

RELATIVE MALCEV COMPLETION

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ABSTRACT. This is a somewhat Orwellian version of my talk on relative Malcev completion and related topics: it is a combination of partially edited real-time notes (taken by Hiro) and my own notes. In particular, these notes include material that I never got to!

I will start with a few words about some results of Richard Hain. These are deep technical theorems, and I must say that my understanding of some of them is a bit nebulous so you should take things with a grain of salt.

Motivation:

We use classical Hodge theory to understand the local moduli of a complex variety X (say, smooth and proper) as follows.

First we look for some topological invariant, like integral homology or cohomology, $H^k(X, \mathbb{Z})$. Then we complexify, look at the Hodge filtration on $H^k(X, \mathbb{C})$, and study how the filtration varies with the complex structure.

Now, $H^k(X, \mathbb{Z})$ is a nice finitely generated abelian group. Its complexification can be described by differential forms, and the filtration is “by the number of dz ’s”. To see how the filtration (on $H^1(X, \mathbb{C})$) varies we look at periods.

Question:

What is the analogue of the above if we want to study a *pointed* variety (X, x_0) ?

For example X could be a moduli space of some objects (e.g., one of our Dolbeault, de Rham or Betti moduli spaces) and x_0 could an object of interest (e.g., a variation of Hodge structures).

First we need some topological invariant which keeps track of the point $x_0 \in X$. A natural candidate is the fundamental group, $\pi_1(X, x_0)$. In general, it is non-abelian, so a better attempt is to take the group ring $\mathbb{Z}\pi_1(X, x_0)$. It is abelian but not finitely generated. A natural way to fix that is to truncate it and consider instead

$$\mathbb{Z}\pi_1(X, x_0)/J^k,$$

where J is the augmentation ideal.

Recall:

Let π be a discrete group, R a commutative ring with 1 and $R\pi$ the group algebra of π , that is, the set of all finite linear combinations $\sum_{g \in \pi} r_g g$, $r_g \in R$. Then the augmentation ideal J is the kernel of the ring homomorphism $\epsilon : R\pi \rightarrow R$, $g \mapsto 1$. The powers J^k determine a topology on $R\pi$ (often non-separated), and the J -adic completion is the topological R -algebra

$$\widehat{R\pi} = \varprojlim R\pi/J^k.$$

Fun Exercise:

In the above notation, the homomorphism

$$\pi^{ab} \rightarrow J/J^2, \quad g \mapsto (g - 1) \bmod J^2$$

induces an isomorphism $H_1(\pi; R) = \pi^{ab} \otimes_{\mathbb{Z}} R \simeq J/J^2$. This is $H_1(X, R)$ if $\pi = \pi_1(X, x_0)$.

So, $\mathbb{Z}\pi_1(X, x_0)$ will be our substitute for the integral cohomology. Next we want a substitute for the Hodge filtration, possibly constructed in terms of differential forms. Notice that for composable paths α, β and a 1-form ω we have $\int_{\alpha\beta} \omega = \int_{\alpha} \omega + \int_{\beta} \omega = \int_{\beta\alpha} \omega$, so usual integrals of 1-forms can see only elements of $\pi_1(X, x_0)$ which are visible in $H^1(X, \mathbb{C})$.

Chen looked at a generalisation of line integrals – the *iterated line integrals* – and found the right substitute for the Hodge filtration. We heard a bit about this in Mike’s talk. Given 1-forms $\omega_1, \dots, \omega_r$, and a loop γ based at x_0 , one looks at

$$\int_{\gamma} \omega_1 \dots \omega_r = \int_{\Delta} \dots \int f_1(t_1) \dots f_r(t_r) dt_1 \dots dt_r$$

where $\gamma^* \omega_i = f_i(t_i) dt_i$ and Δ is the time-ordered simplex $0 \leq t_1 \leq \dots \leq t_r \leq 1$. Sometimes this expression is called a “basic iterated integral”. For this quantity to make sense (and have good properties) it is enough that X be a real manifold, and that ω_i be \mathbb{R} -valued differential forms, or more generally, differential forms with values in some associative \mathbb{R} -algebra, e.g., \mathbb{C} , $\mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{gl}(n, \mathbb{C})$. One could also consider paths γ joining two different points $a, b \in X$.

An iterated integral of length at most s is a linear combination of constants and basic integrals with $r \leq s$. We think of the iterated integral as a map from the space of loops (based at x_0) to, say, \mathbb{C} . Of course, the value of this function could depend on the path γ , so we restrict attention to iterated integrals which are homotopy functionals (depend only on the homotopy class of γ). I will denote the iterated integrals by \mathcal{I}_{x_0} and the ones that are homotopy functionals by $H^0(\mathcal{I}_x)$. The iterated integrals are *filtered* by the length s (they are not graded!) and this induces a filtration on the homotopy functionals. If $s = 1$, an integral is a homotopy functional iff $d\omega = 0$. For iterated integrals of higher length things are more complicated. In fact, one of the ingredients of Chen’s theorem is an explicit description of $H^0(\mathcal{I}_{x_0})$ in terms of the bar construction on the de Rham complex of X .

Aaron: Chen gives a formula for the exterior differential of an iterated integral, it involves shuffles and such; and you can write it in terms of differentials of stuff in the inside. This is in the original iterated integral paper.

Peter: Iterated integrals have been around for a while. For instance, they are used in the proof of the existence and uniqueness theorem for ODE (Picard-Lindelöf) where one uses Picard iterations to write the “evolution operator” as a “time-ordered exponent”. This is the same object that appears in Chen’s and Hain’s work when they discuss parallel transport on vector bundles with flat connection.

Then there are two statements in Chen’s π_1 de Rham theorem.

Theorem 0.1 (Chen’s Theorem). (1) *For every $s \geq 0$, integration induces an isomorphism*

$$H^0(\mathcal{I}_{x_0})_s \simeq \text{Hom}_{\mathbb{Z}}(\mathbb{Z}\pi_1/J^{s+1}, \mathbb{C})$$

(2) *Integration induces a Hopf algebra isomorphism $H^0(\mathcal{I}_{x_0}) \simeq \mathcal{O}(\mathfrak{U})$, where $\mathcal{O}(\mathfrak{U})$ is the coordinate ring of the unipotent completion \mathfrak{U} of $\pi_1(X, x_0)$.*

I am going to recall all the relevant definitions in a minute.

One of the main applications of this theorem is a direct construction of Morgan's MHS on $\mathcal{O}(\mathfrak{U})$.

R.Hain took many of these results much further. He generalised the notion of iterated integrals, and obtained a Hopf algebra isomorphism between the coordinate ring of a certain relative Malcev completion and the Hopf algebra of "locally constant iterated integrals" (which can be constructed algebraically using a suitable bar construction). However, the first big theorem of R.Hain that I want to talk about is Theorem 13.1 from *The Hodge-de Rham Theory of the relative Malcev completion*.

Theorem 0.2. *Let X be smooth, complex algebraic variety, or the complement of a divisor with normal crossings in a compact Kähler manifold, and let $x_0 \in X$. Let V be an admissible VHS on X , V_0 the fiber of V at x_0 and $S := \text{Aut}(V_0, \langle, \rangle)$. Suppose that the monodromy $\rho : \pi_1(X, x_0) \rightarrow S$ is Zariski-dense.*

Let $\tilde{\rho} : \pi_1(X, x_0) \rightarrow \mathcal{G} := \mathcal{G}(x, x_0)$ be the completion of $\pi_1(X, x_0)$ relative to ρ .

Then the coordinate ring $\mathcal{O}(\mathcal{G})$ of \mathcal{G} has a canonical real MHS with non-negative weights.

Moreover, the canonical map $\mathcal{O}(S) \rightarrow \mathcal{O}(\mathcal{G})$, the multiplication, comultiplication and antipode on $\mathcal{O}(\mathcal{G})$ are all morphisms of MHS. Finally, the canonical homomorphism $\mathcal{G} \rightarrow S$ induces an isomorphism $Gr_0^W \mathcal{O}(\mathcal{G}) \simeq \mathcal{O}(S)$.

I will discuss now the various notions that appear in the statement of this theorem.

I will not explain what an "admissible" V(M)HS is. For a V(M)HS on a punctured disk (around x_0) admissibility is a certain condition on the behaviour at x_0 (the monodromy has to be quasi-unipotent, etc. etc.); for higher-dimensional X admissibility can be tested on curves through x_0 . Consult Steenbrink-Zucker and Kashiwara for the (very technical) definition.

The above theorem states that the coordinate ring $\mathcal{O}(\mathcal{G})$ of a certain pro-algebraic group \mathcal{G} carries a MHS which is compatible with the Hopf algebra structure on $\mathcal{O}(\mathcal{G})$.

Hopf algebras

Recall that a Hopf algebra over a field k is a unital k -algebra H (so it is a k -vector space with multiplication $H \otimes H \rightarrow H$ and unit $k \rightarrow H$) which has extra structure: *comultiplication* $\Delta : H \rightarrow H \otimes H$, a *counit* $\epsilon : H \rightarrow k$, and an *antipode* $S : H \rightarrow H$. Here Δ and ϵ satisfy properties which are obtained from the properties of the multiplication and unit by reversing all arrows, and the properties of the antipode are obtained by formalising the first example below. There are also natural compatibility properties between the different structures. Equivalently, one can also say that a Hopf algebra is bialgebra H (i.e., it is both an algebra and a coalgebra in a compatible way) with an antipode.

Example: Functions on a group

Let G be a *finite group* or an affine algebraic group over k , and $\mathcal{O}(G)$ the ring of regular functions on it. Then $H = \mathcal{O}(G)$ is a Hopf algebra. The comultiplication of $\mathcal{O}(G)$ is the dual to the multiplication on G , i.e., $\Delta f(g, h) = f(gh)$. The antipode is dual to inverting elements, i.e., $S(f)(g) = f(g^{-1})$. The counit is given by evaluation at the identity: $\epsilon(f) = f(1)$. This is the kind of Hopf algebra appearing in the right hand side of statement (2) of Chen's theorem – except that there one has the coordinate ring $\mathcal{O}(\mathfrak{U})$ of a pro-algebraic group.

Example: Group Algebra

If G is a group and $H = kG$ its group algebra, then H has is naturally a Hopf algebra with $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$ and $S(g) = g^{-1}$.

Example: Universal Enveloping

Let \mathfrak{g} be a Lie algebra. Then its universal enveloping algebra $\mathcal{U}\mathfrak{g}$ is a Hopf algebra with $\Delta x = x \otimes 1 + 1 \otimes x$, $\epsilon(x) = 0$, $S(x) = -x$.

Example: $H^\bullet(G)$

If G is a Lie group, its cohomology ring $H^\bullet(G)$ is a Hopf algebra with multiplication given by cup product and comultiplication and antipode induced by the operations on G . This was the original example studied by Hopf.

Example: Iterated Integrals

The set of iterated integrals \mathcal{I}_{x_0} on a manifold X forms a Hopf algebra, where the comultiplication is induced by path multiplication and the antipode is induced by reversing the path. In fact, the three operations are described by the formulas

$$\int_{\alpha\beta} \omega_1 \dots \omega_r = \sum_{j=0}^r \int_{\alpha} \omega_1 \dots \omega_j \int_{\beta} \omega_{j+1} \dots \omega_r,$$

$$\int_{\alpha} \omega_1 \dots \omega_r \int_{\alpha} \omega_{r+1} \dots \omega_{r+s} = \sum_{Sh(r,s)} \int_{\alpha} \omega_{\sigma(1)} \dots \omega_{\sigma(r+s)}$$

$$\int_{\alpha^{-1}} \omega_1 \dots \omega_r = (-1)^r \int_{\alpha} \omega_r \omega_{r-1} \dots \omega_1$$

Here α and β are composable paths. The basic properties of iterated integrals are due to K.T.Chen. The Hopf-algebraic formulation is due to R.Hain. This is the kind of Hopf algebra appearing in the left hand side of statement (2) of Chen's theorem.

Fun Exercise:

If G is an affine algebraic group, we can recover $\mathfrak{g} = \text{Lie } G$ from $\mathcal{O}(G)$.

The Relative Malcev Completion

Let π be an abstract group and S a linear algebraic group over a field k of char 0. Let $\rho : \pi \rightarrow S$ be a representation with Zariski-dense image. Look at diagrams of the form

$$\begin{array}{ccccccc} 1 & \longrightarrow & U & \longrightarrow & E & \longrightarrow & S \longrightarrow 1 \\ & & & & \uparrow \tilde{\rho} & \nearrow \rho & \\ & & & & \pi & & \end{array}$$

where E is a linear algebraic group (over k), U is unipotent, and $\tilde{\rho}$ is lifting of ρ with Z-dense image. One checks that the set of such diagrams forms an inverse system. The *completion of π relative to ρ* or the *relative Malcev completion* of π is the projective limit over such diagrams,

$$\mathcal{G} = \lim_{\leftarrow} E.$$

It comes with a canonical homomorphism $\hat{\rho} : \pi \rightarrow \mathcal{G}$ which is sometimes called “the S -completion of π ”.

If $S = \{1\}$ then $\mathcal{G} = \mathfrak{U}$ is the pro-unipotent (classical Malcev) completion of π .

David: Mike said U had to be the unipotent radical of E .

Carlos: Well if S itself wasn't reductive, that definition wouldn't work very well.

So this is a more general definition which works well for other groups.

Being a limit, the proalgebraic group \mathcal{G} is characterised by a universal property. Namely, if G is a pro-algebraic group with a surjection onto S with pro-unipotent kernel

$$1 \longrightarrow U \longrightarrow G \longrightarrow S \longrightarrow 1$$

and $\phi : \pi \rightarrow G$ has a \mathbb{Z} -dense image, then there is a unique map $\mathcal{G} \rightarrow G$ so that the map ϕ factors through it, i.e.

$$\begin{array}{ccc} & \mathcal{G} & \longrightarrow S \\ \nearrow & \downarrow & \nearrow \\ \pi & \xrightarrow{\phi} & G \end{array}$$

There is an explicit construction of the classical Malcev completion ($S = \{1\}$) due to Quillen.

Look at the completion of $k\pi$ at the augmentation ideal. It has a comultiplication $\Delta : \widehat{k\pi} \rightarrow \widehat{k\pi} \widehat{\otimes} \widehat{k\pi}$ induced by the one on $k\pi$. Let \mathcal{G} denote the set of group-like of $\widehat{k\pi}$. These are the elements X which satisfy $\Delta(X) = X \otimes X$ and $\epsilon(X) = 1$. We have natural maps $\pi \rightarrow k\pi$ and $k\pi \rightarrow \widehat{k\pi}$, whose composition lands in \mathcal{G} .

Fact:

If $H_1(\pi, k)$ is finite-dimensional, the natural map $\pi \rightarrow \mathcal{G}$ is the Malcev completion of π .

So this takes care of the case $S = \{1\}$.

Carlos: When S is a linear group, it acts on k^n . Then there's a map from π to $\text{End}(k^n)$, and you could take the kernel of this map and complete with respect to it. That should be the relative version of the above.

So these were the main ingredients in Hain's theorem. . .

Carlos: Maybe something to stress is that this MHS depends on choice of basepoint. It gives you a variation of mixed hodge structure over X . As the basepoint moves around, this $\mathcal{O}(\mathcal{G})$ considered as a mixed hodge structure forms a VMHS over X . It's the universal VMHS whose reductive pieces will look like ρ .

Peter: Hain and Pulte actually proved various Torelli-type results. For instance, if we have two curves, X and Y , with marked points p and q , and if there is a ring isomorphism $\mathbb{Z}\pi_1(X, p)/J_p^3 \rightarrow \mathbb{Z}\pi_1(Y, q)/J_q^3$ which is compatible with the Hodge structure, then, with the possible exception of two points, there is a biholomorphism from X to Y which maps p to q .

Carlos: The point is, we need an invariant which distinguishes the point — an invariant of the pointed variety, not just of the variety.

Carlos: Now instead of taking the Malcev completion, you could take the inverse map of reps to all groups (this is what you get by Tannaka duality). This breaks down into the Malcev completions of all kinds of different groups; these are the local splits.

Carlos: At each piece, the proalgebraic completion of π_1 will have this Malcev completion sitting there.

Sam: Now if X is compact, is this a pure Hodge structure?

Carlos: No. These homotopy groups are never pure.

Peter: In fact, the theorem includes a statement about the weight zero piece of $\mathcal{O}(\mathcal{G})$: it is isomorphic to $\mathcal{O}(S)$, which is definitely not the whole thing.

Carlos: There's MHS on the rational homotopy groups in the simply connected case; if π_3 maps to H_3 , the kernel is something having to do with $H_2 \otimes H_2$, and that's already mixed. That's going to give you the weight filtration.

Carlos: And the filtration of the $\mathbb{Z}\pi_1$ by the augmentation ideal is almost never going to be trivial.

There are "fancier" versions of the previous theorem. For $a, b \in X$, let $\pi(X, a, b)$ be the set of paths between a and b up to homotopy. We define the fundamental groupoid of X to be the category $\pi(X)$ with objects the points of X and morphisms paths up to homotopy. We can think of this as a torsor over $X \times X$, whose fibre over (a, b) is $\pi(X, a, b)$. Since $\pi_1(X \times X, (a, a)) = \pi_1(X, x) \times \pi_1(X, x)$, one can

Check:

The torsor $\pi(X)$ corresponds to the representation $\pi_1(X \times X, (a, a)) \rightarrow \text{Aut } \pi_1(X, a)$, $(\gamma, \mu) \mapsto \{g \mapsto \gamma^{-1}g\mu\}$.

Next, suppose that $\rho : \pi_1(X, x_0) \rightarrow S$ be a representation with \mathbb{Z} -dense image, and let $\pi_1(X, x_0) \rightarrow \mathcal{G}$ be the completion relative to ρ . We can then extend the above representation to a representation $\pi_1(X \times X, (a, a)) \rightarrow \text{Aut } \mathcal{G}$, and get a torsor, $\underline{\mathcal{G}}$, over $X \times X$. Its fibres $\mathcal{G}_{a,b}$ come with "multiplication maps" $\mathcal{G}_{a,b} \times \mathcal{G}_{b,c} \rightarrow \mathcal{G}_{a,c}$. Then R. Hain proved:

Theorem 0.3 (Theorem 13.3). *Keep the notation from Theorem 0.2 and let $a, b \in X$. Then $\mathcal{O}(\mathcal{G}_{a,b})$ carries a canonical MHS with non-negative weights. If $a, b, c \in X$ are three points, the map $\mathcal{O}(\mathcal{G}_{a,c}) \rightarrow \mathcal{O}(\mathcal{G}_{a,b}) \otimes \mathcal{O}(\mathcal{G}_{b,c})$ induced by path multiplication is a morphism of MHS.*

Theorem 0.4 (Theorem 13.10, Corollary 13.11). *If X is a compact Kähler manifold, the local system on $X \times X$ with fibre $\mathcal{O}(\mathcal{G}_{a,b})$ underlies an admissible variation of MHS on $X \times X$.*

It is expected that these VMHS are related to the ones discussed in Simpson-Eyssidieux. There the authors consider a complex VHS of weight zero on a compact Kähler manifold X , and consider its monodromy representation $\rho : \pi_1(X, x_0) \rightarrow GL_N(\mathbb{C})$ as a point in the representation variety $R(\pi_1(X, x_0), GL_N(\mathbb{C}))$. Using Goldman-Millson techniques, one can identify $Spf(\widehat{\mathcal{O}}_\rho)$ with $\widehat{\Omega}_\rho \times T$, where T is the (formal germ at the origin of the) quadratic cone given by the obstruction map, and $\widehat{\Omega}_\rho$ is the germ at ρ of the $GL_N(\mathbb{C})$ -orbit of ρ . The natural map $T \rightarrow Spf(\widehat{\mathcal{O}}_\rho)$ induces representations $\rho_n : \pi_1(X, x_0) \rightarrow GL_N(\widehat{\mathcal{O}}_T/\mathfrak{m}^n)$. The powers of the maximal ideal induce a decreasing filtration on the corresponding local system, and Simpson-Eyssidieux show that the corresponding increasing filtration is a \mathbb{C} VMHS on X .