Chris Elliott

TALBOT NOTES ①

math.mit.edu/talbot/2011/library

C. Simpson | Harmonic Bundles - Introduction | Lecture 1

Overview:

We'll be looking at X an algebraic variety over $\mathbb{C}$.

From X we get a topological space X^{top}. Suppose connected

e.g.:

$X \subseteq \mathbb{P}^n$, $X^{top} \subseteq \mathbb{CP}^n$ some subspace with analytic topology

We'll be talking about local systems on X^{top}, after choosing base point $x \in X^{top}$.

These are the same as representations

$$\rho : \pi_1(X^{top}, x) \longrightarrow GL(n, \mathbb{C}).$$

Historically, this basic theme has been central in maths for ~ 200 years:

- Galois theory - local systems of finite sets over Spec k, k a field.
- Riemann surfaces - local systems of sets over $U \subseteq \mathbb{C}$.
- Differential equations à la Liouville: Riemann-Hilbert.
- Picard-Lefschetz theory - the start of Hodge theory: If you have a family $X \to Y$ of varieties then you get a local system L on Y with

$$L_y = H^i(X_y, \mathbb{C}).$$

It turns out that these things have lots of interesting structure.

topological world		algebraic world
local system	$\longleftarrow$	alg map
X^{top}	$\longleftarrow$	X

topological world $\qquad\qquad$ algebraic world

Local System $\xrightarrow[\text{analytic vb } E^{an}]{\text{via complex}}$ $L = E^{\nabla}$ for (E, ∇)

$$E^{an} = L \otimes_{\mathbb{C}} \mathcal{O}_{X^{an}}, \quad \nabla = 1 \otimes d$$

L associates a vector space $L(U)$ to a small open set U, & $E^{an}|_U = L(U) \otimes_{\mathbb{C}} \mathcal{O}_U$. This analytic object is algebraic by GAGA (cf Deligne in the quasiprojective case). A lot of the modern point of view here is thanks to Hironaka's resolution of singularities

Weil, Grothendieck, Deligne et al produce the analogous local systems in the étale topology for number theory & char p geometry. This motivates a lot of things: For Deligne it motivated Hodge theory.

The study of situations like the family $\begin{matrix} X \\ \downarrow f \\ Y \end{matrix}$ led to Griffiths' theory of variations of Hodge structure (VHS): we get extra structure on things like (E, ∇) associated to $L = Rf_* \mathbb{C}$. This structure is the Hodge filtration

$$H^i(X_y, \mathbb{C}) = \bigoplus_{p+q=i} H^{p,q}(X_y)$$

$$F^p = \sum_{p' \geq p} H^{p',q} \; : \text{ holomorphic sub-bundle.}$$

The $H^{p,q} \subseteq E$ are C^{∞} sub-bundles, & we have flat Hermitian form $\langle , \rangle$.

New in the 1980s

People started looking at the fact that local systems can vary. This is special about the fact that we've taken $\mathbb{C}$-coefficients, not $\mathbb{Q}$ or $\mathbb{Z}$ say.

Look at $\{\rho : \pi_1(X,x) \to GL(n,\mathbb{C})\} \subseteq GL(n,\mathbb{C})^a / \text{conjugations}$

Subset defined by some relations. a: # generators of π_1.

- Donaldson gave a new proof of the Narasimhan-Seshadri theorem, which says
$$\{\text{unitary reps}\} = \{\text{stable bundles}\}.$$
- Lubotzky-Magid wrote a book about $Rep(\pi_1)/_\sim =: \chi(\pi_1)$ character variety & topologists started using it to study the Smith conjecture.
- Mazur-Wiles were studying deformations of $\mathbb{Q}_\ell$-local systems; generalising Ihara's work on the rank 1 case

So if our local systems are varying, how does the structure of such families interact with the algebraic structure of the variety?

- Now Hitchin comes in. He generalised the Narasimhan-Seshadri theorem to more general complex groups, along with Corlette & Donaldson.

X smooth projective

$$\text{Representations } \{\rho : \pi_1(X,x) \to GL(n,\mathbb{C})\} \leftrightarrow \{\text{semistable Higgs bundles } (E,\theta)\}$$

In a Higgs bundle, E is a rank n holo vector bundle on X,

$\Theta : E \longrightarrow E \otimes \Omega^1_X$ with $\Theta \wedge \Theta : E \longrightarrow E \otimes \Omega^2_X = 0$

$\longrightarrow$ direction: Corlette & Donaldson

$\longleftarrow$ direction: Hitchin & Simpson

The Narasimhan-Seshadri case is the case $\rho : \pi_1(X) \longrightarrow U(n)$, which corresponds to the $\Theta = 0$ subset on the right-hand side.

Why is this non-abelian Hodge theory?

The space of representations

$$\{\rho : \pi_1(X, x) \longrightarrow GL(n, \mathbb{C})\}/\sim = H^1(X^{top}, GL(n, \mathbb{C}))$$

Why? A local system is given by a cocycle $(g_{ij} \in GL(n, \mathbb{C}))_{\underset{\text{contractible}}{U_i \cap U_j \neq \emptyset}}$, with $g_{ij}\, g_{jk} = g_{ik}$ on triple overlaps, modulo a suitable notion of coboundary $(g_{ij}) \sim (g'_{ij})$ if $g'_{ij} = a_j\, g_{ij}\, a_i^{-1}$ or something like that.

$$\begin{array}{cc} \vdots & \vdots \\ \uparrow & \uparrow \\ A^{0,1} & A^{1,1} \\ \bar\partial \uparrow & \uparrow \bar\partial \\ A^{0,0} & A^{1,0} \\ & H^1 \end{array}$$

Observation: an (E, Θ) can be viewed as a non-abelian Dolbeault cohomology class:

recall $H^1_{Dol}(X) = H^1(X, \mathcal{O}_X) \oplus H^0(X, \Omega^1_X)$

$= H^1(A^{p,q}(X), \bar\partial)$

This is the abelian case $G = \mathbb{G}_a$

Given (E, Θ), E is a vector bundle, so $\in H^1(X, GL(n, \mathcal{O}_X))$.

Θ is a section $\in H^0(\underbrace{End(E) \otimes \Omega^1_X}_{\substack{\text{twisted form of} \\ \mathfrak{gl}(n, \mathcal{O}) \otimes \Omega^1_X}})$

Aaron

Flat to Harmonic

Lecture 2

Main goal will be to talk about the paper

Corlette: "Flat G-bundles with Canonical metrics"

Motivating picture: In finite dimensions, **Kempf & Ness** proved the following. Let $G/\mathbb{C}$ be a semisimple algebraic group, V $\mathbb{C}$-vector space with hermitian form h, $G \curvearrowright V$ linearly.

Say a G-orbit is *stable* if it is closed of maximal dimension among closed orbits. **Observe** Gv is stable $\Leftrightarrow$ Gv possesses a shortest vector. (length wrt h.)

GIT Perspective

Consider $K \leq G$, e.g. $SU(n) \leq SL(n, \mathbb{C})$, such that K is the stabiliser of h, K compact.

$\exists$ a moment map for the K-action

$$\mu : V \longrightarrow \mathfrak{k}^*$$

Theorem:

$$\mu(v) = 0 \Leftrightarrow v \text{ is the shortest vector in } Gv.$$

Corlette's Setup G as above, $K \leq G$ maximal compact. M a compact Riemannian manifold, $\begin{smallmatrix} P \\ \downarrow \\ M \end{smallmatrix}$ a principal G-bundle with fixed reduction of structure group to K (e.g if $K \leq G = SU(n) \leq SL(n, \mathbb{C})$ such a reduction is a hermitian structure).

$ad(P) = \mathfrak{g} \times_G P$ with adjoint representation

Definition

$$\mathcal{C}^\infty = \{ \text{Smooth } SL(n,\mathbb{C}) \text{ connections on } P \}$$
$$\subseteq \Omega^1(P) \otimes \mathfrak{g} \approx \Omega^1(M, ad(P)) \text{ non-canonically as affine spaces.}$$

$\exists$ an L^2-metric on $\mathcal{C}^\infty$ coming from integrating $g^* \otimes h$, where g^* is the metric on T^*M, h the metric on $ad(P)$

This $\mathcal{C}^\infty$ will play the role of V above, & this metric will play the role of h.

Definition:

$\mathcal{G}^k$ = group of ~~k-times differentiable~~ gauge transformations of P
$Aut(P) \approx \Gamma(G \times_G P)$

This is the analogue of $G \curvearrowright V$

Consider $\mathcal{U}^k$ = k-times differentiable gauge transformations preserving the hermitian metric on P. This is the analogue of $K \subseteq G$

Corlette: $\exists$ a moment map for the action of $\mathcal{U} \curvearrowright \mathcal{C}$

$$\Phi : \mathcal{C} \to \Xi^* := \text{"}Lie(\mathcal{U})\text{"}$$
$$= \text{skew adjoint sections of } ad(P)$$

Slight Digression

If D is a connection on P, D always splits
$D = D^+ + \Theta$ where D^+ preserves k-structure (skew-adjoint
we use this to define Φ.

$$\Phi_D = D^{+\,*}(\Theta)$$

Theorem (Corlette)

$d\|\Theta_D\|^2_{L^2}(\xi) = 2i\langle \Phi_D, \xi\rangle$ when restricted to a $\mathcal{G}$ orbit $\mathcal{G}D$

So in particular if the moment map vanishes, the length of Θ_D is minimised.

Theorem: (Corlette)

If D is stable & flat then $\exists!$ $\mathcal{U}$-orbit in $\mathcal{G}D$ on which $\Phi_D = 0$. Equivalently, $\exists!$ hermitian metric on P for which $\Phi_D = 0$.

D stable $\Leftrightarrow$ the holonomy of D is not entirely contained in a parabolic subgroup of G.

We say this metric is harmonic! In the second formulation K-structures on P are sections in $\Gamma(G/K \times_G P)$. Let $\tilde{M}$ be the universal cover of M, & pull back the bundle $Q = G/K \times_G P$ to $\tilde{M}$. This is a trivial bundle because it has a flat connection coming from D. So sections ρ are just maps

$$\rho : \tilde{M} \longrightarrow G/K.$$

Such a map descends iff it is $\pi_1(M)$-equivariant.

Furthermore, the condition $\Phi_D = 0$ implies ρ is harmonic as a map of Riemannian manifolds (& conversely)

The method of developing this correspondence is to solve the following. Let A_t be a family of connections. Solve

$$\frac{dA_t}{dt} = -d_A \Phi_A \qquad d_A \text{ covariant derivative.}$$

One solves this locally for short time & gluing, then pass to a limit. This limit connection is inside the orbit $\Leftrightarrow$ D is stable

Sushmita

Eells - Samson

If $f: M \longrightarrow N$ is a map of Riemannian manifolds, & N has non-positive sectional curvature, then any map can be deformed to a harmonic map. The proof is a heat flow method.

The situation above is similar, with $N = G/K$. To compute curvature at the identity, get something like $R_{ijkl} = -([x_i, x_j], [x_k, x_l])$, which is non-positive, so this method works.

Bochner Formula:

If $u: M \longrightarrow \mathbb{R}$, then Bochner formulae are things like

$$\Delta \tfrac{1}{2} |\nabla u|^2 = |\nabla^2 u|^2 + \mathrm{Ric}(\nabla u, \nabla u)$$

Siu modified this to produce something like

$$\partial\bar{\partial} \langle g, \bar{\partial} f \wedge \partial \bar{f} \rangle = \langle R, \bar{\partial} f \wedge \partial \bar{f} \wedge \partial f \wedge \overline{\partial f} \rangle - \langle g, D\bar{\partial} f \wedge \bar{D} \overline{\partial f} \rangle$$

For any map $f: N \longrightarrow M$ of Riemannian manifolds, where g is the metric on N with curvature R

Siu uses this to show: If f is a map between Kähler manifolds $f: N \longrightarrow M$, & N has strongly negative curvature, i.e. $\exists\, \alpha, \beta$ s.t $\forall\, \xi^{\alpha,\beta} \in \mathbb{C}^2$,

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\beta\gamma\delta}\, \xi^{\alpha\beta}\, \overline{\xi^{\gamma\delta}} > 0,$$

Then If furthermore f is harmonic and $\exists\, q \in N$ s.t $df(q)$ has rank ≥ 4 then f is holomorphic or antiholomorphic.

Corlette proves a similar result.

Theorem:

If P is a $SU(n,1)$ - flat bundle, & Θ has rank ≥ 4 at some point, then Θ is either holomorphic or antiholomorphic.

Locally $\Theta: M \longrightarrow P$, where $g = k \oplus p$ & the theorem says this is either holomorphic or antiholomorphic.

Josh

Higgs to Harmonic

Lecture 3

History: Yang-Mills

Have a 4-manifold M with a metric g & principal $SU(n)$ bundle P. Let V be the assoc. vector bundle with its natural Hermitian metric. We'll look at connection A, & curvature F_A.

Abuse notation sometimes to write $F_A = dA + A \wedge A$

Yang-Mills functional

$$\|F_A\|^2_{L^2} = \int_M |F_A|^2 \, d\mathrm{vol}_g$$

$$= -\int_M \mathrm{Tr}(F \wedge * F)$$

These are the same as, if $m \in su(n)$,

$|m|^2 = \mathrm{Tr}\, m m^\dagger = -\mathrm{Tr}\, m^2$.

To find the critical points, we use Euler-Lagrange methods: $A \longmapsto A + \delta A$

$$\text{So} \quad \delta F_A = d\delta A + \delta A \wedge A + A \wedge \delta A$$

$$= d_A \delta A \quad \text{covariant derivative}$$

$$\Rightarrow \delta(F_A \wedge * F_A) = \delta F \wedge * F + F \wedge * \delta F$$

$$= 2\, d_A \delta A \wedge * F$$

$$\Rightarrow \delta(YM) = -2 \int \mathrm{Tr}\, d_A(\delta A \wedge * F).$$

$$= 2 \int \mathrm{Tr}\, \delta A \wedge d_A * F$$

Hence one produces **Yang Mills equation**

$$d_A * F_A = 0$$

Now, $* : \Lambda^2 \to \Lambda^2$, $*^2 = 1$, So one can decompose into ± 1 eigenspaces $F = F^+ + F^-$: $F^+ = \frac{1}{2}(\mathrm{id} + *)F$ e.g.

If you're self dual $F = *F$

then $d_A * F_A = d_A F_A = 0$ by Bianchi

Similarly if you're anti-self-dual, so these give solutions of Yang-Mills.

Furthermore, $-\int \mathrm{Tr}\, F \wedge * F = \|F^+_A\|^2 + \|F^-_A\|^2$ ①

$8\pi^2 c_2 = \int \mathrm{Tr}\, F \wedge F = \|F^-_A\|^2 - \|F^+_A\|^2$ ②

clearly ① ≥ ②, & equality holds iff F^+ vanishes, i.e. in the anti-self dual case. So clearly there we reach a natural topological lower bound.

Aaron

Suppose X is a complex surface. $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ $SU(2)$ bundle with associated vector bundle E given by standard rep of $SU(2)$ $\mathbb{C}^2 \times_{\substack{std \\ rep}} P$, with standard hermitian structure h.

Remarks:

① Fact: If E is a smooth vector bundle over a complex manifo with hermitian form h, & if we have any operator

$$\bar\partial_E : \Omega^0(E) \to \Omega^{0,1}(E)$$

such that $\bar\partial_E(fv) = (\bar\partial f)v + f\,\bar\partial_E v$ & $\bar\partial_E^2 = 0$.

Then $\exists$ a holomorphic structure on E with $\bar\partial$-operator $\bar\partial_E$ on sections.

② Any hermitian vb $(E, h, \bar\partial_E)$ admits a unique hermitian connection D such that the projection D'' onto 0,1 forms is just $\bar\partial_E$.

③ The curvature of such a connection is a $(1,1)$ form.

④ If (E,h) is a smooth hermitian vb & D is a smooth hermitian connection with $F_D \in \Omega^{1,1}(E)$ then $\exists!$ holomorphic structure on E making D the connection described above.

Hermite - Einstein connections / metrics

Let $\tilde{D}$ be an $SU(2)$ connection on $P \longrightarrow X$ a principal $SU(2)$-bundle over a compact Kähler surface. Let (E,h) be the assoc vector bundle as before. Let D be the connection on E induced by $\tilde{D}$.

Proposition: If $\tilde{D}$ is anti-self-dual (ASD) then

① F_D is of $(1,1)$-type.

② $F_D \wedge \omega = 0$. (ω Kähler form)

In this case, ② is the Hermite - Einstein condition.

Definition:

Let $E \longrightarrow X$ be a hermitian holo vb on a Kähler surface. $F_h \wedge \omega = \mu \cdot \omega^2 \cdot rk(E)/2 \cdot \mathbb{1}$ (These are top forms, so this is a definition of μ).

End valued ~~2-forms~~

In this case we say h is Hermite - Einstein.

~~If dim $X \geq 2$,~~ e.g. for $F_h \wedge \omega^{n-1} = c \cdot \omega^n \mathbb{1}$

Example:

Say X is Kähler, $E = TX$, h a Hermitian structure. Then H-E is the condition $Ric = \mu \cdot g$

In general, one can compute

$$\mu = \frac{1}{\mathrm{Vol}(X)} \frac{\deg(E)}{\mathrm{rk}(E)}, \quad \deg(E) = \int_X c_1(E) \wedge \omega_X$$

Main Theorem (Donaldson for surfaces, Yau-Uhlenbeck in general)

Any stable holomorphic vb over X compact Kähler admits a unique Hermite-Einstein connection.

In our simple situation $\rightarrow$ (Here stable = slope stable. $L \subseteq E \Rightarrow \deg L < \frac{1}{2} \deg E$)

Jack

Self-dual: $F_A = \sum F_{ij}\, dx^i \wedge dx^j$ say

So self-duality means

$$F_{12} = F_{34} \quad F_{13} = F_{42} \quad F_{14} = F_{23}$$

Write A as $A_1 dx^1 + A_2 dx^2 + \phi_1 dx^3 + \phi_2 dx^4$

dimensional reduction $\rightarrow$ & allow only A_1 & A_2 to vary. One has

$$F_{ij} = [\partial_i + A_i, \partial_j + A_j]$$

We've reduced to a connection $A_1 dx^1 + A_2 dx^2$ in 2-dimensions + Higgs Fields ϕ_1, ϕ_2.

Hitchin derives, for $\phi = \phi_1 + i\phi_2$, self-duality becomes

$$F = \tfrac{1}{2} i [\phi, \phi^*]$$

$$[\nabla_1 + i\nabla_2, \phi] = 0 \qquad \text{where } \nabla_i = d + A_i$$

Rewriting further, put
$$\Phi = \tfrac{1}{2}\phi\, dz \in \Omega^{1,0}(\mathbb{R}^2, ad(P) \otimes \mathbb{R})$$
$$\Phi^* = \tfrac{1}{2}\phi^*\, d\bar{z} \in \Omega^{0,1}(\mathbb{R}^2, ad(P) \otimes \mathbb{R})$$

& self-duality equations become

① $F_A = -[\Phi, \Phi^*]$

② $D_A'' \Phi = 0$

Note: Immediately, we see this ensures F_A is a (1,1)-form, so solutions (A, Φ) give holomorphic structure on E, wrt which $\Phi \in \Omega^{1,0}(End(E))$

Examples

① $\Phi = 0$: just gives flat connection

② K_X canonical on a Riemann surface

$K = K^{1/2} \oplus K^{1/2}$. Consider $K^{1/2} \oplus K^{-1/2}$ a vector bundle.

Then there's a tautological Higgs field

$$\Phi = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Self-duality becomes $F = -2 \overbrace{h\, dz\, d\bar{z}}^{\omega}$

constant -ive sectional curvature.

In the notation of last talk $D_A = D^+$, $\Phi + \Phi^* = \theta$

One can form a new connection

$$D = D_A + \Phi + \Phi^*$$

$$F_D = F_A + \underbrace{\Phi^2}_{0} + \Phi\Phi^* + \Phi^*\Phi + \underbrace{\Phi^{*2}}_{0}$$

$$= F_A + [\Phi, \Phi^*] = 0 \quad \text{if self-duality is satisfied}$$

Note D may no longer be unitary, but is some flat connection. We'll see every flat connection arises in this way.

Theorem:

If $L \subseteq V$ is Φ invariant then $\deg L \leq \frac{1}{2} \deg V$, i.e. V here is stable. (If V satisfies the equations)

Furthermore, given V^{stable}, rk 2 holomorphic Φ Higgs field, then $\exists A$ such that the system is self-dual.

$$(V, \Phi) \longrightarrow (V, D, h)$$

Jesse Wolfson

Principal Bundles

Lecture 4

Throughout, X is a compact Kähler manifold
G is a reductive Lie group.

We can generalise the non-abelian Hodge correspondence to give

$$\{\pi_1(X) \to G\} \xleftrightarrow[\text{Corlette}]{\text{Simpson}} \{G\text{-Higgs bundles}\}$$

with some stability.

1. G - Higgs Bundles

First, assume G is complex & reductive (adjoint representation is completely reducible). e.g $GL(n, \mathbb{C})$, but not (upper triangular matrices).

For $GL(n,\mathbb{C})$, have $E \in H^1(X, GL(n, \mathcal{O}_X))$

$$\phi \in H^0(X, \underset{\underset{ad(P_E)}{\shortparallel}}{End(E)} \otimes \Omega^1_X) \mid \quad \phi^2 = 0$$

Definition:

A G-Higgs bundle is a holomorphic principal G-bundle $P \to X$ & a Higgs field $\phi \in H^0(X, ad(P) \otimes \Omega^1_X)$ with $[\phi, \phi] = 0$.

Example:

$G = SL(n, \mathbb{C})$, so G-Higgs is a holo rank n vb E with $\det E \cong \mathcal{O}_X$, & traceless Higgs field.

Real Forms

Consider $GL(n, \mathbb{C})$ with $\mathfrak{gl}(n,\mathbb{C}) = Mat_n(\mathbb{C})$

We can find subgroups to corresp subalgebras

$U(n) \sim \mathfrak{u}(n)$

$GL(n,\mathbb{R}) \sim \mathfrak{gl}(n,\mathbb{R})$

giving splittings $\mathfrak{gl}(n,\mathbb{C}) \cong \mathfrak{u}(n) \oplus \mathfrak{m}$

$\mathfrak{gl}(n,\mathbb{R}) \oplus \mathfrak{m}'$

Both real forms recover the original groups when complexified. But they have different rep theory, e.g. $U(n)$ is compact, $GL(n,\mathbb{R})$ is not.

Furthermore, while $\mathfrak{u}(n)$ is its own maximal compact, we can further split $\mathfrak{gl}(n,\mathbb{R}) = \mathfrak{o}(n) \oplus \mathfrak{m}''$.

Definition

A split real group G is one such that for every (Cartan) decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{m}$ contains a maximal abelian subalgebras.

Examples:

- $GL(n,\mathbb{R})$
- $SU(p,q)$ $p,q>0$. (maybe. There was some discussion...)
- NOT: any compact group.

Definition

For G split real with maximal compact K, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ a G-Higgs bundle is a holomorphic principal bundle for $K^{\mathbb{C}}$ equipped with a holomorphic

$$\phi \in H^0(X, \Omega^1_X \otimes P(\mathfrak{m}^{\mathbb{C}})) \text{ w } [\phi,\phi]=0$$

$$P \times_{\text{isotropy}} \mathfrak{m}^{\mathbb{C}}$$

isotropy action of $K^{\mathbb{C}}$.

Examples:

- $GL(n, \mathbb{C}) \supseteq U(n)$

  $\mathfrak{gl}(n, \mathbb{C}) = \mathfrak{u}(n) \oplus \mathfrak{m}$, $\quad U(n)^{\mathbb{C}} = GL(n, \mathbb{C})$

  So P is a holo $GL(n, \mathbb{C})$-bundle, $\phi \in H^0(X, ad(P) \otimes \Omega^1_X)$

  as before.

- $U(n)$. $\mathfrak{u}(n)^{\mathbb{C}} = \mathfrak{gl}(n, \mathbb{C})$

  So a $U(n)$-Higgs bundle is just a rank n holomorphic vector bundle. Thus NAHT for $U(n)$ will recover Narasimhan-Seshadri.

- $SL(n, \mathbb{R})$. $\mathfrak{sl}(n, \mathbb{R}) = \mathfrak{so}(n) \oplus \mathfrak{m}$

  $\mathfrak{so}(n)^{\mathbb{C}} = \mathfrak{sl}(n, \mathbb{C})$

  So an $SL(n, \mathbb{R})$-Higgs bundle is a holo rank n vector bundle E, with $\det E \cong \mathcal{O}_X$, with

  - $\omega \in H^0(X, S^2 E^*)$ non-degenerate
  - $\phi \in H^0(X, End(E) \otimes \Omega^1_X)$ traceless symmetric

Definition (Stability) Assume V has vanishing Chern classes.

A G-Higgs bundle V is polystable if the associated Higgs bundle is polystable, for some faithful representation of G.

or the other way around!

Remark:

Stability is not so nice to define: criterion involves checking something on all antidominant characters of all parabolic subgroups.

2. Nonabelian Hodge Correspondence

Theorem: (Simpson)

There is a homeomorphism of moduli spaces

$$\{\text{Reps } \pi_1(X) \to G\} \longleftrightarrow \{\text{polystable } G\text{-Higgs bundles with vanishing generalised Chern classes}\}$$

There are two approaches to showing this. We'll concentrate on a gauge-theoretic approach

Assume X is a Riemann surface, G a connected semisimple real Lie group.

Let (E, ϕ) be a G-Higgs bundle. Say

$$\tau : \Omega^1_X(E(\mathfrak{g}^{\mathbb{C}})) \longrightarrow \Omega^1_X(E(\mathfrak{g}^{\mathbb{C}}))$$

is conjugation on Ω^1_X and $\mathfrak{g}^{\mathbb{C}}$.

Denote a K-structure h (reduction of structure group)

Theorem (Hitchin, BGG)

$\exists h$ satisfying the equation

$$F_h - [\phi, \tau(\phi)] = 0$$

if and only if (E, ϕ) is polystable.

F_h curvature of unique compatible conn.

3. Toledo Invariant & Milnor-Wood

X Riemann surface, G reductive real with maximal compact K. If (E, ϕ) is a G-Higgs bundle, by the short exact sequence

$$1 \to \pi_1(K^{\mathbb{C}}) \to \tilde{K}^{\mathbb{C}} \to K^{\mathbb{C}} \to 1$$

we get a map from L.e.s on cohomology

$$[X, BK] \longrightarrow [X, K(\pi_1 K^{\mathbb{C}}, 2)]$$

Fact: If G is connected split real then

$$\pi_1 G = \pi_1 K = \pi_1 K^{\mathbb{C}} = H^2(BK^{\mathbb{C}}; \mathbb{Z}) = H^2(K(\pi_1 K^{\mathbb{C}}, 2); \mathbb{Z})$$

Thus $\exists$ a class $\tau(E) \in H^2(E; \mathbb{Z})$ pulled back from a class in $H^2(X, \pi_1 G)$.

Definition:

G is hermitian if for any maximal compact K, G/K is a G-kähler manifold: G acts on it preserving the kähler form.

Definition:

$\operatorname{rank}(G/K) = \max \{ \dim T_p G/K : \text{curvature vanishes at } p \}$

Frequently, one writes just $\operatorname{rank}(G)$.

Fact: G/K is irreducible $\Rightarrow \pi_1(K^{\mathbb{C}}) / \text{torsion} \cong \mathbb{Z}$.

This is the setting we'll care about, as here $\tau(E)$ can be identified as an integer: the Toledo invariant:

The interesting thing is that polystability bounds this.

Proposition (Milnor–Wood)

Suppose X is a Riemann surface of genus g, G is hermitian, G/K irred. Given (E, ϕ) polystable, we have

$$|\tau(E)| \leq \operatorname{rank}(G/K)(g-1)$$

Example: (Goldman's Theorem)

Reps corresponding to maximal toledo invariant Higgs bundles into $SL(2,\mathbb{R})$ are precisely the Fuchsian reps.

Burger, Iozzi, Weinhard: Maximal reps are discrete embeddings of π_1

Hitchin: For $G = PSL(2,\mathbb{R})$, we can explicitly describe the subspaces of the moduli of reps corresponding to each Toledo invariant. Result:

$$\begin{array}{c} V \\ \downarrow \\ \mathrm{Sym}^4 \Sigma \end{array}$$

Hitchin components

$$\underset{2-2g}{\bullet} \quad \bullet \quad \cdots \quad \underset{2g-2-4}{\bullet} \quad \cdots \quad \bullet \quad \underset{2g-2}{\bullet}$$

Xia

Stacks & Parabolic Bundles

Lecture 5

Definition:

Let $\mathcal{C}$ be a site with fibre products. A stack $\mathcal{S}$ over $\mathcal{C}$ is a category with a functor $p: \mathcal{S} \to \mathcal{C}$ satisfying

i) $\mathcal{S}$ is a fibred groupoid / $\mathcal{C}$.
i.e. each object of $\mathcal{S}$ can be viewed as a functor $\mathcal{C}^{op} \to$ Groupoids.

ii) "descent of morphisms". For all $U \in ob\, \mathcal{C}$, $x, y \in \mathcal{S}(U)$, $Hom_U(x, y)$ defines a sheaf on $\mathcal{C}/U$.

iii) "descent of objects". For all $U \in ob\, \mathcal{C}$, any descent datum of objects wrt $(U_i \to U)$ is effective, i.e. "sections (s_i, U_i)" satisfying a cocycle condition come from a section (s, U).

Definition:

A Deligne-Mumford (DM) stack is a stack satisfying

i) $\mathcal{S} \xrightarrow{\Delta} \mathcal{S}^2$ diagonal is representable

ii) $\exists\, X \xrightarrow{\text{étale}} \mathcal{S}$ for X a scheme, where X is a stack via $h_X(T) = Hom(T, X)$.

i) means for all pullback diagrams

$$\begin{array}{ccc} Y' & \dashrightarrow & Y \\ \downarrow & & \downarrow \\ \mathcal{S} & \xrightarrow{\Delta} & \mathcal{S}^2 \end{array}$$

for Y a scheme, Y' is also a scheme.

Equivalently if X, Y are schemes so is the pullback $X \times_{\mathcal{S}} Y$.

2) means roughly that if $X \xrightarrow{\pi} S$ has a property (P), then so does $X \times_S Y \xrightarrow{\pi^-} Y$ as a morphism of schemes.

Example:

Let S be a scheme, G/S an algebraic group, étale of finite type over S. Then for a scheme X with action $G \circlearrowright X$ we can define the quotient stack $[X/G]$ by

$$[X/G](T) = \left\{ \begin{array}{c} Y \to X \\ \downarrow \\ T \end{array} : \begin{array}{l} Y/T \text{ locally trivial } G\text{-torsor} \\ Y \to X \ G\text{-equivariant} \end{array} \right\}$$

with morphisms

$$\begin{tikzcd} & X & \\ Y \arrow[ur] \arrow[rr, dashed] \arrow[d] & & Y' \arrow[ul] \arrow[d] \\ T \arrow[rr, dashed] & & T \end{tikzcd}$$

commutative diagrams.

Fact: $X \longrightarrow [X/G]$ is étale.

$$x \longmapsto \begin{array}{c} x \times G \\ \downarrow \\ x \end{array}$$

Topological Realisation

Definition:

Δ the category of finite orders has

- objects: $[0], [1], \ldots, [n], \ldots$ $\quad [n] = \{0, 1, 2, \ldots, n\}$
- morphisms: $[n] \longrightarrow [m]$ nondecreasing maps between sets

$$[0] \rightleftarrows [1] \;\substack{\longrightarrow\\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; [2] \quad \cdots$$

Definition:

The category of simplicial objects of a category $\mathcal{C}$ is the category of functors $\Delta^{op} \longrightarrow \mathcal{C}$.

We know the data of a stack is a functor $\mathcal{C}^{op} \to Gpd$ satisfying some conditions. So form the composition

$$\mathcal{C}^{op} \xrightarrow{\text{stack}} Gpd \xrightarrow{\text{Nerve}} sSet \underset{\text{singular complex}}{\overset{\text{realisation}}{\rightleftarrows}} Top$$

To take the <u>nerve</u> of a category, we consider

$[0] \rightsquigarrow \{\bullet\}$ objects

$[1] \rightsquigarrow \{\bullet \to \bullet\}$ morphisms

$[2] \rightsquigarrow \{\bullet \to \bullet \to \bullet\}$ composable morphisms

etc.

Given a <u>simplicial space</u> $X_\bullet$, one can produce a topological space by letting R^k be the standard k-simplex & $\forall\ \phi : [n] \to [m]$

$$\begin{array}{c} X_n \times R^n \\ \phi^* \uparrow \quad \downarrow \phi_* \\ X_m \times R^m \end{array}$$

defines a relation $\sim$

So we can produce a space

$$\left(\coprod X_n \times R^n \right) / \sim$$

This composition $\mathcal{C}^{op} \to Top$ is called the <u>topological realisation</u> of the stack.

<u>Example</u>

$\mathcal{C}$ = Schemes with the étale site, $\mathcal{S}$ a DM-stack over $\mathcal{C}$, so $\exists$ étale $Z \twoheadrightarrow \mathcal{S}$ then consider

$$\underbrace{Z^3 \substack{\rightarrow\\\rightarrow\\\rightarrow} Z^2 \rightrightarrows Z}_{\text{simplicial scheme}} \to \mathcal{S}$$

Then one can take the topological realisation of Z^{top}, $|Z^{top}|$. This is weakly equivalent to the topological realisation of the stack.

<u>Idea</u>:

We already know how to assign to a <u>scheme</u> a Top space, & we'd like to extend this to an assignment from <u>DM stacks</u> to top spaces in a universal way: this is precisely what we saw: form $\mathfrak{Z}$ as a homotopy colimit of schemes & take the topological realisation of that diagram.

Daniel

<u>Root stacks</u>:

For simplicity, let X be a smooth proper variety, $D = \cup D_i$ a normal crossings divisor, cut out by $z_1, \dots, z_p$.

We'll construct a stack

$$Z = X[D_1/n_1, \dots, D_k/n_k] \xrightarrow{p} X$$

that will locally look like a cover of X branched along D: Zariski-locally take $Y \subseteq U \times \mathbb{A}^b \to U$ to be $Y = \{ f_i = u_i^{n_i} \}$. There is a natural action of $G = \mathbb{Z}/n_1 \times \dots \times \mathbb{Z}/n_b$, so take the quotient to give a local model

$$Y/G \longrightarrow U$$

Fitting together to give a stack $Z \to X$, the <u>root stack</u>.

Given a finite Galois cover $Y \longrightarrow X$, étale away from D, then it lifts

$$\begin{tikzcd} Z \arrow[d] & \\ X & Y \arrow[l] \arrow[ul, dashed] \end{tikzcd}$$

IFF the kernel of the monodromy representation is contained in the product $n_1\mathbb{Z} \oplus \cdots \oplus n_k\mathbb{Z} \leq \mathbb{Z}^k$.
"sufficiently branched".

X is a coarse moduli space for Z.
Z comes with canonical divisors (D_i/n_i) such that $p^* D_i = n_i \cdot (D_i/n_i)$.

Facts: • $H^i(Z; \mathbb{Q}) \cong H^i(X; \mathbb{Q})$

• $CH^i(Z; \mathbb{Q}) \cong CH^i(X; \mathbb{Q})$

These are general facts about coarse moduli spaces.

Parabolic Bundles:

Let (X, D) be as above. The idea is we want a family of subsheaves indexed by $(\alpha_1, \dots, \alpha_k) \in \mathbb{Q}^k$, with $F_\alpha \subseteq F_\beta$ if $\alpha \leq \beta$ in all degrees, & if $\delta^i = (0, \dots, 0, \underset{\substack{\uparrow \\ i^{th}\text{ place}}}{1}, 0, \dots, 0)$, then

1) $F_{\alpha+\delta^i} = F_\alpha(D_i)$ (twist of sheaves)

2) $F_\alpha = F_{\alpha+\varepsilon}$ for ε small enough

Subsheaves of what? Well, they all sit inside
$j_*(F_\alpha|_{X \setminus D})$, where j is the inclusion.

Example:

- Take a vector bundle E on X, & define $E_\alpha^{triv} = E(\sum a_i D_i)$ where a_i is the smallest integer $\leq \alpha_i$

- Alternatively, consider $E(\sum \beta_i D_i)_\alpha = E(\sum a_i D_i)$ where a_i is the smallest integer $\leq \alpha_i + \beta_i$

Theorem:

Given a root stack $Z \longrightarrow X$, if one can choose a denominator large enough for the jumps then there's an equivalence of stack between vector bundles on Z & parabolic bundles, via push forward.

C. Simpson

Lecture 6

Non-abelian Hodge Analogy

We saw $\{\text{reps } \pi_1(X) \to G\}/\sim \; = H^1_B(X, G)$

$\{\text{Higgs bundles}\}/\sim \; = H^1_{Dol}(X; G)$

where $H^i_{Dol}(X; \mathbb{C}) = \bigoplus_{p+q=i} H^q(X, \Omega^p_X)$, the E_1 page of the Hodge-to-de-Rham spectral sequence, which degenerates.

In between, we can put something analogous to algebraic de Rham cohomology: i.e. look at the complex of sheaves $(\Omega^\bullet_X, d)$ on X^{zariski}. Take its hypercohomology

$$\mathbb{H}^i(X, \Omega^\bullet_X) = \mathbb{H}^i(X^{an}, \Omega^\bullet_{X^{an}}) \quad \text{by GAGA}$$

$$\overset{\shortparallel}{H^i(A^\bullet(X), d)}$$

Note $(\Omega^\bullet_X, d) \to (A^\bullet(X), d)$ provides a flasque resolution.

$\mathbb{H}^i(X, \Omega^\bullet_X)$ varies naturally in families over a scheme X, in a purely algebro-geometric way.

Have $H^i_{dR}(X_k) \subseteq H^i_{dR}(X)$ for $k \subseteq \mathbb{C}$ while we have $H^i_B(X, \mathbb{Q}) \subseteq H^i_B(X, \mathbb{C})$. The relationship between them is given by periods

In the non-compact setting, have

$$\{\text{Flat bundles}\}/\sim \; = H^1_{dR}(X; G)$$

i.e. E an algebraic vector bundle over E, $\nabla : E \to E \otimes_{\mathcal{O}_X} \Omega^1_X$ an algebraic flat connection.

Riemman - Hilbert Correspondence says

$$H^1_{dR}(X;G) \cong H^1_B(X;G)$$ for G a linear algebraic group

$G = \mathbb{G}_m$ was Serre's example of 2 analytically isomorphic varieties which were not algebraically isomorphic.

There is a similar phenomenon of periods here:
If X is defined $/k$, $G/\overline{\mathbb{Q}}$, then $H^1_{dR}(X, G)$ is also defined over k, so can look at its $\overline{k}$ points.
On the other hand $H^1_B(X, G)$ is defined over $\overline{\mathbb{Q}}$, so look at its $\overline{\mathbb{Q}}$ points.

e.g: if $k \subseteq \overline{\mathbb{Q}}$ we get 2 sets of points

$$H^1_B(X;G)(\overline{\mathbb{Q}}) \subseteq H^1_B(X,G) \cong H^1_{dR}(X;G) \supseteq H^1_{dR}(X,G)(\overline{\mathbb{Q}})$$

Conjecture:

The intersection of these two sets of points consists exactly of the motivic points: monodromy reps of families $Z \to X' \overset{\text{open}}{\subseteq} X$.

Transcendental number theory $\Rightarrow$ this is true for $G = \mathbb{G}_m$, but otherwise it's unknown.

Similarly, suppose we have a smooth family of varieties,

$\mathfrak{X}$ $\quad X = X_s$ then we get a family of $H^1_{dR}(X_s, G)$

$\downarrow$ $\quad$ $\downarrow$ over $\mathcal{S}$ varying algebraically over $\mathcal{S}$. We have a

$\mathcal{S}$ $\quad s$ Gauss-Manin connection = The isomonodromic deformation equation.

Example:

The Painleve VI equation is this for $SL(2)$ on $X_s = \mathbb{P}^1 \setminus 4$ points$_s$.

What about viewing this as a variation of Hodge structure? We would like to say that each $H^1_{dR}(X_s, G)$ has a Hodge Filtration: a filtration should be a $\mathbb{G}_m$-equivariant family over $\mathbb{A}^1$, so the idea is to define

$$\begin{array}{ccc} H^1_{Hodge}(X_s; G) & & \\ \downarrow & \circlearrowleft & \mathbb{G}_m \\ \mathbb{A}^1 & & \end{array}$$

with fibre H^1_{Dol} over 0, H^1_{dR} over 1: this is the moduli space of λ-connections.

Hitchin says $H^1_{Dol} = H^1_{dR}$ provides 2 complex structures forming part of a quaternionic structure (I, J, K) preserving a metric giving a hyper Kähler structure. Such a manifold M gives a twistor space $M \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ with complex structures coming from the hyperkähler family of complex structures.

Deligne: $H^1_{Hodge} = \{\lambda\text{-connections}\} \subseteq M \times \mathbb{P}^1$

$$\begin{array}{ccc} H^1_{Hodge} & \subseteq & M \times \mathbb{P}^1 \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & \subseteq & \mathbb{P}^1 \end{array}$$

In fact you can use this to construct the twistor space. There is a natural $\mathbb{G}_m$-action giving the Hodge Filtration i.e. get Griffiths transversality wrt Gauss-Manin connection.