + 2\xi(\omega_1) - \xi^2(\omega_2 - \omega_3)$

$\underline{\omega}(0) = \omega_2 + i\omega_3$

The sections are just $X \times \{u\} \subseteq X \times S^2$.

Theorem:

Let Z be a $2n+1$ dimensional complex manifold satisfying (1) $\to$ (3). Then the space of sections as in (4) is a $\dim_{\mathbb{R}} = 2n$ hyperkähler manifold.

Hyperkähler quotients

Let G be a complex reductive group with maximal compact $K \subseteq G$. Suppose $K \curvearrowright X$ hyperkähler, preserving the symplectic forms & the metric.

Then we have moment map

$$\mu : X \longrightarrow \mathfrak{k}^* \otimes \mathbb{R}^3$$

Combining the various moment maps.

Now, $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Consider $\mu_1 : X \to \mathfrak{k}^*$ by fixing I

$$\mu_2 + i\mu_3 : X \quad \mathfrak{g}^*$$

Claim:

The preimage $\mu^{-1}(\xi)/K$ is hyperkähler, for $\xi \in \mathfrak{z} \otimes \mathbb{R}^3$, $\mathfrak{z} \in (\mathfrak{k}^*)^K$.

Consider $\alpha \in \mathfrak{z} \otimes \mathbb{C}$. Then kähler quotient $\mu_1^{-1}(\alpha)$ by K in I agrees with this hyperkähler quotient.

Hyperkähler Structure on the Moduli Space

This is due to Hitchin in dimension 1, Fujiki in general.

Let X be a compact kähler manifold. Fix a C^∞ G-bundle $P \longrightarrow X$, with reduction P_K to K. Pick an $ad(G)$ invariant form on $\mathfrak{g}$.

Note: We'll consider sections in some Sobolev class, but probably write L^2 for simplicity.

Let $\mathcal{G} = \Gamma(X, \mathrm{Ad}\, P)$ of class H_{k+1}

$\mathcal{K} = \Gamma(X, \mathrm{Ad}\, P_K)$ of class H_k

Let $\mathcal{A} = \{\text{connections on } P\} \supset \mathcal{G}$

$\mathcal{A}_K = \{\text{connections on } P_K\} \supset \mathcal{K}$

Claim:

$\mathcal{A}$ is a Hilbert hyperkähler manifold.

Proof:

First consider $\mathcal{A}_K$. $T_D \mathcal{A}_K \cong \Omega^1_{\mathbb{R}}(X; \operatorname{ad} P_K)$,
so we have complex structure here I coming from X. Define a metric

$$(\alpha, \beta)_{\mathcal{A}_K} = \int_X (\alpha, \beta) \frac{\omega_X^n}{n!}$$ making $\mathcal{A}_K$ kähler.

Now on $\mathcal{A}$, $T_D \mathcal{A} \cong \Omega^1_{\mathbb{C}}(X, \operatorname{ad} P)$
So we have $\operatorname{ad} P = \operatorname{ad} P \oplus i \operatorname{ad} P_K$, & hence

$T_D \mathcal{A} \cong T_D \mathcal{A}_K \otimes \mathbb{C}$ naturally.

So Extend I antilinearly to get hyperkähler structure. $\square$

The holomorphic symplectic form wrt I is

$$\omega(\alpha, \beta) = \int \langle \alpha \wedge \beta \rangle \wedge \frac{\omega_X^{n-1}}{(n-1)!}.$$

Here, the holomorphic moment map is given by

$$\mu(D) = \Lambda(F_A)$$

where $\Lambda = L^*$, $L = \wedge \omega$ Lefschetz map.

Also $\mu_1(D) = iD$.

$$D = D^+ + \Theta$$

Decompose $D = D^+ + \Theta \leftarrow$ orthogonal $\leftarrow$ along $\mathfrak{u}$

$D^+ = D' + D''$

$\Theta = \Theta' + \Theta''$ Harmonic means $D^+\Theta = D^{+,*}\Theta = 0$

$\partial = D' + \Theta'$ weakly harmonic: $D^{+,*}\Theta = 0$

$\bar{\partial} = D'' + \Theta''$

Denote $T = \mathcal{A} /\!/\!/ \mathcal{G}$ to be the hyperkähler quotient.
T is Einstein $(\Lambda F_D = 0)$ & weakly harmonic.
In general it is ∞-dimensional, & we'd like to reduce.

Let
- $\tilde{T}_J \subseteq T$ s.t. F_D has type $(1,1)$
- $T_I \subseteq$ s.t. • $F_{D''}$ has type $(1,1)$
  - Θ'' is $\bar{\partial}$-holomorphic
  - $[\Theta \wedge \Theta] = 0$

So $T_I \cong \{$stable Higgs bundles$\}$: Dolbeault model
$\tilde{T}_J \longrightarrow T_J$: De Rham model.
& there's a bijection $T_I \longleftrightarrow T_J$.

Deligne's Construction

Recall the Hodge-moduli: $\mathcal{M}_{Hod}(X, GL_n) \to \mathbb{A}^1$.
Construct $\tau : \mathcal{M}_B(X,n) \to \mathcal{M}_B(X,n)$ antilinear involution
$\tau(\rho) = \overline{(\rho^{-1})^T}$

So construct antilinear
$$\mathcal{M}_{dR}(X,n) \xrightarrow{RH} \mathcal{M}_B(X,n) \xrightarrow{\tau} \mathcal{M}_B(X,n) \xrightarrow{RH} \mathcal{M}_{dR}(X,n)$$
So we have antilinear
$$\mathcal{M}_{dR}(X,n) \times \mathbb{G}_m \longrightarrow \mathcal{M}_{dR}(X,n) \times \mathbb{G}_m$$
$$(\rho, \lambda) \longmapsto (\tau(\rho), -\bar{\lambda}^{-1}).$$
We also have $\mathcal{M}_{Hod}(\bar{X}, GL_n) \longrightarrow \mathbb{A}$,
say with parameter $-\bar{\lambda}^{-1}$.
Define
$$\mathcal{M}_{Del}(X, n) = \mathcal{M}_{Hod}(X, n) \cup_{\mathbb{G}_m} \mathcal{M}_{Hod}(\bar{X}, n).$$

<u>Claim</u>: $\mathcal{M}_{Dol}$ satisfies all the axioms to be a twistor space.

Consider $\bar{\partial}_\lambda = \bar{\partial} + \lambda\theta''$
$D_\lambda = \lambda\partial + \theta'$
Then we can check $(P, \bar{\partial}_\lambda, D_\lambda)$ is a flat G-bundle with λ-connection.

<u>Note</u>: $\mathcal{M}_{Dol}$ satisfying twistor space axioms means we have identifications of all the fibres. In particular $\mathcal{M}_{Dol}$ & $\mathcal{M}_{dR}$