APPLICATIONS OF GOLDMAN-MILLSON THEORY TO NAHT

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ABSTRACT. These are my notes for the talk on Goldman-Millson theory.

First we fix notation. Let G be a complex reductive Lie group, $\mathfrak{g} = \mathrm{Lie}G$, $X-$ smooth projective variety. Many of the statements that follow will actually work for a compact Kähler X.

1. Some examples

Hiro told us that every deformation problem in characteristic zero is goverened by a dgla and discussed the Betti deformation space. Let us look at two other relevant examples.

Example
Let (P, φ) be a principal G-Higgs bundle, $\varphi \in H^0(X, \mathrm{ad}P)$, $\varphi \wedge \varphi = 0$. Then the deformations of (P, φ) are controlled by

$$L^\bullet_{Dol}(P, \varphi) = \left(A^\bullet(\mathrm{ad}P), \overline{\partial}_P + \mathrm{ad}\varphi\right).$$

For simplicity, we will usually write $L^\bullet_{Dol}$ for $L^\bullet_{Dol}(P, \varphi)$ and $\overline{\partial}$ for $\overline{\partial}_P$, the Dolbeault operator of P.

If we know P and φ reasonably well, we can be fairly explicit in working with this complex. Notice that $L^\bullet_{Dol}$ is the total complex of the Dolbeaul resolution of certain complex of holomorphic vector bundles.

In particular, for $\dim X = 1$, we have

$$L_{Dol}(P, \varphi) = A^{0,0}(\mathrm{ad}P) \bigoplus A^{1,0}(\mathrm{ad}P) \oplus A^{0,1}(\mathrm{ad}P) \bigoplus A^{1,1}(\mathrm{ad}P),$$

which is the total complex of

$$\begin{array}{ccc} A^{0,1}(\mathrm{ad}P) & \xrightarrow{\mathrm{ad}\varphi} & A^{1,1}(\mathrm{ad}P)\ . \\ \uparrow{\scriptstyle \overline{\partial}} & & \uparrow{\scriptstyle \overline{\partial}} \\ A^{0,0}(\mathrm{ad}P) & \xrightarrow{\mathrm{ad}\varphi} & A^{1,0}(\mathrm{ad}P) \end{array}$$

This is Dolbeault resolution of the complex $\mathrm{ad}P \xrightarrow{\mathrm{ad}\varphi} \mathrm{ad}P \otimes \Omega^1_X$ introduced by Biswas and Ramanan.

Example
Let (P, D) be a principal G-bundle with a flat connection $D = d' + d''$. As usual, if we want to stay entirely in the holomorphic/analytic category, we treat this data as a holomorphic flat connection d'' on the holomorphic principal G-bundle (P, d'').

Then the controlling dgla is $L^\bullet_{DR}(P,D) = (A^\bullet(\mathrm{ad}P), D)$. For instance, for a curve this is

$$\begin{array}{ccc} A^{0,1}(\mathrm{ad}P) & \xrightarrow{d'} & A^{1,1}(\mathrm{ad}P) \\ {\scriptstyle d''}\uparrow & & \uparrow{\scriptstyle d''} \\ A^{0,0}(\mathrm{ad}P) & \xrightarrow{d'} & A^{1,0}(\mathrm{ad}P). \end{array}$$

If the Higgs and flat bundle are related by a harmonic metric, we have, as usual,

$$D = d' + d'' = \overline{\partial}_P + \partial + \mathrm{ad}\varphi + \mathrm{ad}\overline{\varphi} = D^+ + \theta = D' + D'',$$

where $D^+ = \overline{\partial}_P + \partial$ is the metric-compatible piece of D, $\theta = \varphi + \overline{\varphi}$, $D'' = \overline{\partial} + \mathrm{ad}\varphi$, $D' = \partial + \mathrm{ad}\overline{\varphi}$.

If we fix a point $x \in X$, we can look at the evaluation map $\epsilon : A^0(\mathrm{ad}P) \to \mathfrak{g}$. This gives us a *deformation diagram*, also known as $\mathfrak{g}$-*augmented dgla*, i.e., a pair $(L^\bullet, \epsilon)$ where $\epsilon : L^0 \to \mathfrak{g}$ is a morphism of dglas. The deformation diagram is *rigid* if $\epsilon : H^0(L^\bullet) \hookrightarrow \mathfrak{g}$. We are also going to assume that $L^\bullet$ is non-negatively graded and $\dim H^i(L^\bullet) < \infty$.

2. Prorepresentability Results

As Hiro explained, to a dgla we can assign the corresponding functors from $\mathrm{Art}_\mathbb{C}$ to groupoids (DGM) and to Sets ($IsoDGM$). In the terminology of Marco Manetti, these are the functors MC and Def_L. One then is interested in the prorepresentability of the deformation functor or, if less lucky, in having a hull for $IsoDGM$ or pro-representing a different functor (see below).

Theorem 2.1. *If $(L^\bullet, \epsilon)$ is formal, rigid and $\epsilon : L^0 \to \mathfrak{g} \to 0$, then $IsoDGM(\ker\epsilon)$ is prorepresented by the germ at the origin of*

$$C_{H(L)} \times \mathfrak{g}/\epsilon(H^0),$$

where $C_{H(L)}$ is the cone in H^1 given by the cup product.

Corollary 2.1. *Given a formal dgla $L^\bullet$ with $H^0(L^\bullet)$, $IsoDGM(L^\bullet)$ is prorepresented by the germ at the origin of $C_{H(L)}$.*

The corollary follows from the Theorem by taking $\mathfrak{g} = 0$ and $\epsilon = 0$.

Notation: If we have a transformation groupoid (X, G) and a set Y with G-action, we define a new transformation groupoid, $(X,G) \bowtie Y := (X \times Y, G)$, where G acts on the product by the diagonal action.

The proof of the theorem uses the following Lemma, which is also often used in applications.

Lemma 2.1. *If $((L^\bullet, d = 0), \epsilon)$ is rigid, then $C_{H(L)} \times \mathfrak{g}/\epsilon(L^0)$ prorepresents the functor $IsoDGM(L, \varepsilon,\ ) := IsoDGM(L) \bowtie \exp(\mathfrak{g} \otimes \mathfrak{m})$.*

It is *not* true that $IsoDGM(L^\bullet)$ is always prorepresentable! However, every *splitting*, δ, of $L^\bullet$ gives a hull $Kur^\delta \to IsoDGM(L^\bullet)$.

Intuitively: In abelian Hodge theory $\delta = Gd^*$, where G is the Green operator.

Formally, $\delta \in Hom^{-1}(L^\bullet, L^\bullet)$, such that $\delta^2 = 0$, $\delta d \delta = \delta$, $d\delta d = d$. Then this gives a decomposition $L^i = B^i \oplus \mathcal{H}^i \oplus C^i = im d \oplus \ker d \cap \ker \delta \oplus im\delta$. One then shows that the functor Kur^δ is prorepresented by the germ of the origin of the space

$$\mathcal{K} = \ker\left(\eta \mapsto pr_{\mathcal{H}}[\Phi^{-1}(\eta), \Phi^{-1}(\eta)]\right) \subset \mathcal{H}^1,$$

where Φ is the Kuranishi map $\Phi(x) = x + \frac{1}{2}\delta[x,x]$.

$\mathcal{K}$ is invariant under quasi-isomorphisms, in particular, if $(L^\bullet, d)$ is *formal*, $\mathcal{K} = C_{H(L)}$.

A version of the above statements is proved in Simpson-Eyssidieux:
If $(L^\bullet, \epsilon)$ is *rigid*, $IsoDGM(L, \epsilon, \ )$ is prorepresented by the germ at the origin of $\ker\left(\mathcal{H}^1 \ni \eta \mapsto pr_{\mathcal{H}}[\eta,\eta] \in \mathcal{H}^2\right)$.

3. First Applications

Theorem 3.1. *Suppose (P,p) is a point in a closed orbit of $R_{DR/Dol}(X,x,G)$ and $C \subset H^1(L^\bullet_{DR/Dol})$ is the corresponding quadratic cone. Then the functor $IsoDGM(L^\bullet_{DR/Dol}, \epsilon, \ )$ is prorepresented by $C \times h^\perp$, where $h^\perp$ is the orthogonal complement to $\epsilon(A^0(adP)) \subset \mathfrak{g}$. Consequently, the completions of the moduli spaces at the origin satisfy:*

$$R_{Dol}\widehat{(X,x},G) \simeq R_{Dol}\widehat{(X,x},G),\ \ M_{Dol}\widehat{(X},G) \simeq M_{DR}\widehat{(X},G)$$

Idea of Proof:
Prove formality and then apply the GM theorem. For example, starting with a Higgs bundle, pick an HYM metric on (P,p), and write the usual decomposition:

$$D = D' + D'' = \overline{\partial}_P + \partial + \mathrm{ad}\varphi + \mathrm{ad}\overline{\varphi},\ \ D' = \partial + \mathrm{ad}\overline{\varphi},\ \ D'' = \overline{\partial} + \mathrm{ad}\varphi.$$

There are natural morphisms of dgla's

$$(L^\bullet_{Dol}, \epsilon) \longleftarrow ((\ker D', D''), \epsilon) \longrightarrow ((\mathcal{H}^\bullet, 0), \epsilon)\,.$$

By the "principle of two types" ($\partial\overline{\partial}$-lemma), this is a quasi-isomorphism. A similar argument works for the De Rham deformation theory.

4. Relative GM theory

Next we need a relative version of GM theory in order to deal with R_{Hod}. Consider $(L^\bullet, d)$: a dgla over $\mathbb{C}[\lambda]$, where the L^i are flat $\mathbb{C}[\lambda]$-modules. We can assign to it a functor $\mathrm{Art}_{\mathbb{C}[\lambda]} \to Gpds$, using the same formulas as before. (Here $\mathrm{Art}_{\mathbb{C}[\lambda]}$ denotes the category of local Artin $\mathbb{C}[\lambda]$-algebras.) That is, for a given $B \in \mathrm{Art}_{\mathbb{C}[\lambda]}$, the objects of our groupoid are elements $\eta \in L^1 \otimes_{\mathbb{C}[\lambda]} \mathfrak{m}_B$, satisfying $d\eta + \frac{1}{2}[\eta,\eta] = 0$, and the morphisms between two objects η, η' are determined by $\eta' = ds + e^s \eta e^{-s}$.

If P is a flat principal G-bundle with harmonic K-reductions P_K (where $K \subset G$ is the maximal compact), we consider the dgla

$$L^\bullet_{Hod}(P) = (A^\bullet(\mathrm{ad}P) \otimes_{\mathbb{C}} \mathbb{C}[\lambda], \lambda D' + D'')$$

One then shows:

- $L^\bullet_{Hod}$ gives the deformation theory of $\mathcal{M}_{Hod}$ near the twistor line of P. For that one has to check that if $B \in \mathrm{Art}_{\mathbb{C}[\lambda]}$, then $L^\bullet_{Hod}(B)$ is the groupoid with objects maps $Spec(B) \to \mathcal{M}_{Hod}$ plus an isomorphism of the induced map $Spec(B/\mathfrak{m}) \to \mathcal{M}_{Hod}$ with $Spec(B/\mathfrak{m}) \to \mathbb{A}^1 \to \mathcal{M}_{Hod}$ (the second arrow being the preferred section determined by P).
- $L^\bullet_{Hod}$ is formal over $\mathbb{C}[\lambda]$: this involves the same strategy as above. By proving a statement about harmonic elements being a product, one gets

$$\widehat{R_{Hod}}(X,x,G) \simeq \widehat{R_{Dol}}(X,x,G) \widehat{\times} \mathbb{A}^1,$$

  and similary for M_{Hod} and M_{Dol}.

- In fact (using Artin's Approximation Theorem) one shows that the above holds etale locally trivially. Namely, given $P \in M_{Hod}(X,G)_\lambda$, there exists an etale neighbourhood $U \to M_{Hod}(X,G)$ of P, and and etale morphism $U \to M_{Hod}(X,G)_\lambda \times \mathbb{A}^1$.
- The above implies the *isosingularity principle*: Given a point $y \in R_{Dol}(X,x,G)$, there exist $z \in R_{DR}(X,x,G)$, together with isomorphic etale neighbourhoods. The local systems corresponding to y and z have isomorphic semisimplifications.