NOTES FROM TALBOT, 2010

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Contents

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1. Overview, Constantin Teleman UC Berkeley

This goes way back to Frobenius with character theory of finite groups. So let G be a finite group and

$$\sum_{irreps\ V} \mathrm{dim} V^2 = \#G.$$

Then $Z = \mathbb{C}[G]^G \hookrightarrow \mathbb{C}[G]$ is an algebra with convolution.

$$\phi \star \psi(g) = \sum_{h \in G} \phi(gh^{-1})\psi(h).$$

We also get a trace, $t : Z \to \mathbb{C}$,

$$t(\phi) = \frac{\phi(1)}{\#G}$$

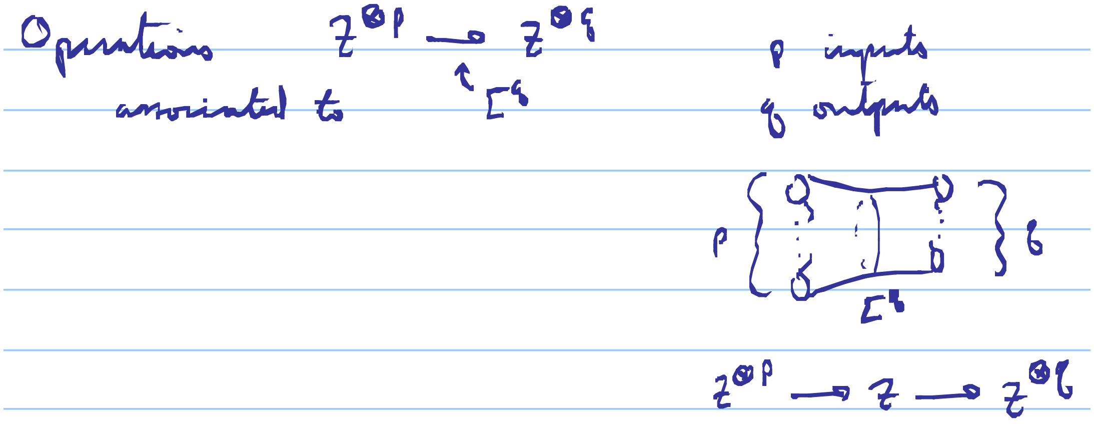

FIGURE 1. The bordism ${}^p\Sigma^q$.

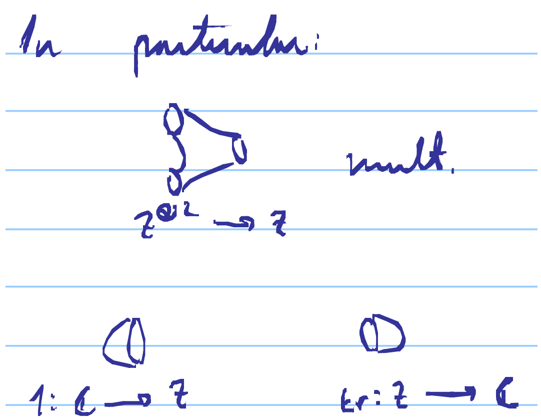

FIGURE 2. Bordisms determining a Frobenius algebra structure..

Get a nondegenerate pairing $Z \times Z \to \mathbb{C}$, $(z_1, z_2) \mapsto t(z_1 \times z_2)$. Then $Z \cong \oplus \mathbb{C} P_i$, $t(P_i) = \frac{dim \chi_i^2}{(\#G)^2}$,

$$P_i = \frac{\chi_i dim \chi_i}{\#G}.$$

This is the structure of a commutative frobenius algebra, and from a more modern perspective we have

Theorem 1.1. *Commutative frobenius algebras are the same as 2-dimensional TQFTs*

We can think of operations $Z^{\otimes p} \to Z^{\otimes q}$ as being associated to surfaces with p inputs and q outputs. By looking at pairs of pants and discs with different source and target data, it is easy to see a map from 2-d TQFTs to Frobenius algebras.

Theorem 1.2. *The TQFT given by the Frobenius algebra $\mathbb{C}[G]^G$ is "pure gauge theory with structure group G"* *given a weighted count of principle G-bundles (with holonomy on boundary in prescribed conjugacy classes.*

For example, we can compute the value of the TQFT on S^2. We get the trace of the identity

$$t(1) = t(\sum P_i) = \frac{\sum \dim\chi_i^2}{\#G^2}.$$

Note the fact that

$$Z = Z(\mathbb{C}[G])$$

allows an enhancement of the TQFT structure to surfaces with corners.

We can also add twistings to this story:

$$H^3(BG;\mathbb{Z}) \cong H^3_G(\mathbb{Z}) \cong H^2_G(U(1))$$

where the second isomorphism comes from the exponential sequence

$$1 \to \mathbb{Z} \to \mathbb{R} \to U(1) \to 1$$

and $H^2_G(\mathbb{R})$ and $H^3_G(\mathbb{R})$ vanish for a finite (or more generally a compact) group.

The "2" above comes from the dimension of the TQFT. Note also that

$$H^2_G(U(1)) \cong H^2_G(\mathbb{C}^\times).$$

This is the group of units of $\mathbb{C}$ of cohomology with complex coefficients. We call $\tau \in H^3(BG;\mathbb{Z})$ a twisting.

Proposition 1.3. *τ parametrizes central extensions of G by $\mathbb{C}^\times$, twisted by the convolution algebra of this group.*

We can associate to a central extension of G a line bundle over G so that nonzero elements have a group structure. Then

$${}^\tau\mathbb{C}[G] = \{algebra\ of\ sections\ with\ convolution\}.$$

The representations of G give projective G-representations with cocycle determined by τ.

Now if $P \to \Sigma$ is a principle G bundle, this is classified (up to homotopy) by a map $[p] : \Sigma \to BG$, and we can pullback the twisting $\tau \in H^2(BG;U(1))$, and $[p]^*\tau \in H^2(\Sigma;U(1))$ can be integrated,

$$\int_\Sigma [p]^*\tau \in \mathbb{C}^\times.$$

Then we can a count principle G-bundles weighted by the above number. These weights play well with the TQFT structure, e.g.

$$weight(\Sigma_1)weight(\Sigma_2) = weight(\Sigma_1\Sigma_2)$$

Now we generalize a bit. Let

(1) G be a compact group.
(2) replace H^* by K^*.
(3) replace twistings accordingly
(4) notice that K^* contains the group of lines, under $\otimes$ among its units.

In the above, "accordingly" means that twistings are in $H^2(BG; BU(1)) = H^2(BG; \mathbb{CP}^\infty) = H^2(BG; K(\mathbb{Z}, 2))$. Then

$$H^2(BG; K(\mathbb{Z}, 2)) \cong H^3(BG; U(1)_{cts})$$

twists the K-theoretic gauge theory of a compact group.

For the geometric picture: before we had the moduli stack of principle G-bundles (with finite group G) and twistings were functions with values in $\mathbb{C}^\times$. The number associated to a surface was in integral cohomology (which is the toy example of a path integral in physics). Usually in physics one integrates exponential of something purely imaginary, i.e. we integrate a $U(1)$-valued function. This is precisely what we had.

Now in the generalization, have the moduli of (flat) principle G-bundles denoted $\mathcal{M}$. Twistings are line bundles $\mathcal{L}$, and the invariant we get for a surface is some "integral element in K-theory." We can intepret this as holomorphic Euler characterisitic:

$$\sum (-1)^q h^q(\mathcal{M}; \mathcal{L}).$$

These numbers are controlled by a particular Frobenius algebra, namely the Verlinde algebra of G. So now we'd like to examine the algebraic side of this story.

Again, let G be a compact group. Our first attempt is to construct the convolution algebra of G, $(Co(G), \star)$ and its center as a frobenius algebra. Basically, this works and gives some physical theory described by Witten a long time ago, namely the topological limit of 2-dimensional Yang-Mills theory with group G. The involves the character theory of G. But this is not the topological theory descibed previously! What we're computing on this algebraic side is the "symplectic volume" of $\mathcal{M}$,

$$\int_{\mathcal{M}} exp(\omega)$$

where ω is a distinguished 2-form, the curvature of $\mathcal{L}$. What we need to do is pass to K-theory; the philosophy behind this is that the move from cohomology to K-theory corresponds to passing from spaces to loop-spaces.

Theorem 1.4. *As Frobenius algebras, projective representations of LG are isomorphic to something like twisted $K(LBG)$, which we will later define as twisted $K_G(G)$.*

Both twisted $Rep(LG)$ and $K_G(G)$ have products and traces. On representations, this is the fusion product. On K-theory this is the Pontryagin product.

$$m_* : {}^\tau K_G(G) \times {}^\tau K_G(G) \to {}^\tau K_G(G)$$

as a shriek map from $m : G \times G \to G$.

We remark that there is a cup product in $K_G(G)$ (which initially is actually zero!) and the tensor product on $Rep(LG)$. When we look at elliptic cohomology, the cup product structure becomes the interesting part. Notice that this gives some vague connection between elliptic cohomology and Chern-Simons, since the fusion product is related to CS and the cup is related to elliptic cohomology.

This also gives one interesting example of string topology that works! So let's say a few words about string topology.

Let X be a (pointed) closed oriented manifold. The relation to our story is that X will be the classifying stack of the group G, $X = BG$, $\Omega X = G$ as a group. Homology and cohomology will be with rational coefficients below.

String topology is an attempt to make $A := H_*(LX)$ into a 2-dimensional TQFT, i.e. a Frobenius algebra. Any (easy) attempt is bound to fail because $H_*(LX)$ is not self-dual:

$$H_*(LX)^* \cong H^*(LX) \neq H_*(LX).$$

Frobenius algebras have a pairing, so must be isomorphic to their dual.

Chas and Sullivan gave a partial Frobenius algebra structure, which defines a "positive output" 2-dimensional TQFT. One can define operations

$$A^{\otimes p} \to A^{\otimes q}$$

from surfaces with p incoming and q outgoing boundard components, dentoed ${}^p\Sigma^q$, so long as $q > 0$. These compose correctly. We can even define this for surface bundles ${}^p\Sigma^q \to B$ giving operations

$$H_*(B) \otimes A^{\otimes p} \to A^{\otimes q}.$$

For example we have the string product, given by maps of figure eights into X. We can either restrict to each side of the figure eight, giving a map to $LX \times LX$, denoted r_+, or we can conncatentate the loops giving a map to LX denoted r_-. We'd like to define operation $(r_+)_* \circ (r_-)^*$ on homology. We have a diagram

$$\begin{array}{ccc} Maps(8, X) & \hookrightarrow & LX \times LX \\ \downarrow & & \downarrow \\ X & \overset{\Delta}{\hookrightarrow} & X \times X \end{array}$$

which allows one to define this map by cap product with diagonal.

Would like to define the string topology operation by a correspondence

$$Map({}^p\Sigma^q; X) \overset{r_+}{\to} LX^{\times q}$$

$$Map({}^p\Sigma^q; X) \overset{r_-}{\to} LX^{\times p}.$$

We need some good choice of a relative cycle to define $(r_+)_* \circ (r_-)^*$, and there really isn't one as yet. It seems that we're secretly looking at the Floer theory of T^*X, i.e. holomorphic maps into T^*X.

For $X = BG$, a stack, this works! So here we think of LX as the stack of flat G-bundles on S^1. This is classified by G/G, where G acts on itself by the adjoint action. Then $Maps(\Sigma; G)$, which is the stack of flat G-bundles on Σ, which is isomorphic to $G^{\# of loops}$/conjugation-action. We have maps from $Maps(\Sigma; G)$ to stacks $(G/G)^p$ and $(G/G)^q$.

These maps are proper and smooth so long as $p, q \neq 0$, so we can define operations by correspondence diagrams, and one gets a nondegenerate trace *provided there is a twist.*

Now it turns out for open surfaces, the map from the stack of flat G-bundles to the stack of all G-bundles with connection is a homotopy equivalence, which is what allows us to work with that above.

2. Introduction to K-Theory, Jesse Wolfson, Northwestern

2.1. **Generalized Cohomology Theories.** We begin with the definition of ordinary cohomology due to Eilenberg and Steenrod:

Def. 1. An *ordinary cohomology theory* is a collection $\{H^i\}_{i\in\mathbb{Z}}$ such that:

- For each $n \in \mathbb{Z}$, H^n is a contravariant functor from the category of pairs of spaces to abelian groups.[1]
- (Homotopy Invariance) If $f \simeq g$ through maps of pairs, then $H^n f = H^n g$ for all n.
- (Preserves Products) $H^n(\coprod X_\alpha) = \prod H^n(X_\alpha)$ for all n.
- (LES of the Pair) For each pair (X, A), there exists a long exact sequence

$$\cdots \to H^i(X, A) \to H^i(X) \to H^i(A) \to^{\delta} H^{i+1}(X, A) \to \cdots$$

  such that the boundary map δ is natural.
- (Excision) If $Z \subset A \subset X$ and $\overline{Z} \subset Int(A)$ then the induced map

$$H^i(X, A) \to H^i(X - Z, A - Z)$$

  is an isomorphism for each i.
- (Dimension) $H^i(*) = 0$ for $i \neq 0$.

An *extraordinary cohomology theory* satifies all of the above except the dimension axiom. Complex K-theory was one of the first extraordinary cohomology theories to be discovered and studied in depth. My aim here is to present it as such and develop some of the key structures of K-theory as a cohomology theory. Whenever going through the gory details would obscure this development, I'll refrain and refer interested readers to other sources instead.

2.1.1. *K-Theory Take 1.* As a disclaimer, assume all spaces X are compact and Hausdorff.

Def. 2. Given a space X, let $Vect_{\mathbb{C}}(X)$ denote the semiring of isomorphism classes of (finite dimensional) complex vector bundles over X with addition given by $\oplus$ and multiplication by $\otimes$.

Def. 3. We define $K^0(X)$ to be the group completion of $Vect_{\mathbb{C}}(X)$.

Example 1. All vector bundles over a point are trivial, so $Vect_{\mathbb{C}}(*) = \mathbb{N}$ and $K^0(*) = \mathbb{Z}$.

Let X_* denote a space with basepoint $*$. For any space X, let X^+ denote the union of X with a disjoint basepoint. Let $S^n(X_*)$ denote the n-fold reduced suspension of X_*. With this notation, we define the negative K-groups as follows:

[1] The assignment $X \mapsto (X, \emptyset)$ makes this into a functor on spaces as well, and this is what is meant by $H^i(X)$.

Def. 4. Letting $i : * \to X_*$ denote the inclusion of basepoint, we define

$$\widetilde{K}^0(X_*) := ker(i^* : K(X_*) \to K(*))$$

This is called the *reduced K-group*. Observe that $K^0(X) = \widetilde{K}^0(X^+)$. Now, for $n \in \mathbb{N}$, let,

$$\begin{aligned} \widetilde{K}^{-n}(X_*) &:= \widetilde{K}^0(S^n(X_*)) \\ K^{-n}(X) &:= \widetilde{K}^0(S^n(X^+)) \\ K^{-n}(X,Y) &:= \widetilde{K}^0(S^n(X/Y)) \end{aligned}$$

To extend our definition of the K-groups to the positive integers, we use Bott Periodicity.

Theorem 1. *(Bott Periodicity v. 1) Let $[H]$ denote the class of the canonical bundle in $K^0(\mathbb{CP}^1)$. Then, identifying $\mathbb{CP}^1$ with S^2, and letting $*$ denote the reduced exterior product, the map*

$$\begin{aligned} \widetilde{K}^0(X_*) &\to \widetilde{K}^0(S^2(X_*)) \\ [E] &\mapsto ([H] - 1) * [E] \end{aligned}$$

is an isomorphism for all compact, Hausdorff spaces X. We call $[H] - 1$ the Bott class.

Periodicity allows us to define the positive K-groups inductively, setting $K^n(-) := K^{n-2}(-)$, and similarly for the reduced groups.

In order to verify that K-theory gives a cohomology theory, we need two last facts:

Prop. 2. *If X is compact and Hausdorff, and E is any vector bundle on Y, then a homotopy of maps $f \simeq g : X \to Y$ induces an isomorphism of bundles $f^*E \cong g^*E$.*[2]

Prop. 3. *To every pair of compact, Hausdorff spaces (X, Y), there exists an infinite exact sequence*

$$\cdots \to K^n(X,Y) \longrightarrow K^n(X) \longrightarrow K^n(Y) \longrightarrow K^{n+1}(X,Y) \to \cdots$$

which is natural in the usual sense.[3]

Now, checking our definitions against the axioms:

- Pullback of bundles makes the K-groups into contravariant functors so Axiom 1 is satisfied.
- Homotopy invariance follows from the proposition we just stated.

[2]The proof is an application of the Tietze extension theorem formulated for vector bundles. c.f. Atiyah [**?**] L.1.4.3.

[3]The proof of this requires the most work, after Bott periodicity, in setting up K-theory as a cohomology theory. Both Atiyah [**?**] (P.2.4.4) and Hatcher[**?**] provide a detailed construction, but I recommend just taking this as a given when first getting a handle on K-theory, and coming back to the details later.

- For products, a quick check shows that the map
$$Vect_{\mathbb{C}}(\coprod X_\alpha) \to \prod Vect_{\mathbb{C}}(X_\alpha)$$
given by pullback along the inclusions is an isomorphism, and that this isomorphism is preserved under group completion.
- The LES of the pair was given above.
- Excision is satisfied because $X/Y \cong (X-Z)/(Y-Z)$ for any $Z \subset Y \subset X$.

so we see that K-theory is indeed a cohomology theory.

2.1.2. *A Quick Note on K-classes.* From the definitions we've given, every K-class is an element of $K^0(X)$ for some compact space X. We can say more than this:

(1) Two vector bundles E and F define the same K-class if there exists a trivial bundle ϵ^n such that $E \oplus \epsilon^n \cong F \oplus \epsilon^n$. This is known as stable isomorphism, so we see $K^0(X)$ is the group completion of the semiring of vector bundles modulo stable isomorphism.
(2) Every K-class can be written as $[H] - [\epsilon^n]$ for some vector bundle H over X.
(3) A vector bundle E is in the kernel of $K^0(X) \to \widetilde{K}^0(X)$ if and only if it is stably isomorphic to a trivial bundle.

The upshot of this is that when we want to make arguments in K-theory, we can actually make arguments using vector bundles and then check that these arguments behave well when we pass to K-classes. This is one of the main techniques for making constructions in K-theory.

These conclusions follow from two facts:

Prop. 4. *Every vector bundle on a compact space is a direct summand of a trivial bundle.*

This follows from a partition of unity argument, and the finiteness of the cover; in particular, this can fail for paracompact spaces. See Hatcher [?] P.1.4.

Prop. 5. *Given a commutative monoid A, with group completion $K(A)$, $K(A) \cong A \times A/\Delta(A)$ and $x \mapsto (x, 0)$ gives the canonical map $A \to K(A)$.*

Since $K(A)$ is defined by a universal property (that it's a left adjoint to the forgetful functor from groups to monoids), it's sufficient (and straightforward) to check that $A \to A \times A/\Delta(A)$ satisfies the universal property.

Putting these together, we see every K-class is of the form $[E] - [F]$ for two bundles E and F. 1 follows because $[E] = [F] \Leftrightarrow \exists\, G$ s.t. $E \oplus G \cong F \oplus G$. Given such a G, let G' be a bundle such that $G \oplus G' \cong \epsilon^n$ for some n. Then
$$[E] = [F] \Leftrightarrow E \oplus G \oplus G' \cong F \oplus G \oplus G'$$
i.e. $E \oplus \epsilon^n \cong F \oplus \epsilon^n$. The proofs of 2 and 3 are similarly straightforward applications of the two propositions above.

2.1.3. *K-Theory Take 2.* We can give another characterization of K-theory that is frequently useful, and which illuminates the definitions. Recall that a *spectrum* is a sequence of spaces (CW-complexes) $\{E(n)\}$ and connecting maps $f_n : E(n) \to \Omega E(n+1)$. A *loop spectrum* is one where the connecting maps are homotopy equivalences.

Theorem 6. *(Brown Representability) Every reduced cohomology theory $\tilde{h}$ on the category of pointed CW complexes has a representing loop spectrum $\{H(n)\}$, unique up to homotopy, such that $\tilde{h}^n(X) = [X, H(n)]_*$ (where $[-,-]_*$ denotes based maps up to homotopy).*[4]

Since we recover an unreduced theory h by adding in a disjoint basepoint, i.e. $h^*(X) := \tilde{h}^*(X^+)$, we see Brown also says that unreduced cohomology theories correspond to unbased maps up to homotopy.

We can start to identify the spectrum KU of complex K-theory using the following:

Prop. 7. *For any compact, Hausdorff space X_*,*

$$\widetilde{K}^0(S^n(X_*) \cong [X_*, \mathbf{U}]$$

where the unitary group $\mathbf{U} := \varinjlim U(n)$.[5]

This together with periodicity and the suspension-loop adjunction shows that

$$\begin{aligned}
\widetilde{K}^0(X_*) &= \widetilde{K}^{-2}(X_*) \\
&= \widetilde{K}^0(S^2(X_*)) \\
&= [S(X_*), \mathbf{U}] \\
&= [X_*, \Omega\mathbf{U}]
\end{aligned}$$

Thus, we can give a homotopy theoretic definition of complex K-theory as

$$\widetilde{K}^{-n}(X_*) = [X_*, \Omega^{n+1}\mathbf{U}]$$

and Brown Representability, plus periodicity, again shows that this gives a cohomology theory.

In particular, periodicity can be restated as

Theorem 8. *(Bott Periodicity v.2) $\Omega\mathbf{U} \simeq B\mathbf{U} \times \mathbb{Z}$, and since $G \simeq \Omega BG$ for any topological group G, we see $\Omega^2\mathbf{U} \simeq \mathbf{U}$. Moreover, For all $n \in \mathbb{N}$,*

$$\begin{aligned}
\pi_{2n+1}(\mathbf{U}) &= \mathbb{Z} \\
\pi_{2n}(\mathbf{U}) &= 0
\end{aligned}$$

This is in fact the original form in which Bott proved it.[6]

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[4]For a fuller discussion and proof of Brown Representability, see Hatcher [**?**] § 4.E.

[5]The proof follows from considering a clutching argument and that if X is compact, [X,-] preserves filtered colimits (e.g. direct limits).

[6]Both homotopy equivalences have concrete implementations which Bott was able to formulate and study using Morse theory. The interested reader should see Milnor [**?**] for the full proof. Alternatively, both Hatcher [**?**] and Atiyah [**?**] provide proofs of version 1 in terms of bundle constructions.

We can view this version as a calculation of the reduced K-groups of spheres. Passing to unreduced, we see that the values of K-theory for spheres are:

$$K^0(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$$
$$K^1(S^{2n}) = 0$$

and

$$K^0(S^{2n+1}) = \mathbb{Z}$$
$$K^1(S^{2n+1}) = \mathbb{Z}$$

Since the spectrum of a cohomology theory is only specified up to homotopy, it's possible to give several equivalent spectra, each of which can shed light on the theory.

For example we can interpret K-theory in terms of operators on a separable infinite dimensional Hilbert space H. Recall that a bounded operator on a Hilbert space is *Fredholm* if it has a closed image, and its kernel and cokernel are finite. To each operator T, we can assign an index

$$Index(T) := dim\ ker(T) - dim\ coker(T)$$

It turns out that this is the restriction to a point of an isomorphism

$$index : [X, Fred(H)] \to K^0(X)$$

Appendix A to Atiyah [**?**] spells this out in detail. Note that this interpretation of the spectrum K provides a link between K-theory and index theory of elliptic operators. Other representing spectra also exist and they illuminate deep connections between complex K-theory and areas of interest to mathematical physics and analysis,[7] but this is beyond my scope right now.

2.2. **Computational Tools.** As with any cohomology theory, we have the usual computational tools of Mayer-Vietoris sequences, and the LES of the pair. However, these are often not very useful in K-theory, because periodicity means we rarely have enough zero entries to reduce the long exact sequences to a series of isomorphisms. However, K-theory, and in fact any extraordinary cohomology theory, comes with two additional tools which relate its values to those of ordinary cohomology. These are the Atiyah-Hirzebruch spectral sequence, which is a special instance of a generalized Serre spectral sequence, and the Chern character, which relates K-theory to rational cohomology.

2.2.1. *A Quick Recap of Spectral Sequences.* Spectral sequences can seem quite daunting at first, at least they did to me.[8] However, once you get comfortable using them, they open up an incredible number of calculations which look nearly impossible without them. Recall that a spectral sequence is an infinite sequence of "pages" consisting of a grid of groups and of differentials between

[7]For example, see Atiyah-Bott-Shapiro [**?**].

[8]If you haven't worked with them before, I highly recommend Ch.1 of Hatcher [**?**], available online. I recommend taking the construction of spectral sequences as a given, and focusing first on how use them. Hatcher has a great set of examples, sample computations and exercises and I really enjoyed this.

them. We write $E_r^{p,q}$ for the (p,q)th entry of the rth page, and $d^r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ for the differential. You pass from one page to the next by taking the homology with respect to the differentials, and cooking up a new set of differentials from the data of the previous ones. A spectral sequence *converges* if for for each (p,q), $E_r^{p,q} = E_{r+1}^{p,q}$ for $r \gg 0$; we write $E_\infty^{p,q}$ for this stable group.

In a wonderful variety of situations, we can cook up a spectral sequence such that E_2 page starts with something that we know, and the E_∞ page is closely related to something we want to calculate, e.g. the cohomology ring of a space X. As a shorthand, we say the spectral sequence converges to the thing we want, and we write something like

$$E_2^{p,q} \Rightarrow h^{p+q}(X)$$

However, this notation is shorthand and should not be taken literally. Spectral sequences **do not** converge to the groups written on the right hand side of the arrow, they converge instead to the associated graded objects of a filtraton of these groups. Whether or not we can recover the groups we care about depends on an extension problem, and is often nontrivial.

Alright, with these disclaimers in place, let's lay out the main tools:

2.2.2. *The Atiyah-Hirzebruch Spectral Sequence.* Recall that given a fibration $F \to E \to B$, we have the Serre spectral sequence:

$$H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

In fact, the proof and construction carry over to any cohomology theory h giving us a generalized Serre spectral sequence:

$$H^p(B, h^q(F)) \Rightarrow h^{p+q}(E)$$

Taking the trivial fibration $id : X \to X$, we get the Atiyah-Hirzebruch spectral sequence

$$H^p(X, h^q(*)) \Rightarrow h^{p+q}(X)$$

This spectral sequence should be seen as reiterating for generalized cohomology what we already know from ordinary cohomology: namely, that cohomology theories are largely determined by their values on the point.[9] This spectral sequence, along with the generalized Serre SS for K-theory, provides one of the main tools for computing $K^*(X)$.

Note also, that the generalized Serre SS for K-theory allows us to prove a K-theory version of Kunneth (as the same proof in ordinary cohomology using the Serre SS carries over here), and its formulation is precisely the one we're used to.

[9]More precisely, two cohomology theories h and h' are equivalent if there exists a natural transformation $\alpha : h \to h'$ such that α induces an isomorphism $h(*) \cong h'(*)$. c.f. Adams [**?**].

2.2.3. *The Chern Character.* The Chern Character is a ring homomorphism

$$ch : K^*(-) \to H^*(-;\mathbb{Q}) \otimes K^*(*)$$

induces an isomorphism $K^*(-) \otimes \mathbb{Q} \cong H^*(-;\mathbb{Q}) \otimes K^*(*)$.[10] In other words, for any space X,

$$\begin{aligned} K^0(X) \otimes \mathbb{Q} &\cong \bigoplus_{n\in\mathbb{N}} H^n(X;\mathbb{Q}) \otimes K^{-n}(*) \\ &\cong \bigoplus_{n\in\mathbb{N}} H^{2n}(X;\mathbb{Q}) \end{aligned}$$

and similarly,

$$K^1(X) \otimes \mathbb{Q} \cong \bigoplus_{n\in\mathbb{N}} H^{2n+1}(X;\mathbb{Q})$$

2.3. **Sample Computations.** The following set of spaces are suggestions for good examples to apply these tools to calculate the K-theory of spaces.

- Riemann Surfaces
- CP^n
- $SO(3)$
- $O(4)$

The first two can be computed immediately from the Atiyah-Hirzebruch SS. For $SO(3)$, you'll need to combine the isomorphism from the Chern character with the Atiyah-Hirzebruch SS. You can use this to calculate the K-theory of $O(3)$ and then use this plus Kunneth to calculate the K-theory of $O(4)$.

2.4. **Theoretical Tools.** As we observed above, since every K-class can be represented as formal difference of actual bundles, we can usually make our arguments for K-theory in terms of actual bundles, and then observe that these arguments behave well when we pass to K-classes. Frequently these constructions involve reducing the structure group, for example, in decomposing a bundle as sum of line bundles, or reducing its dimension by one. I sketch the general process in the next section; we will use it frequently. Following this, the theoretical tools discussed are:

- Adams operations
- The Thom Isomorphism and Applications
- A construction of the Chern character

These are largely independent, and you should feel free to tackle them in any order.

2.4.1. *Splitting Principles.* In the most general case, a splitting principle refers to an operation in which a G-bundle $E \to X$ is pulled-back along a map $f : F \to X$, such that f^*E has a reduced structure group, and the map induced by f on cohomology is injective.

The generic splitting principle arises from a fiber sequence

$$H \hookrightarrow G \to G/H$$

[10] Such an isomorphism is actually a general property of cohomology theories, and the map $h \to H^*(-;\mathbb{Q}) \otimes K^*(*)$ is called the *character* of the theory.

where H and G are topological groups.

Give a G bundle $E \to X$, the splitting principle is the action of pulling back the G-bundle along the projection of the associated bundle

$$E \times_G G/H \to X$$

This pullback gives us a bundle with structure group H, and for nice enough fiber sequences, the projection induces the desired injective map on cohomology.[11]

Some common examples of splitting principles for complex bundles are:

- $SU(n) \to U(n) \to S^1$ which corresponds to orienting the vector bundle.
- $U(n-1) \to U(n) \to S^{2n-1}$ which allows us to make arguments/constructions by inducting on the dimension of the bundle.
- $\mathbb{T}^n \to U(n) \to Fl_n$ which corresponds to reducing a vector bundle to a direct sum of line bundles.[12]

This last principle is the one most commonly referred to as the splitting principle, and we will use it repeatedly going forward. In fact, the moral for basic K-theory constructions seems to be:

- Define the construction for (direct sums of) line bundles.
- Extend to arbitrary bundles.
- Use the splitting principle to show that properties which hold for line bundles hold in general.

2.4.2. *Adams Operations.* The Adams operations $\{\psi^k\}$ are cohomology operations on complex K-theory, analogous to Steenrod Squares in mod 2 cohomology.

Their basic properties are:

Prop. 9. *For each compact, Hausdorff space X, and each $k \in \mathbb{N}$, there exists a ring homomorphism*

$$\psi^k : K^0(X) \to K^0(X)$$

satisfying:

(1) *Naturality, i.e. $\psi^k f^* = f^* \psi^k$ for all maps $f : X \to Y$.*
(2) *For any line bundle L, $\psi^k([L]) = [L]^k$.*
(3) *$\psi^k \psi^l = \psi^{kl}$.*
(4) *$\psi^p(x) \equiv x^p$ (modp).*

[11] The cleanest way to see that this construction does reduce the structure group is to pass to universal bundles over classifying spaces. Recall that over paracompact spaces X, isomorphism classes of bundles E correspond to homotopy classes of maps $f_E : X \to BG$, where BG is the classifying space of G-bundles. We recover the bundle E on X by pulling back the universal G-bundle $EG \to BG$ along f_E. Up to homotopy, EG is defined as a contractible space with a free G-action, and BG is defined as EG/G. Now, given an injection $H \to G$, EG also has a free H-action, so $BH \simeq EG/H$. But then $BH \to BG$ is the universal G-bundle with fibre G/H. Forming an associated bundle $E \times_G G/H$ over X corresponds to pulling back the universal bundle BH along the map f_E, and pulling back EG along $f_E^*(BH) \to BH \to BG$ gives the "splitting" of E. See Segal [**?**] for the basics of classifying spaces.

[12] Fl_n denotes the complete flag variety over $\mathbb{C}^n$.

Property 2 characterizes the Adams operations, and we can use this, plus the splitting principle, to give a general construction. Observe that if $E = \bigoplus_{i=0}^{n} L_i$, then property 2 and being a homomorphism says

$$\psi^k(E) = \sum_{i=0}^{n} L_i^k$$

We can extend this to a general definition using exterior powers. Given any bundle E, let $\lambda^i E$ denote the ith exterior power of E. From linear algebra, we know

- $\lambda^k(E \oplus E') = \bigoplus_{i=0}^{k} \lambda^i(E) \otimes \lambda^{k-i}(E')$
- $\lambda^0(E) = 1$, the trivial line bundle.
- $\lambda^1(E) = E$.
- $\lambda^k(E) = 0$ for $k > dim E$.

There exists a useful class of integral polynomials called the Newton polynomials, denoted s_k (for $k \in \mathbb{N}$). With a little work,[13] one can show that if $E = \bigoplus_{i=0}^{n} L_i$ as above, then $\sum_{i=0}^{n} L_i^k = s_k(\lambda^1(E), \ldots, \lambda^k(E))$. We can then take

$$\psi^k(E) := s_k(\lambda^1(E), \ldots, \lambda^k(E)$$

as the general definition. By applying the splitting principle associated with $\mathbb{T}^n \to U(n) \to Fl_n$, it's enough to verify for (direct sums of) line bundles that the Adams operations satisfy the specified properties, and this is a straightforward check from our definitions.[14]

2.4.3. *The Thom Isomorphism and Applications.* Given the dependence of K-theory on vector bundles, we might expect that those features of ordinary cohomology related to vector bundles also arise in K-theory (e.g. the Thom Isomorphism, characteristic classes, and Gysin maps). All of these rely on the orientability of the vector bundles in ordinary cohomology, and formulating these for K-theory will similarly require a suitable notion of K-orientable bundle and manifold.

2.4.4. *Orientability in K-theory.* Orientations for vector bundles or manifolds are defined formally in any cohomology theory analogously to how they are defined in ordinary cohomology. First, some notation: Given a bundle V on X, let $B(V)$ denote the unit ball bundle, and $S(V)$ the unit sphere bundle (under any metric).

Def. 5. *A bundle V is orientable in the cohomology theory E^* if there exists a class $\omega \in E^*(B(V), S(V))$ such that $\omega|_p$ is a generator of $E^*(B(V)_p, S(V)_p)$ as a module over $E^*(*)$ for all $p \in X$.*

Despite the formal similarity between ordinary orientability and orientability in a general theory, we should not expect these to be closely related. First, while orientability in ordinary integral cohomology is equivalent to orientability in a (differential) geometric sense, there is no guarantee that this is the

[13] See Hatcher [**?**] §2.3 for the details of this construction.

[14] For the proof of this splitting principle, see Hatcher [**?**] §2.3.

case in general, or that integral orientability is in any way related to E^* orientability. In general, the question of whether an integrally orientable bundle or manifold is E^*-orientable, for some theory E, is a question of whether the orientation class survives to the E_∞ page of the Atiyah-Hirzebruch SS and then whether we can recover the cohomology from the associated graded. In general, this is not trivial, and it still leaves us with the question of interpreting the meaning of the E-orientability of a space.

In complex K-theory, Atiyah-Bott-Shapiro [**?**] shows that a vector bundle is K-orientable if and only if it admits a *spin*^{c} structure. This has applications for mathematical physics, but is beyond the scope of my introduction. On the other hand, every complex bundle admits a *spin*^{c} structure, and is thus K-orientable. More directly, we can construct the orientation class of the bundle from its exterior algebra. Thus, for the remainder of this section, I will assume that any bundle or manifold is almost complex, and avoid worrying about the more general case.

2.4.5. *The Isomorphism.* In preparation for the Thom isomorphism, we reformulate our test space for orientations of bundles as follows:

Def. 6. Given a vector bundle E on X, we define its *Thom space*, X^E, to be the one point compactification of E. [15]

Notice that $(X^E/\infty) \cong B(E)/S(E)$, so we can replace $K^*(B(E), S(E))$ with $K^*(X^E)$ in our definition of orientability above.

Prop. 10. *Every complex vector bundle E is K-orientable, with a canonical orientation class $\lambda_E \in \widetilde{K}^0(X^E)$ satisfying*

(1) *Naturality, i.e.* $\lambda_{f^*E} = f^*\lambda_E$
(2) *Sums, i.e.* $\lambda_{E\oplus E'} = (\pi_1^*\lambda_E) \cup (\pi_2^*\lambda_{E'})$

For the construction of λ_E from the exterior algebra of the bundle E, and a verification that it satisfies the desired properties, see Atiyah [**?**], §2.6, p. 98-99. λ_E is known as the *Thom class* of the bundle. We can now state:

Theorem 11. *(Thom Isomorphism) If E is a complex vector bundle over X, then $\widetilde{K}^*(X^E)$ is a free module of rank 1 over $K^*(X)$ with generator ω_E. In other words, the map*

$$\Phi_K : K^*(X) \to \widetilde{K}^*(X^E)$$

given by $\Phi_K(x) = \lambda_E \cdot x$ is an isomorphism.

The theorem can be proven via showing it in the case for (direct sums of) line bundles and then using the splitting principle associated to $\mathbb{T}^n \to U(n) \to Fl_n$ as usual.[16]

[15]Equivalently, if E is a complex vector bundle, we can define $X^E := P(E \oplus 1)/P(E)$. A quick check will show these are the same. Namely, $P(E \oplus 1)$ amounts to compactifying the fibers of E by gluing in projective hyperplanes at ∞, and modding out by $P(E)$ sends all these hyperplanes to a point.

[16]c.f. Atiyah [**?**] §2.7.

2.4.6. *Gysin Maps.* Gysin, or wrong-way, maps, are a useful tool in cohomology theories. Using either Duality or the Thom Isomorphism, we can associate covariant maps to orientation preserving maps between manifolds, in addition to the contravariant mappings guaranteed by the theory. As a matter of notation, given such a map $M \to M'$, let N_M denote the normal bundle of M in M', and ϵ_M denote a tubular neighborhood of M in M' isomorphic to $B(N_M)$ with boundary $\partial(\epsilon_M)$ isomorphic to $S(N_M)$.[17] Let ϵ_M° denote the interior of ϵ_M, i.e. $\epsilon_M^\circ = \epsilon_M - \partial(\epsilon_M)$.

Def. 7. (Gysin Maps) Given an immersion $f : M \to M'$, of almost complex manifolds, we define $f_! : K^*(M) \to K^*(M')$ as the composite:

$$K^*(M) \to \widetilde{K}^*(M^{N_M}) \to K^*(\epsilon_M, \partial(\epsilon_M)) \to K^*(M', M' - \epsilon_M^\circ) \to K^*(M')$$

where the first map is the Thom isom., the second is the isom. $(B(N_M), S(N_M)) \cong (\epsilon_M, \partial(\epsilon_M))$ given by the tubular neighborhood theorem, the third is the isom. due to excision, and the last map corresponds to the canonical map $(M', \emptyset) \to (M', M' \setminus \epsilon_M^\circ)$.

Note that in ordinary cohomology, the Gysin map raises the degree by $dim(M') - dim(M)$. However, if M and M' are almost complex, they are of even dimension, so the difference is even as well. The periodicity of K-theory then ensures that Gysin maps between almost complex manifolds preserve degree.

2.4.7. *The Thom Class and Characteristic Classes.* We use the Thom class to construct characteristic classes in K-theory. There are two equivalent constructions for these classes, one using a splitting principle and the other using the Adams operations. I will sketch both and the interested reader can check that they are equivalent by doing the calculations with the universal bundle on BU.

Def. 8. (The Euler Class) Given a complex vector bundle E on X, let ζ denote the 0-section. Then the Euler class, $e(E) \in K^0(X)$, is given by $e(E) = \zeta^* \lambda_E$.

Def. 9. (Chern Classes) Given a complex vector bundle E on X, we define the Chern classes classes $c_1(E), \ldots, c_n(E) \in K^0(X)$ inductively as follows:

(1) $c_n(E) = 0$ if $n > rk(E)$
(2) $c_n(E) := e(E)$
(3) $c_i(E) := \pi_*^{-1}(c_i(\widehat{E}))$ where $\widehat{E}$ is the vector bundle corresponding to the splitting principle given by $U(n-1) \to U(n) \to S^{2n-1}$.[18]

We can also define $c_i(E)$ by $c_i(E) := \Phi_K^{-1} \circ \psi^i(\lambda_E)$ where ψ^i is the i^{th} Adams operation.

[17] The existence of such a neighborhood is an immediate consequence of the usual tubular neighborhood theorem.

[18] We show this to be a splitting principle via the Gysin sequence coming from the Thom isomorphism.

An obvious question to ask is how the Chern classes in K-theory behave relative to those in ordinary integral cohomology. Recall that in integral cohomology, if L and L' are line bundles, then $c_1(L \otimes L') = c_1(L) + c_1(L')$.[19] In K-theory, a similar calculation shows that $c_1(L \otimes L') = c_1(L) + c_1(L'_) + c_1(L)c_1(L')$. Jacob Lurie includes a nice discussion of this in [?]. In particular, he considers the notion of formal group laws, and observes that in this language, ordinary chern classes are governed by the formal additive group, whereas K-theoretic chern classes are governed by the formal multiplicative group. We can exploit this to motivate both the definition and the existence of the Chern character.

2.5. **Constructing the Chern Character.** I asserted the existence of the Chern character above, and this section outlines a concrete construction of it. Morally, the Chern character for K-theory arises from the observation that, over $\mathbb{Q}$, exponentiation gives an isomorphism between the formal additive group and the formal multiplicative group. We can make this concrete as follows:

Given a K-class $[L]$ represented by a line bundle $L \to X$, define

$$ch([L]) := exp(c_1(L)) = 1 + c_1(L) + \frac{c_1(L)^2}{2} + \ldots + \frac{c_1(L)^k}{k!} + \ldots$$

where $c_1(L)$ is the first Chern class of L in ordinary cohomology. Given a K-class $[E]$ represented by $E = \bigoplus_{i=0}^n L_i$, define

$$ch([E]) := \sum_{i=0}^{n} ch(L_i) = n + (t_1 + \ldots t_n) + \ldots + \frac{t_1^k + \ldots t_n^k}{k!} + \ldots$$

where $t_i = c_1(L_i)$. By the same algebra that we used in the construction of the Adams operations, we can convert this expression into one purely in terms of the ordinary chern classes of E.[20] If c_j denotes the j^{th} Chern class $c_j(E)$ of E, and s_k denotes the k^{th} Newton polynomial, then we can rewrite this as

$$ch(E) = dim(E) + \sum_{k>0} \frac{s_k(c_1, \ldots, c_k)}{k!}$$

and take this as a general definition for arbitrary bundles. The interested reader can check that this is well defined and that it lands in the correct dimensions in $H^*(X;\mathbb{Q})$. Moreover, a straightforward check from our definitions shows that for any line bundles L and L', $ch(L \otimes L') = ch(L)ch(L')$ and $ch(\sum_i L_i) = \sum_i ch(L_i)$, so ch gives a ring homomorphism as desired.

[19]We show this by doing the computations with the chern class of the universal line bundle over $\mathbb{CP}^n$).

[20]From the axioms of characteristic classes, the total chern class

$$c(E) = (1 + t_1) \ldots (1 + t_n) = 1 + \sigma_1 + \ldots + \sigma_n$$

where σ_j denotes the jth symmetric polynomial in the t_i, and considering cohomological degrees, we see $c_j(E) = \sigma_j$. As noted in the construction of the Adams operations, there exists a class of polynomials s_k called the *Newton polynomials* with the property that $t_1^k + \ldots t_n^k = s_k(\sigma_1, \ldots, \sigma_k)$. This allows us to rewrite the formula above as $ch([E]) = dim(E) + \sum_{k>0} \frac{s_k(\sigma_1,\ldots,\sigma_k)}{k!}$. See Hatcher [?] §.2.3 for more details on the Newton polynomials.

Now apply the splitting principle associated to $\mathbb{T}^n \to U(n) \to Fl_n$ to conclude that any cohomology relation which holds for line bundles under this reduction also holds for arbitrary bundles.[21] This completes the construction of the Chern character. For the proof that it induces the isomorphism claimed above, see Hatcher [**?**], Ch.4, P.4.3 and 4.5.

2.6. **Hirzebruch-Riemann-Roch.** The Hirzebruch-Riemann-Roch Theorem was the first in a series of generalizations of the classical Riemann-Roch theorem, which eventually culminated in the Grothendieck-Hirzebruch-Riemann-Roch Theorem in algebraic geometry, and in the Atiyah-Singer Index Theorem in the differential case. I include it here, both for historical purposes, and because some of the deeper applications of K-theory occur in its implications for index theory, and Hirzebruch-Riemann-Roch is a first sign of these results.

Before we can state the theorem, we need to define another characteristic class $Td^*(-)$ called the *Todd class.* We can think of the Todd class as a formal reciprocal of the Chern character, and its construction is similar.

2.6.1. *The Todd Class.* In the usual manner, we define the Todd class for direct sums of line bundles, use some algebra to massage this into a formula for general bundles, and then prove it has the desired properties by using the splitting principle. We can characterize the Todd class as follows:

Prop. 12. *Given a vector bundle $E \to X$, there exists a unique class $Td^*(E) \in H^*(X;\mathbb{Q})$ satisfying:*

(1) *Naturality, i.e. $f^*Td^*(E) = Td^*(f^*E)$.*
(2) $Td^*(E \oplus F) = Td^*(E)Td^*(F)$
(3) $Td^*(L) = \frac{c_1(L)}{1-e^{-c_1(L)}}$ *for all line bundles L.*

Using the Bernoulli numbers $\{B_{2i}\}_{i=1}^{\infty}$, we can expand the righthand side of number 3 as a formal power series. Explicitly:

$$Q(x) := \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}x^{2i}$$

For any finite dimensional basespace X, number 3 describes a polynomial in the chern classes with rational coefficients, and thus it does specify an element of $H^*(X;\mathbb{Q})$. This, together with number 2 then dictate the formula for direct sums of line bundles, and though we are unable to massage out a general expression using the Newton polynomials, there exists another class which does the job and allows us to get get a general expression for $Td^*(E)$ as a power series in the Chern classes of E. The first few terms of this expansion are

$$Td^*(E) = 1 + \frac{1}{2}c_1 + \frac{1}{12}(c_1^2 - 2c_2) + \frac{1}{24}c_1c_2 - \frac{1}{720}(c_1^4 - 4c_1^2c_2 - 3c_2^2 - c_1c_3 + c_4) + \ldots$$

[21]This is why we need the injectivity of the map $p : Fl(E) \to X$ on cohomology. For a proof of this splitting principle, see Hatcher [**?**], §3.

where the c_i are the Chern classes of E. With this definition, naturality is a consequence of the naturality of the chern classes, and the splitting principle associated to $\mathbb{T}^n \to U(n) \to Fl_n$ shows that 2 is satisfied in general.

2.7. **The Theorem.** Recall that the euler characteristic of a coherent sheaf F on a space X is the alternating sum of its Betti numbers, i.e.

$$\chi(F) := \sum (-1)^i rk(H^i(X, F))$$

Theorem 13. *(Hirzebruch-Riemann-Roch) Given a vector bundle E on a compact complex manifold M, let $\mathcal{E}$ denote the sheaf of holomorphic sections of E. Then*

$$\chi(\mathcal{E}) = \int_M ch(E) \cdot Td^*(TM)$$

I don't give the proof here, but if you're interested, see Hirzebruch [**?**], or for a nice motivational discussion, see Griffiths and Harris [**?**] §3.4.

As an illustration of the theorem, we can quickly derive the classical Riemann-Roch formula for curves, which I go through below. There's a similarly nice derivation of Noether's Theorem for surfaces, and the interested reader should Hartshorne's [**?**], Appendix A.

2.8. **Riemann-Roch for Curves.** Let X be a curve of genus g and let $L(D)$ be a line bundle X corresponding to divisor D. We need to quickly recall several facts from algebraic geometry:

- $\chi(\mathcal{O}_X) = 1 - g$.
- The first Chern class gives the isomorphism between divisors and line bundles, so $c_1(L) = D$ and $ch(L) = 1 + D$.
- The tangent bundle TX is the dual of the canonical bundle of X, and so if K is the canonical divisor of X, TX corresponds to $-K$, and $Td^*(TX) = 1 - \frac{1}{2}K$.
- Since D and K define elements of $H^2(X; \mathbb{Z})$, $DK = 0$ for dimension reasons.
- For a divisor D, $\int_M D = degD$ (this is tautological from the definitions, but worth recalling).

. Putting these together, we see

$$\begin{aligned} ch(L)Td^*(TX) &= (1 + D)(1 - \frac{1}{2}K) \\ &= 1 + D - \frac{1}{2}K \end{aligned}$$

and plugging this into the formula from the theorem, we get

$$\begin{aligned} \chi(\mathcal{L}(D)) &= \int_M (1 + D - \frac{1}{2}K) \\ &= deg(D - \frac{1}{2}K) \end{aligned}$$

Now, taking $D = 0$, this says $1 - g = \chi(\mathcal{O}_X) = -\frac{1}{2} degK$, and plugging this in we get the classical formula:

$$\chi(\mathcal{L}(D) = degD + 1 - g$$

3. More K-Theory, Chris Kottke, MIT

As the token analyst here, I'll tell you a bit about how to think of the pushforward in K-theory. We'll start with the Gysin map as an index of an elliptic differential operator; proceed to clifford algebras and $spin^{\mathbb{C}}$ as an orientation and Dirac operators; then $spin^{\mathbb{C}}$ Dirac operators as a fundamental class; and finally we'll talk about the higher index theorems (everything else is about K^0).

3.1. K-Theory Via Bundles of Operators.

Let $V \to X$ be a vector bundle. Then we can define compactly supported K-theory as

$$K_c^*(V) = K^*(\overline{V}, \partial\overline{V}).$$

In particular,

$$K_c^0(V) = \{\pi^*E, \pi^*F, \sigma \ s.t. \ \sigma|_{V-0} : \pi^*E \overset{\simeq}{\to} \pi^*F\}$$

To compare, recall that

$$K^0(X, A) = \{[V], [W], \sigma \ s.t. \ \sigma : V \to W, \ \sigma|_A : V \overset{\simeq}{\to} W\}/\{\text{stable homotopy}\}$$

Now let H be a separable infinite dimensional Hilbert space. Let $F \in \mathcal{B}(H)$, be a bounded operator. We say that F is Fredholm if it is invertible modulo compact operators:

$$\exists G \in \mathcal{B}(H) \ s.t. \ FG - I \ and \quad GF - I \text{ compact}$$

Note that this implies that the kernel and cokernel of F are finite dimensional.

Theorem 3.1 (Atiyah)**.** $[X, \mathrm{Fred}(H)] \cong K^0(X)$.

Say we have a map $X \to \mathrm{Fred}(H)$, $x \mapsto P(x)$. Then the element of K-theory we want in the above theorem is $[ker \ P(x)] - [coker \ P(x)] \in K^0(X)$.

3.2. Differential Operators.

Now recall that $P \in \mathrm{Diff}^k(X; V, W)$ if locally

$$P = \sum_{|\alpha| \leq k} a_\alpha(x) \partial_x^\alpha$$

where $a_\alpha(x) \in Hom(V_x, W_x)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\partial_k^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}$.

To get something independent of coordinates, we want the principal symbol of P, defined as

$$\sigma(P) = \sum_{\alpha = k} a_\alpha(x) \xi^\alpha \in C^\infty(T^*X; Hom(\pi^*V, \pi^*W))$$

where (x, ξ) are local coordinates for T^*X. We say that P is elliptic if $\sigma(P)(\xi)$ is invertible for all $\xi \neq 0$. Notice that an elliptic operator defines a class, namely $[[\pi^*V], [\pi^*W], \sigma(P)] \in K_c^0(T^*X)$.

3.3. **Families of Differential Operators.** Say $Y \to X \to Z$ is a fibration. We may regard X as a bundle associated to a principle Diffeo(Y)-bundle Q:

$$X = Q \times_{\mathrm{Diffeo}(Y)} Z$$

and

$$\mathrm{Diff}^k(X/Z; V, W) := Q \times_{\mathrm{Diffeo}(Y)} \mathrm{Diff}^k(Y, V, W),$$

so

$$\sigma(P) \in C^\infty(T^*X/Z; Hom(V, W)).$$

Really one should think of these as differential operators in the fiber direction.

Now if P is elliptic, it is also Fredholm as an operator on $L^2(X/Z; V) \to L^2(X/Z; W)$ over Z.

Theorem 3.2 (Kupier). *The group of unitary operators on Hilbert bundles are trivializable, in particular, $P \in Map(Z, \mathrm{Fred}(H))$ for an elliptic operator P.*

Now, define the analytic index as

$$Ind(P) = [Ker\ P(x)] - [Coker\ P(x)].$$

Then we find that $Ind(P) \in K^0(Z)$. In particular, when $Z = pt$ and we have the bundle $X \to pt$, then

$$Ind(P) = Dim(Ker(P)) - Dim(Coker(P)) \in \mathbb{Z} = K^0(pt)$$

3.4. **Gysin Maps for Fibrations.** Say $M \to B$ a fibration for B compact. Then

$$M \overset{i}{\hookrightarrow} B \times \mathbb{R}^N$$

over B gives a normal bundle $\nu \to M$ with respect to i and

$$\tilde{h}^*(M^\nu) \hookrightarrow \tilde{h}^*(\Sigma^N B) = h^*(B).$$

(The inclusion above is just excision.) Then if M is oriented then the Thom isomorphism $h^*(M) \to \tilde{h}^*(M^\nu)$ composes with the above to give a shriek map $h^*(M) \to h^*(B)$. Notice there are degree shifts in each separate map, but they cancel in the composition.

Now consider for any fibration of manifolds $X \to Z$,

$$\begin{array}{ccc} T^*(X/Z) & \hookrightarrow & Z \times \mathbb{R}^{2N} \\ \downarrow & & \downarrow \\ X & \hookrightarrow & Z \times \mathbb{R}^N \\ \downarrow & & \downarrow \\ Z & = & Z \end{array}$$

Have $\nu(T^*(X/Z)) = \nu(X) \oplus \nu(X) \cong \nu(X) \otimes \mathbb{C}$, so $T^*(X/Z)$ is canonically complex, and hence canonically oriented. So we have a map (the topological index):

$$K^0_c(T^*(X/Z)) \to K^0(Z).$$

Theorem 3.3 (Atiyah-Singer). *The topological index equals the analytic index.*

When can we get a pushforward from X to Z? First we need some more background.

3.5. **Clifford Algebras.** Let (V, q) be a vector space with a nondegenerate quadratic form. Then consider maps $f : V \to \mathcal{A}$ to an algebra $\mathcal{A}$ such that $f(v) \cdot f(v) = -q(v) \cdot 1$. The Clifford algebra $Cl(V, q)$ is the universal object of these.

More concretely as a vector space $Cl(V, q)$ is the exterior algebra of V, but they are not isomorphic as algebras. Let $\{e_i\}$ be a basis for V. Then products

$$\{e_{i(1)} \cdots e_{i(k)} | i(1) < \cdots i(k)\}$$

is a basis for $Cl(V, q)$. If the basis happened to be orthonormal with respect to q, then $e_i e_j = -e_j e_i$. In general

$$e_i e_j = -e_j e_i - q(e_i, e_j).$$

Moreover, we see that $Cl(V, q)$ is $\mathbb{Z}/2\mathbb{Z}$-graded, where

$$Cl^0(V) = \Lambda^{even}(V), \quad Cl^1(V) = \Lambda^{odd}(V),$$

We can also complexify this

$$\mathbb{C}l(V) = Cl(V \otimes \mathbb{C}) = Cl(V) \otimes \mathbb{C}.$$

We define

$$\mathbb{C}l_k := \mathbb{C}l(\mathbb{R}^k, \cdot)$$

where $\cdot$ is the standard dot product on $\mathbb{R}^k$. The representation theory of complex Clifford algebras is very simple, as we have

$$\mathbb{C}l_k \cong M(2^k, \mathbb{C}) \quad k \text{ even}$$

$$\mathbb{C}l_k \cong M(2^{k-1}, \mathbb{C}) \oplus M(2^{k-1}, \mathbb{C}) \quad k \text{ odd}$$

(i.e. there is only one irrep if k is even and two if k is odd). Let $Spin(V)$ be defined as the subset of $Cl(V)$ (viewed as an algebra of endomorphisms of $Cl(V)$) given by

$$Spin(V) := \{u \in \mathbb{C}l(V)^* \subset Aut(\mathbb{C}l(V)) | u \text{ fixes } V \text{ and } u \in SO(V)\}$$

We can also define $Spin(\mathbb{R}^n)$ as the universal cover of $SO(\mathbb{R}^n)$:

$$0 \to \mathbb{Z}/2 \to Spin_n \to SO_n \to 1$$

but it's nice to have $Spin(V)$ as a subset of the Clifford algebra. We have the "spin representation:"

$$\rho : \mathbb{C}l_{2n} \to \mathfrak{gl}(W)$$

where $dim(W) = 2^{2n}$, and $\rho|_{Spin_{2n}} : Spin_{2n} \to GL(W)$. In fact this is a $\mathbb{Z}/2$-graded representation of $\mathbb{C}l_n$,

$$W = W^0 \oplus W^1.$$

3.6. **Clifford Algebras on Manifolds.** Let (X, g) be a Riemannian manifold. We have a bundle $\mathbb{C}l(X) \to X$ whose fiber at a point p is

$$\mathbb{C}l(X)_p = \mathbb{C}l(T_pX, g(p)).$$

We get Clifford modules over X by considering vector bundles on which the bundle $\mathbb{C}l(X) \to X$ acts. When do (complex) Clifford modules decompose globally into irreducibles? This happens when X is $Spin^{\mathbb{C}}$.

First let's explain ordinary spin manifolds. A *spin structure* on (X, g) is a lift $P_{Spin_n}(X) \to P_{SO_n}(X)$ over X (this is a 2-sheeted cover) with compatible actions from SO_n and Cl_n. The obstruction to this is $w_2(X) \in H^2(X; \mathbb{Z}/2)$.

Proposition 3.4. *X spin implies that for E a Clifford module, we have*

$$E \cong P_{Spin_n}(X) \times_\rho F$$

In particular we have that sections of $S^+ \oplus S^- \cong S = P_{Spin_n}(X) \times_\rho W$ are called the "spinors."

Theorem 3.5. $[\pi^*S^+, \pi^*S^-, Cl] \in K_c^0(T^*X)$ *is an orientation class for K^**

There is a group

$$Spin_n^{\mathbb{C}} \cong Spin_n \times_{\mathbb{Z}/2} U(1).$$

Then a $Spin^{\mathbb{C}}$-structure on (X, g) is a complex line bundle $L \to X$ and a lift

$$P_{Spin_n^{\mathbb{C}}(X)} \to P_{SO(n)(X)} \times P_{U(1)}(L),$$

another 2-sheeted cover. The obstruction to this is

$$w_2(X) + c_1(L) \ mod\ 2 \in H^2(X; \mathbb{Z}/2)$$

Again we have

$$S = S^+ \oplus S^- = P_{Spin_n^{\mathbb{C}}} \times_\rho W,$$

and

$$[\pi^*S^+, \pi^*S^-, Cl] \in K_c^0(T^*X)$$

is an orientation.

3.7. **Dirac Operators.** Let's say we have a Clifford module $E \to X$, $\mathbb{C}l(V) \to \tilde{U}(E)$, and a *Clifford connection*, which is a connection ∇ on E, such that

$$\nabla(cl(v) \cdot s) = cl(\nabla^{LC} v) \cdot s + cl(v) \nabla s,$$

where ∇^{LC} is the Levi-Civita connection. From this we can construct a Dirac operator

$$D \in \mathrm{Diff}^1(X; E)$$

where

$$D_p = \sum_i cl(e_i) \cdot \nabla_{e_i}$$

for $\{e_i\}$ an orthonormal basis for T_pX. Now

$$\sigma(D)(\xi) = i \cdot cl(\xi)$$

is clearly invertible. Also notice that

$$\sigma(D^2) = |\xi|^2.$$

If X is $Spin^{\mathbb{C}}$ then we have an operator $D_{Spin^{\mathbb{C}}} \in \mathrm{Diff}^1(X, S)$

It turns out D is formally self adjoint, and with respect to grading $S = S^+ \oplus S^-$

$$D_{Spin^{\mathbb{C}}} = \begin{bmatrix} 0 & D^1 \\ D^0 & 0 \end{bmatrix}.$$

Then $[\sigma(D_{Spin^{\mathbb{C}}})] \in K_c^0(T^*X)$ is a Thom class.

As an example, for $T^*X \to X \to pt$ we have

$$K^0(X) \cong K_c^0(T^*X) \overset{ind}{\to} K^0(pt)$$

So for $[E], [F] \in K^0(X)$, we get $ind(D_E - D_F) = ind(D_E^0) - ind(D_F^0) \in K^0(pt)$.

As a final remark, if X happens to be complex, there is a canonical $Spin^{\mathbb{C}}$ structure and

$$S^{\pm} \cong \Lambda_{\mathbb{C}}^{even/odd} T^*X.$$

In this case, $D = \overline{\partial} + \overline{\partial}^*$.

4. Twisted K-Theory, Mehdi Sarikhani Khorami, Wesleyan

4.1. Twists of (Co)homology Theories. A "good" cohomology theory can be twisted by its units. Let R be a "highly structured" ring spectrum (say E_∞, A_∞, etc.). Then the units of R are defined as the pullback

$$\begin{array}{ccc} Gl_1(R) & \to & \Omega^\infty R \\ \downarrow & & \downarrow \\ \pi_0(R)^\times & \to & \pi_0(R) \end{array}$$

where $\Omega^\infty R = \lim_\to \Omega^n R(n)$ so that we have an honest Ω-spectrum in the upper right.

One finds that

$$[X_+, Gl_1(R)] \cong R^0(X)^\times,$$

hence the name "units."

In the example of K-theory, $R = K(n)$ and then $Gl_1(K) = \mathbb{Z}/2 \times BU$:

$$\begin{array}{ccc} Gl_1(K) & \to & BU \times \mathbb{Z} \\ \downarrow & & \downarrow \\ \{-1, 1\} & \to & \mathbb{Z} \end{array}$$

Unfortunately, in general $Gl_1(R)$ need not be a topological group. However, if R is E_∞ (or even A_∞) we can form $BGl_1(R)$. Then for a map to $BGl_1(R)$ we get a bundle via pullback,

$$\begin{array}{ccc} & & Gl_1(R) \\ & & \downarrow \\ P & \to & EGL_1(R) \\ \downarrow & & \downarrow \\ X & \overset{\tau}{\to} & BGL_1(R) \end{array}$$

and can form the associated Thom space (in fact an R-module)

$$X^\tau = \Sigma_+^\infty P \wedge^L_{\Sigma_+^\infty GL_1(R)} R$$

and define the τ-twisted R-homology of X as

$$R^\tau_*(X) = \pi_*(X^\tau) = \pi_0(Hom_R(\Sigma^n R, X^\tau))$$

The associated τ-twisted R-cohomology of X is

$${}^\tau R^*(X) = \pi_0(Hom_R(X^\tau, \Sigma^* R)).$$

In fact we have that GL_1 is a functor

$$GL_1 : (E^\infty - spectra) \to (certain\ E_\infty\ spaces)$$

and there is an adjoint functor Σ^∞_+.

4.2. **Twists for K-Theory.** We would like to use this to define twisted K-theory. Twists can be given by

$$X \to K(\mathbb{Z}, 3).$$

We have a (functorial) action

$$Pic(X) \times K(X) \to K(X),$$

so that for a map $f : X \to Y$ we get

$$\begin{array}{ccc} Pic(Y) \times K(Y) & \to & K(Y) \\ \downarrow f^* & & \downarrow f^* \\ Pic(X) \times K(X) & \to & K(X) \end{array}$$

By the above reasoning twistings are given by maps to $B(BU \times \mathbb{Z}/2)$. The $\mathbb{Z}/2$ part says certain twistings are real line bundles on X. So concentrating on the BBU-part, a twisting is a BU bundle on X. So what we might do is start with a class $\tau \in H^3(X;\mathbb{Z})$ which determines a homotopy class of a map

$$\tau \in [X, K(\mathbb{Z}, 3)]$$

giving a $K(\mathbb{Z}, 2)$-bundle on X. Then we would like to form an associated bundle

$$P \times_{\mathbb{CP}^\infty} BU,$$

to recover an honest twist of K-theory. Unfortunately this doesn't make sense because we can't form a point-set action of $\mathbb{CP}^\infty$ on BU: all maps in the game are only determined up to homotopy. However, with K-theory we are lucky and can model our maps on the nose rather than up to homotopy, and this is precisely the content in the Atiyah-Segal paper.

So what does this construction actually do? It's enough to see that we have a morphism of ∞-ring spectra

$$\Sigma^\infty_+ \mathbb{CP}^\infty \to K,$$

because in particular, such a morphism would induce a map

$$K(\mathbb{Z}, 2) = \mathbb{CP}^\infty \to GL_1 K$$

which in turn induces a map

$$K(\mathbb{Z}, 3) \cong BK(\mathbb{Z}, 2) \xrightarrow{\alpha} BGL_1 K.$$

Morally, this is a generalization of "a map from a group ring to an algebra is induced by a map from the group to the units of the algebra."

So then for a twisting τ we have

$$X \xrightarrow{\tau} K(\mathbb{Z},3) \xrightarrow{\alpha} BGL_1K.$$

Then

$$\pi_*(X^{\alpha\tau}) = K_*^\tau(X)$$

It is interesting to note that from the short exact sequence

$$0 \to BU(1) \to BSpin^{\mathbb{C}} \to BSO \to 0$$

we can "Thom-ify" to get

$$\Sigma_+^\infty \mathbb{CP}^\infty \cong MBU(1) \to MSpin^{\mathbb{C}}$$

and using the Atiyah-Bott-Shapiro orientation (lifted to the spectrum level) we get $MSpin^{\mathbb{C}} \to K$.

4.3. **K-Theory of Categories.** Let S be a small additive category. Then we can form the K-theory of this category, if we look at isomorphism classes of objects and use the Grothendieck construction:

$K(S) :=$ {Grothendieck group of the set of isomorphism classes of objects of S}

There are many interesting examples that arise in this way: say X is a compact space and let S be vector bundles over X. Then $K(S) = K(X)$. If instead we let S be the finitely generated projective modules over $A = C(X)$ the continuous functions on X, we recover the algebraic K-theory, which is isomorphic to the topological one in this case.

Furthermore, if you have an additive functor $\phi : S \to S'$ you can get a map on the K-theory, $K(\phi)$. But how do we define this? Well, we look at triples (E, F, α), E, F objects of S and $\alpha : E \to F$ an isomorphism. Two of these triples are isomorphic if there are morphisms $f : E \to E'$, $g : F \to F'$ such that

$$\begin{array}{ccc} \phi(E) & \xrightarrow{\phi(f)} & \phi(E') \\ \alpha\downarrow & & \downarrow\alpha' \\ \phi(F) & \xrightarrow{\phi(g)} & \phi(F') \end{array}$$

commutes. Then define

$$K(\phi) = \{[(E, F, \alpha)]\}/\sim$$

where

$$(E, F, \alpha) \sim (E', F', \alpha')$$

if there exists a triple (G, G, σ) such that $(E+G, F+G, \alpha+\sigma) \cong (E'+G, F'+G, \alpha'+\sigma)$.

Let A be a graded finite dimensional $\mathbb{C}$-algebra. Take A to be central simple, i.e. the center of A is $\mathbb{C}$. (For example, let A be a Clifford algebra.) An A-bundle over X is a locally trivial bundle over X with fibers A and transition functions respecting the algebra structure. Fix an A-bundle $\mathcal{A}$ over X. Let $\epsilon^A(X)$ be the category of graded $\mathbb{C}$-vector bundles which are projective as A-modules with morphisms of degree zero. Then let $\bar{\epsilon}^{\mathcal{A}}(X)$ be the same category, but with morphisms of all degrees. There is an inclusion $\iota : \epsilon^{\mathcal{A}}(X) \to \bar{\epsilon}^{\mathcal{A}}(X)$

The K-theory of this functor is independent of the class of the bundle $\mathcal{A}$ in the Brauer group over X.We can then define K-theory with local coefficients by

$$K^{\mathcal{A}}(X) := K(\iota).$$

Alternatively, we can take $\alpha \in Brauer(X)$ and write

$$K^{\alpha}(X) := K(\iota).$$

If X is compact, a theorem of Serre shows that $Brauer(X) \cong Tor(H^3(X;\mathbb{Z}))$.

5. Geometric Twistings of K-Theory, Braxton Collier, University of Texas

There are really two aspects to discuss:

Firstly, we would like to understand formal properties of spaces with twistings. For example, how do we generalize the Eilenberg-Steenrod axioms. Since usually computations in cohomology can be very formal (for example using Mayer-Vietoris and spectral sequences), we might actually be able to read these off from the formal properties alone.

Second we want some (geometric) models for twistings. It turns out that the whole zoo of twistings can be understood in the language of groupoids.

Twistings are classified by a central extension,

$$1 \to H^1(X;\mathbb{Z}) \to F(X) \to H^1(X;\mathbb{Z}/2) \to 1,$$

so as a set the twistings are just the product, but the twistings have a product structure and it is not the direct product. We won't be getting into this issue now, but let it be known that it is lurking in the background.

Let X be a space, and $\tau(X)$ be twistings of the K-theory of X. First let's actually define a *category* of twists rather than just a group. So let the objects of $Twist_X$ be $\tau(X)$, and let the morphisms be

$$Hom(\tau, \tau') = \{\text{“geometric maps” from } \tau \to \tau'\}/\cong .$$

It turns out that this is a symmetric monoidal groupoid. Now say $f : Y \to X$ is a map and $\tau \in \tau(X)$. Then we get $f^{-1}(\tau) \in \tau(Y)$. In fact, we can beef this up to a functor

$$f^{-1} : Twist_X \to Twist_Y.$$

So now we form the category of twists, whose objects are pairs (X, τ_X) with $\tau_X \in Twists_X$ and morphisms are maps $f : Y \to X$ and an isomorphism $\phi : \tau_Y \cong f^{-1}\tau_X$. Notice that even for fixed (X, τ_X), we may well get interesting automorphisms. In fact the automorphisms of τ are complex line bundles on X, which in turn are isomorphic to $H^2(X;\mathbb{Z})$. One way to say this is that any two automorphisms differ by a line bundle, in the appropriate sense.

5.1. **Twisted K-Theory.**

Twisted K-theory is a functor $Twist^{op} \to Ab$ where for $(Y, \tau_Y) \overset{(f,\phi)}{\to} (X, \tau_X)$, $(f, \phi)^* : K^{\tau_X}(X) \mapsto K^{\tau_Y}(Y)$. This functor is homotopy invariant in the sense that $f \simeq g$ if and only if we have a diagram

$$\begin{array}{ccc} (Y \times I, \pi^* Y) & \leftarrow & (Y, \tau_Y) \\ \uparrow & \searrow & \downarrow^{g} \\ (Y, \tau_Y) & \to^{f} & (X, \tau_X) \end{array}$$

There is also a multiplication

$$K^{\tau}(X) \otimes K^{\tau'} \to K^{\tau+\tau'}(X)$$

As an aside, we want a way to canonically identify $K^{\tau=0,1}(X)$ with $K^{0,1}(X)$, and we do this via the automorphisms of τ. Specifically, if $[L] = \phi \in Aut_{Twist_X}(\tau)$, then $[L]$ acts on $K^{\tau}(X)$ via $\alpha \mapsto [L] \cdot \alpha$. I claim this is all we need to know.

Now if we have

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ g \downarrow & & \downarrow \\ Y & \to & X \bigsqcup_Z Y, \end{array}$$

as a pushout of spaces, what do the twistings do? We need to know how to glue twistings to get local things, like Mayer-Vietoris. Morally, the twists should be something sheaf-like.

The twists we'll be considering are classified by $H^3(X; \mathbb{Z})$. Let's do an example, say $X = S^3$. Then $H^3(S^3; \mathbb{Z}) \cong \mathbb{Z}$. We'll use the fact that S^3 is the gluing of two hemispheres $U_{\pm}$ along the equator S^2. Then

$$\begin{array}{ccc} (U_+ \bigcap U_-, \epsilon^{-1}\tau) & \xrightarrow{\beta} & (U_+, \alpha^{-1}\tau) \\ \gamma \downarrow & \searrow \epsilon & \downarrow \alpha \\ (U_-, \delta^{-1}\tau) & \xrightarrow{\delta} & (S^3, \tau) \end{array} \tag{1}$$

Now we compute,

$$K^0(U_{\pm}) \cong \mathbb{Z}, \quad K^1(U_{\pm}) = K^1(U_+ \bigcap U_-) \cong 0, \quad K^0(U_+ \bigcap U_-) \cong a\mathbb{Z} \oplus b\mathbb{Z}.$$

Then we have an exact sequence

$$0 \to K^{\tau+0}(S^3) \to K^{\alpha^{-1}\tau}(U_+) \oplus K^{\delta^{-1}\tau}(U_-) \to K^{\epsilon^{-1}\tau}(U_+ \bigcap U_-) \to K^{\tau+1}(S^3) \to 0.$$

Now we need to get explicit trivialization over the open hemispheres:

$$(U_{\pm}, 0) \xrightarrow{1_{U_{\pm}}, T_{\pm}} (U_{\pm}, \alpha^{-1}\tau / \delta^{-1}\tau).$$

There are two trivializations of $(U_+ \bigcap U_-, \epsilon^{-1}\tau)$, determined by $\beta^{-1}T_+$ and $\gamma^{-1}T_-$, which gives us (under some inverting and composition)

$$[L^{\otimes k}] = [\beta^{-1}T_+]^{-1} \circ [\gamma^{-1}T_-] \in Aut(0_{U_+ \bigcap U_-}).$$

This is a line bundle on S^3 which is classified by integers k, so we get the equals sign above.

We find $K^{0+\tau}(S^3) = 0$ and $K^{1+\tau}(S^3) = \mathbb{Z}/k\mathbb{Z}$.

This computation is done in FHT I.

6. Equivariant Twisted K-Theory, Mio Alter, University of Texas

First we'll go through equivariant K-theory via vector bundles and C^*-algebras, then review the completion theorem, then do a twisted equivariant example.

6.1. Equivariant K-theory via vector bundles.

Definition 6.1. If X is compact space and G is a compact Lie group, an equivariant vector bundle $\pi : E \to X$ is a vector bundle such that π is equivariant and $E_x \mapsto E_{gx}$ is linear for each $x \in X$.

Proposition 6.2. *If G acts freely on X, $Vect_G(X) \cong Vect(X/G)$.*

We see this as

$$\begin{array}{ccc} E & \mapsto & E/G \\ \downarrow & & \downarrow \\ X & & X/G \end{array} \tag{2}$$

Proposition 6.3. *If G acts trivially on X and $F \to X$ is a equivariant vector bundle, then*

$$F \cong \oplus_{i=1}^n V_i \otimes E_i$$

where $V_i = X \times V_i$, V_i an irreducible representation of G and

$$E_i \cong Hom_G(V_i, F)$$

is a equivariant vector bundle with a trivial G-action.

Definition 6.4. $K_G^*(X)$ is the K-t=heory of G-equivariant vector bundles over X, i.e. $K_G^*(X) = K(Vect_G(X))$.

Proposition 6.5. *For X a trivial G-space, $K_G^*(X) \cong R(G) \otimes K(X)$, and $R(G) \cong K_G(pt)$.*

For example if $X = G/H$ and we have a bundle $E \to G/H$, then $G \times_H E_H \to E$ sits over G/H. I MISSED THE STATEMENT HERE,(JESSE: SO DID I) BUT I THINK THE K-THEORY IS COMPLETELY DETERMINED.

Note that

$$Vect_G(G/H) \cong Vect_H(pt),$$

so we find that

$$K_G^*(G/H) \cong K_H^*(pt),$$

and we see that $K_H^0(pt) = R(H)$ and $K^*(pt) = 0$ otherwise.

There is a localization theorem for equivariant K-theory which is analogous to the localization theorem for equivariant cohomology. Won't go into it much.

One might also start with the Borel construction for equivariant K-theory, as just the ordinary K-theory of

$$X_G \cong EG \times_G X.$$

However, the equivariant bundle picture is much richer. In fact,

Theorem 6.6 (Atiyah-Segal Completion)**.**

$$\widehat{K_G^*(X)}_I \xrightarrow{\cong} K^*(X_G).$$

where I is the augmentation ideal, i.e. I is the kernel of the augmentation map $\epsilon : R(G) \to \mathbb{Z}$ is. Notice that the augmentation ideal is the ideal generated by virtual representations of virtual dimension zero.

6.2. **Equivariant K-theory via C^*-algebras.** Recall that a C^*-algebra is a Banach algebra with a $\mathbb{C}$-linear involution. All C^*-algebras have embeddings into $\mathcal{B}(H)$ for some Hilbert space H. An example of a C^*-algebra is the ring of continuous functions $C(X, \mathbb{C})$, where X is a compact space.

Definition 6.7. For G a compact Lie group and A a C^*-algebra, a continuous action of G on A is a map $\alpha : G \to Aut(A)$ where for a sequence $g_n \to g$, $\alpha(g_n)(a) \to \alpha(g)(a)$ for all $a \in A$.

Definition 6.8. A (G, A, α)-module is an A-module in which G acts compatibly with the A-action.

Definition 6.9. $K_0^G(A)$ is the Groethendieck group of isomorphism classes of projective (G, A, α)-modules.

Definition 6.10. The crossed product $G \times_\alpha A$ is a completion of the twisted C^*-algebra, $C(G) \otimes A$ where the product (convolution) and involution are defined in terms of α.

Theorem 6.11. $K_0^G(A) \cong K_0(G \times_\alpha A)$

Note the difference between this theorem and the case for spaces and the Atiyah-Segal completion theorem.

Also, it's worth noting that if your C^*-algebra happens to be the algebra of functions on a compact Hausdorff space, the (equivariant) K-theory of the space agrees with the (equivariant) K-theory of the C^*-algebra.

6.3. **Atiyah-Segal Construction of Twisted Equivariant K-Theory.**

Proposition 6.12. *Classes in $H^3(X; \mathbb{Z})$ are in bijection with projective Hilbert bundles $P \to X$ up to isomorphism.*

Proof. To go from a bundle to a class, consider a cover of X on which $P_\alpha \cong \mathbb{P}(E_\alpha)$. Then there are transition functions

$$g_{\alpha\beta} : P_\alpha \to P_\beta,$$

and we can get a lift

$$\tilde{g}_{\alpha\beta} : E_\alpha \to E_\beta,$$

but there is a cocycle condition on these, namely

$$\tilde{g}_{\alpha\beta}\tilde{g}_{\beta\gamma}\tilde{g}_{\gamma\alpha} : X_{\alpha\beta\gamma} \to U(1),$$

and this defines a class $\eta \in H^2(X; U(1)) \cong H^3(X; \mathbb{Z})$ where the isomorphism follows from the exponential sequence, and $H^i(X; \mathbb{R}) = 0$ for $i > 0$.

Less concretely (but for a quick proof of both directions in the theorem) $PU(H)$ is a $K(\mathbb{Z}, 2)$, so $BPU(H)$ is a $K(\mathbb{Z}, 3)$, and then we have that

$$H^3(X; \mathbb{Z}) \cong [X, BPU(H)],$$

which in turn is projective Hilbert bundles, up to isomorphism. □

Now given $\tau \in H^3(X; \mathbb{Z})$ Let ${}^\tau\mathbb{P} \to X$ be the corresponding projective Hilbert bundle. Let $Fred({}^\tau\mathbb{P}) \to X$ be the associated bundle with fiber $Fred(H)$. Similarly let ${}^\tau\mathbb{K} \to X$ be the associated bundle with fiber compact operators on H.

Now we define twisted K-theory as

$$ {}^\tau K^0(X) \cong \pi_0(\Gamma(X; Fred({}^\tau\mathbb{P}))) $$
$$ {}^\tau K^0(X) \cong \widetilde{K}_0(\Gamma(X; {}^\tau\mathbb{K} \oplus I \cdot \mathbb{C})) $$

where these are sections that are a compact operator plus the identity.

We remark that the automorphisms of these projective Hilbert bundles come from tensoring with a line bundle.

6.4. **Twisted Equivariant Story.**

Theorem 6.13 (Atiyah-Segal)**.** *There is a bijection between G-equivariant projective Hilbert bundles over X and projective Hilbert bundles over X_G, the homotopy quotient.*

So G-equivariant twists are in bijection with $H^3_G(X; \mathbb{Z})$.

Then we define

$$ {}^\tau K^0_G(X) := \pi_0(\Gamma(X; Fred({}^\tau\mathbb{P}))) $$

We can also take as our definition $\widetilde{K}_0$ of the category of continuous projective $(G, \Gamma(X; {}^\tau\mathbb{K} \oplus I \cdot \mathbb{C}))$-modules. This is also equal to

$$ \widetilde{K}_0(G \times_\alpha \Gamma(X; {}^\tau\mathbb{K} \oplus I \cdot \mathbb{C})). $$

6.5. **A Computation.** $H^1(X; \{G\text{−Line Bundles}\}) \cong H^1_G(X; \{\text{Line Bundles}\}) \cong H^3_G(X; \mathbb{Z})$

Let's consider the example where $X = U(1)$ acts on itself by conjugation (i.e. trivially). We still get interesting vector bundles, since the fibers are representations of $U(1)$. So take a line bundle given by the representation $z \mapsto z^n$ and another given by $z \mapsto z$. Call these line bundles $L^\pm$. These line bundles determine a Cech-1-cocycle, and given any Cech-1-cocycle, modulo a boundary, it is in this form.

We have the Mayer-Vietoris sequence

$$\begin{array}{ccccc}
{}^\tau K^0_{U(1)}(U(1)) & \to & {}^\tau K^0_{U(1)}(U) \oplus {}^\tau K^0_{U(1)}(V) & \to & {}^\tau K^0_{U(1)}(U \bigcap V) \\
(\uparrow 3) & & & \downarrow & \\
{}^\tau K^1_{U(1)}(U \bigcap V) & \leftarrow & {}^\tau K^1_{U(1)}(U) \oplus {}^\tau K^1_{U(1)}(V) & \leftarrow {}^\tau K^1_{U(1)}(U(1)) &
\end{array}$$

but

$$ {}^\tau K^1_{U(1)}(U) \cong {}^\tau K^1_{U(1)}(pt) = 0, $$

so we get a sequence

$$ 0 \to {}^\tau K^0_{U(1)}(U(1)) \to \mathbb{Z}[L^\pm]^2 \to \mathbb{Z}[L^\pm]^2 \to {}^\tau K^1_{U(1)}(U) \to 0. $$

where the middle map is given by $(x, y) \mapsto (x - y, x - y^n)$. This shows

$$^{\tau}K^1_{U(1)}(U(1)) \cong \mathbb{Z}[L^{\pm}]/ < 1 - L^n >,$$

and 0 otherwise.

Remark: when we add $\mathbb{C}{\cdot}I$ in the above, we need to take reduced K-theory. It's like adding a disjoint basepoint.

7. Twisted Equivariant Chern Character, Owen Gwilliam, Northwestern

We'll start with discussing the equivariant Chern character. Then we'll look at the twisted version. Lastly, we'll apply this to $^{\tau}K^*_G(G)$.

7.1. **Equivariant Chern Character.** Recall that the ordinary Chern character gives us a map

$$ch : K_{\mathbb{Q}} \xrightarrow{\cong} H_{\mathbb{Q}}.$$

and that $H^*_G(X) := H^*(EG \times_G X)$.

So say $G = S^1$. Then

$$H^*_{S^1}(pt) = H^1(BS^1) \cong H^1(\mathbb{CP}^{\infty}) \cong \mathbb{Z}[t],$$

where the generator lives in degree 2.

More generally if $G = T = (S^1)^n$ then

$$H^*_T(pt) = \mathbb{Z}[t_1, \ldots, t_n].$$

Now if $G = SU(2)$ we have

$$H^*_{SU(2)}(pt) = \mathbb{Z}[t]$$

where now t lives in degree 4.

Remark: in general, we need to take power series rather than polynomial rings to define the Chern character. For finite dimensional spaces, this is irrelevant, but it's important in the general case, and justified in that we can equally well extract a ring from a graded ring by taking power series rather than the conventional direct sum.

Now suppose we took the naive definition of equivariant K-theory coming from the Borel construction, i.e.

$$\text{“}K_G(X)\text{”} = K(EG \times_G X) \xrightarrow{ch} H(EG \times_G X) = H_G(X).$$

However, we have

Theorem 7.1. *$K(BG)$ is the completition of $K_G(pt)$ at the augmentation ideal.*

so the Chern character, naively defined, would give us incomplete information. Now let G be a compact Lie group. Recall that in the “real” definition,

$$K_G(pt) = Rep(G) \otimes \mathbb{C} =: R(G)$$

Then

$$\begin{array}{ccc} R(G) \otimes \mathbb{C} & \xrightarrow{\cong} & \text{character ring} \\ V & \mapsto & \chi_V \end{array} \tag{4}$$

Now if $G = S^1$ the representation ring is $\mathbb{C}[t, t^{-1}]$ with ring structure giving by the formal multiplicative group.

If instead we take $G = Gl_n(\mathbb{C})$, putting matrices in Jordan canonical form, we see χ_V is a symmetric polynomial in the generalized eigenvalues of the conjugacy class of the matrix, so

$$Rep(G) \cong \mathbb{C}[t_1, t_1^{-1}, \dots, t_n, t_n^{-1}]^{S_n}$$

In general, for $T \subset G$ a maximal torus,

$$Rep(G) \cong R(T)^W.$$

Now $K_G(pt) = R(G)$ is a commutative ring, and $K_G(X)$ is an $R(G)$-module, so we can think of $K_G(X)$ as a quasicoherent sheaf $\mathcal{K}_G(X)$ on $Spec(R(G))$.

Theorem 7.2 (Untwisted)**.** *For a point $p \in Spec(R(G))$,*

$$\widehat{K_G^*(X)_p} \cong H^*_{Z(g)}(X^g)$$

where here $g \in G$ is associated to the conjugacy class of p, $Z(g)$ is the centralizer, and X^g is the fixed point set of $< g >$.

Let's do an example. Take S^2 with S^1 acting on it by rotation in a fixed plane. Then

$$\mathbb{C}[t, t^{-1}] \cong R(S^1) \hookrightarrow \mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^\times.$$

So take $g = e^{i\theta}$ for $\theta \neq 0$. Then $X^g = \{N, S\}$ and $Z(g) = S^1$. So then

$$H^*_{S^1}(N \bigsqcup S) \cong \mathbb{C}[[t_N]] \oplus \mathbb{C}[[t_S]].$$

If instead we take $g = 1$, then $X^g = X$. We'll use Mayer-Vietoris here with $U_S = S^2 - N$ and $U_N = S^2 - S$. Notice these charts play well with the S^1 action. Then $U_S \bigcap U_N = S^2 - \{N, S\}$ and

$$H^*_{S^1}(U_S) = \mathbb{C}[[t_S]], \quad H^*_{S^1}(U_N) \cong (C)[[t_N]], \quad H^*_{S^1}(U_S \bigcap U_N) \cong \mathbb{C}$$

Then we see that $H^*_{S^1}(S^2)$ is the kernel of the map

$$\mathbb{C}[[t_N]] \oplus \mathbb{C}[[t_S]] \to \mathbb{C}$$

that reads off the constant parts of the respective power series.

THERE WAS AN INTERESTING COMMENT BY CONSTANTINE ABOUT WHEN WE ROTATE BY TWICE THE SPEED SO THAT -1 ALSO FIXES THINGS. ESSENTIALLY, HE WAS SAYING THAT THIS DOUBLE ROTATION WOULD BE CAPTURED IN EQUIVARIANT K-THEORY BUT NOT IN EQUIVARIANT COHOMOLOGY.

7.2. **Twisted Chern Character.** Let $(\tau, \epsilon) \in H^3(X; \mathbb{Z}) \oplus H^1(X; \mathbb{Z}/2)$. For now, say $\epsilon = 0$.

Theorem 7.3. *There is a functorial Chern character*

$${}^\tau ch : {}^\tau K(X, \mathbb{C}) \to {}^\tau H(X; \mathbb{X})$$

which is a module isomorphism over

$$ch : K(X; \mathbb{C}) \to H(X; \mathbb{C}).$$

The nice thing is there is a spectral sequence to compute ${}^\tau H(X;\mathbb{X})$.

Let's use a de Rham model, as it's a bit easy to think about and to compute with. So pick a cocycle $\eta \in \Omega^3(X)$ representing τ. Then define

$$D = d - \eta : \Omega^{\pm} \to \Omega^{\mp}.$$

Now notice that

$$D^2 = d^2 - \eta d - d\eta + \eta^2 = 0,$$

so we can define

$$H^0_\eta = \text{even cohomology}, \quad H^1_\eta = \text{odd cohomology}.$$

Let's see how this changes when we take a different representative for τ. Now take η' and $\eta - \eta' = d\xi$. Think of this like a "change of gauge." We get a formula

$$D_{\eta'} e^\xi = e^\xi D_\eta.$$

This gives an isomorphism

$$H_\eta \cong H_{\eta'}.$$

So then we see that $H^2(X;\mathbb{C})$ acts on H_η. We can define ${}^\tau H$ as this, but we need to make a choice for η, and need to remember the action of H^2, i.e. the action

$$Pic(X) \times {}^\tau K(X) \to {}^\tau K(X).$$

There is a spectral sequence $F^p\Omega^* = \oplus_{q\geq p}\Omega^q$ and D preserves it. Then $E_2^{pq}(X) \Rightarrow {}^\tau H^{p+q}(X)$. Furthermore the E_2 page is $H^p(X)$ for q even and 0 elsewhere, and $d_2 = 0$ and $d_3 = \tau \cup -$.

Remark: We've said that this twisted equivariant Chern character exists, but defining it is actually quite a chore.

Let's look at another example. Take $G = SU(2)$ acting on itself by conjugation. Then

$$H^*(SU(2)) \cong \mathbb{C}[v]$$

where v is degree 3 and $v^2 = 0$, so τ is some multiple of v and for $\tau \neq 0$

$${}^\tau H^*(SU(2)) = 0$$

which follows quickly after looking at the spectral sequence: everything get's hit by cupping with v.

Let G be a compact Lie group and $G_\mathbb{C}$ its complexification. For concreteness, think of $SU(n)$ and $SL_n(\mathbb{C})$.

Take $g \in G_\mathbb{C}$ "normal" (i.e. it commutes with hermitian adjoint). Then

$$< g >= G \bigcap < g >_\mathbb{C}$$

is cyclic and

$$Z(g) = G \bigcap Z_\mathbb{C}(g)$$

is the centralizer. Then as before, X^g is the fixed point set.

Now

$${}^\tau \widehat{K_G(X)}_q = \{\text{formal completition at conjugacy class } q\}.$$

Now form a flat $Z(g)$-equivariant line bundle ${}^{\tau}\mathcal{L}(g)$ on $X^g =: Y$. Pick a projective Hilbert bundle

$$\mathbb{P}_Y \xrightarrow{\pi} Y$$

associated to τ and lifting the $Z(g)$ action on Y projectively. At each point $y \in Y$, $< g > \subset Z(g)$ acts projectively in the fiber $\pi^{-1}(y)$ and so we get a central extension, $< \tilde{g} >_y$, coming from lifiting

$$\begin{array}{ccccccccc} 1 & \to & \mathbb{C}^{\times} & \to & GL & \to & PGL & \to & 1 \\ & & \uparrow & & \uparrow & & & & \\ 1 & \to & \mathbb{C}^{\times} & \to & < \tilde{g} > & \to & < g > & \to & 1 \end{array} \tag{5}$$

This give a principle $\mathbb{C}^{\times}$-bundle ${}^{\tau}\mathcal{L}$ on $Y \times < g >_{\mathbb{C}}$

So we have a line bundle, but it is not yet clear that it is flat. So let's see this.

Two splittings differ by a homomorphism

$$< g >_{\mathbb{C}} \xrightarrow{\rho} \mathbb{C}^{\times}$$

and holonomy given by picking a splitting around a closed loop is $\rho(g)$.

Constantine: Flat line bundles are determined by holonomy, so for each loop we want to produce a complex number. A twisting is a 3-cocycle, and a loop is a 1-cycle, and contracting the 3-cocycle of the twisting with the 1-cycle of the loop and a 2-cycle coming from exponentiation a Lie algebra element produces a complex number, which happens to be invertible, giving our flat connection.

With this, we have:

Theorem 7.4.

$${}^{\tau}\widehat{K^*_G(X)}_g \xrightarrow{\cong} {}^{\tau}H^*_{Z(g)}(X^g, {}^{\tau}\mathcal{L}(g))$$

Remark: For a group G with $\pi_1 G$ free, FHT show $\mathcal{K}_G(G)$ is a skyshaper sheaf.

Now let's do the example $SU(2)$.

$$Rep(SU(2)) \otimes \mathbb{C} = \mathbb{C}[t, t^{-1}]^{S_2}$$

There is a map

$$Spec(Rep(SU(2)) \xrightarrow{\pi} \mathbb{A}^1$$

which takes the trace of an element, which looks like $\lambda + \lambda^{-1}$. Now

$$SU(2) \hookrightarrow SL_2\mathbb{C}$$

and the map takes $diag(e^{i\theta}, e^{-i\theta}) \mapsto e^{i\theta} + e^{-i\theta} \in [-2, 2] \subset \mathbb{A}^1$.

So for $\theta \neq 0$,

$$< g >_{\mathbb{C}} = T_{\mathbb{C}}$$

or a cyclic group if $e^{i\theta}$ is a root of unity.

$$Z(g) = T, \quad SU(2)^{<g>} = T.$$

Instead if we choose $\theta = 0, \pi$ then $Z(g) = G$ and $X^g = G$.

It turns out (as Dan H-L will show)

$$H^3_G(G; \mathbb{Z}) \cong \mathbb{Z}.$$

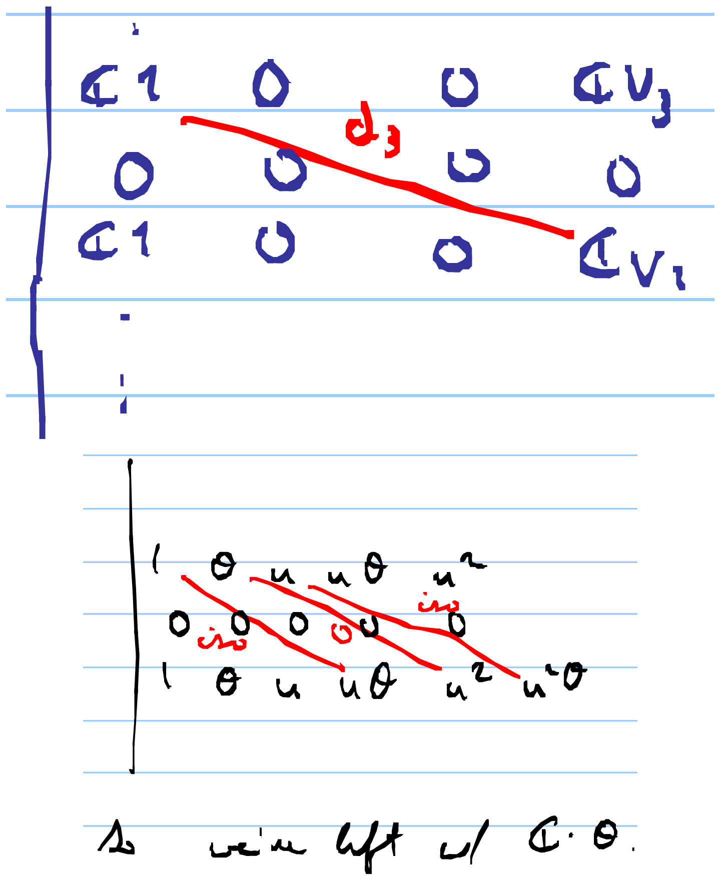


FIGURE 3. The spectral sequences.

Now let's take $\tau \neq 0$ and $g \neq \pm 1$. One can show

$$^{\tau}\mathcal{L}(g) \to S^1$$

a line bundle over the circle has holonomy $\lambda^{2\tau}$. This implies

$$^{\tau}H_{Z(g)}(S^1; {}^{\tau}\mathcal{L}(g)) = 0$$

if $\lambda^{2\tau} \neq 1$. So to be interesting, we should choose λ to be a root of unity,

$$\lambda_k = e^{i\pi k}/\tau, \quad k = 1, \dots, \tau - 1.$$

Now for the circle

$$H_T^*(T) \cong H^*(BT) \otimes H^*(T) \cong \mathbb{C}[[u, \theta]]/\theta^2$$

where u is the generator for BT and θ is the generator for S^1.

(The 2 is coming from some restriction map of $H_G^3(G)$ to $H_T^3(T)$.)

Now let's look at the spectral sequence. The d_3 differential is cupping with $2\tau \cdot u\theta$.

8. $K_G(G)$, Dan Halpern-Leistner, UC Berkeley

We'll start with the example $G = SU(2)$. We'll think of them as the unit quaternions. We have

$$ {}^{\tau}K^*_G(G) = {}^{\tau}K^*(G//G). $$

The twists for this are pretty simple:

$$ H^0(G//G;\mathbb{Z}/2) = 0, \quad H^1(G//G;\mathbb{Z}/2) = 0 $$

where the second one follows from taking the geometric realization of the fibration $G \to G//G \to pt//G$ and then looking at the long exact sequence of the fibration. So we only have twists

$$ H^3(G//G;\mathbb{Z}) \cong \mathbb{Z}. $$

We prove this and set up our calculation for twisted K-theory as follows: there is a map $SU(2) \xrightarrow{Re} [-1,1]$ and

$$ U_+ := SU(2) - \{-1\} \simeq \{+1\}, \quad U_- := SU(2) - \{+1\} \simeq \{-1\}, $$

where the homotopies are equivariant with respect to the conjugation action of $SU(2)$ on itself. Lastly,

$$ U_+ \bigcap U_- \simeq S^2 = \{Re = 0\}. $$

Then Mayer-Vietoris gives:

$$ H^2(BG) \oplus H^2(BG) \to H^2_G(S^2) \to H^3(G//G) \to H^3(U_+) \oplus H^3_G(U_-) $$

and

$$ H^2_G(S^2) \cong H^2_G(SU(2)/U(1)) \cong H^2_{U(1)}(pt) \cong \mathbb{Z}, $$

(this isomorphism comes from general considerations for homogeneous spaces G/H)

$$ H^3_G(U_+) \oplus H^3_G(U_-), \quad H^2(BG) \oplus H^2(BG) \cong 0 $$
$$ H^*(BG) \cong \mathbb{Z}[[t_4]], \quad H^*(BU(1)) \cong \mathbb{Z}[[x_2]]. $$

All of this together proves the assertion about twistings. Now, let $\tau \in H^3_{SU(2)}(SU(2);\mathbb{Z})$ correspond to the integer k under the isomorphism $H^3_{SU(2)}(SU(2);\mathbb{Z} \cong \mathbb{Z}$. Mayer-Vietoris in K-theory gives

$$ \begin{array}{ccccc} K^{\tau+0}_G(G) & \to & K^{\tau+0}_G(U_+) \oplus K^{\tau+0}_G(U_-) & \to & K^{\tau+0}_G(U_+ \bigcap U_-) \\ \downarrow & & & & \downarrow \\ K^{\tau+1}_G(U_+ \bigcap U_-) & \leftarrow & K^{\tau+1}_G(U_+) \oplus K^{\tau+1}_G(U_-) & \leftarrow & K^{\tau+1}_G(G) \end{array} \tag{6} $$

and

$$ K^{\tau+0}_G(pt) \cong R^{\tau}(G) $$

and is zero otherwise. Then since

$$ K^{\tau}_G(U_+ \bigcap U_-) \cong K^{\tau}_G(G/U(1)) \cong K^{\tau}_{U(1)}(pt), $$

$$ Rep(SU(2)) \cong \mathbb{Z}[L^{\pm}]^W $$

where $L^{\pm}$ is coming from the twisting, as we saw before.

Since everything is over $R := K_G(pt) = \mathbb{Z}[L^{\pm}]^W$, the maps must be maps of R-modules. We write $R = span\{e_k = L^k + L^{k-2} + \cdots + L^{-k}\}$. Putting this together, we have

$$0 \to {}^\tau K^0_G(G) \to (\mathbb{Z}[L^{\pm}]^W)^{\oplus 2} \to \mathbb{Z}[L^{\pm}] \to {}^\tau K^1_G(G) \to 0$$

$$\begin{aligned}(f,g) &\mapsto f - L^k g\\ (1,0) &\mapsto (1,0)\\ (0,1) &\mapsto (0,-L^k) = -Le_{k-1} + e_{k-2}\end{aligned}$$

And writing $\mathbb{Z}[L^{\pm}] = R \cdot 1 \oplus R \cdot L$, the middle map is given by

$$\varphi = \begin{pmatrix} 1 & e_{k-2} \\ 0 & -e_{k-1} \end{pmatrix}$$

So we arrive at what will turn out to be a general result (for nice G and nice twistings). For $SU(2)$

$$K^{\tau+0}_G(G) = 0.$$

$$K^{\tau+1}_G(G) = 0 \oplus R/ < e_{k-1} > .$$

In particular, note the shift, i.e. the interesting group is in 1, not 0. Also, note that the twisted G-equivariant K-theory of G is a quotient of $R(G)$.

Theorem 8.1. *There is a canonical isomorphism $K^{\tau+n}_G(G) \cong R(G)/I^\tau$ as $R(G)$-modules, where n is the dimension of G and I is the fusion ideal of τ.*

This is in FHT I for $\pi_1 G$ free, G connected and τ nondegenerate. τ is called nondegenerate when, pushing τ through the following maps

$$H^3_G(G) \to H^3_T(T) \cong H^1(T) \otimes H^2(BT) \cong H^1(T) \otimes H^1(T)$$

we have τ nondegenerate as a bilinear form on $H^1(T)$. For $SU(2)$, $\tau \neq 0$ is nondegenerate.

Note that in the above theorem, $K^\tau_G(G)$ is not a necessarily a ring, and in general it need not be. In particular, it will not give a TQFT. If G is simply connected, however, we do get a ring structure.

There is a map $\pi : G//G \to G/G$. We have a sheaf of spectra on $G//G$, $\pi_0(\Gamma(G//G, Fred^{(n)}(H)))$. On G/G consider the sheafification of $\mathcal{K}^{\tau+q}$ defined by

$$U \mapsto K^{\tau+q}(\pi^{-1}U).$$

This looks like the pushforward of the sheaf on $G//G$.

The generalized Leray-Serre spectral sequence gives us

$$E_2^{p,q} = H^p(G/G; \mathcal{K}^{\tau+q}) \implies K^{\tau+p+q}(G//G).$$

The stalk of $[g] \in G/G$ is

$$K^{\tau+q}_G([g]) \cong K^{\tau+q}_{Z(g)}(pt),$$

which is 0 when $q = 1$ and $R^\tau(Z(g))$ when $q = 0$. The challenge is in describing $\mathcal{K}^{\tau+0}$. Have $T \subset G$ with $\mathfrak{t}$ its Lie algebra. Then there is the cocharacter lattice

$$\Pi = ker(exp : \mathfrak{t} \to T)$$

The extended affine Weyl group, W^e_{aff} is the group of isometries of $\mathfrak{t}$ generated by translations in Π and W.

So we almost know the sheaf $\mathcal{K}^{\tau+0}$ because we know the stalks. Now we need to know how they glue together. Since G is connected, we have

$$exp : \mathfrak{t}/W^e_{aff} \overset{\cong}{\to} G/G.$$

We restrict τ to $*//T$ and then we denote the set of splittings of the sequence

$$1 \to \Pi \to T^\tau \to T \to 1$$

by Λ^τ. Then consider

$$f : \mathfrak{t} \times_{W^e_{aff}} \Lambda^\tau \to \mathfrak{t}/W^e_{aff}.$$

It is a fact that

$$\mathcal{K}^{\tau+0} \cong f^c_*(\mathbb{Z})$$

if τ is nondegenerate. So, the fiber is a compactly supported function $\Lambda^\tau \to \mathbb{Z}$ that is W^e_{aff} invariant in a suitable sense.

Now we get from the Leray-Serre spectral sequence

$$H^p(G/G; f^c_*(\mathbb{Z}) \cong H^p_c(\tilde{\mathfrak{t}}; \mathbb{Z})$$

where $\tilde{\mathfrak{t}} \cong \mathfrak{t} \times_{W^e_{aff}} \Lambda^\tau$. After a calculation, we have that

$$E_2^{p,q} = \begin{cases} 0 & p \neq rk(G) \\ \mathbb{Z}\langle free\ W^e_{aff}\ orbits\ in\ \Lambda^\tau\rangle & p = rk(G) \end{cases}$$

$$= \begin{cases} 0 & p \neq rk(G) \\ Map_{W^e_{aff}}(\Lambda^\tau, \mathbb{Z}) & p = rk(G) \end{cases}$$

where there is some nontrivial action of W^e_{aff} on $\mathbb{Z}$.

So, the spectral sequence looks like:

and we see it degenerates at E_2 giving ${}^\tau K^{rk(G)}_G(G) = Map_{W^e_{aff}}(\Lambda^\tau, \mathbb{Z})$.

Now let's think about the field theory side of things and see why the twisting is important for actually obtaining one. Given some bordism $X_0 \overset{\Sigma}{\to} X_1$, we assign some moduli space to all these guys (X_0, X_1, Σ). Then we want to consider

$$\begin{array}{ccc} K^\tau(\mathcal{M}_\Sigma) & \overset{r_+}{\to} & K^\tau(\mathcal{M}_{X_1}) \\ r_- \downarrow & & \\ K^\tau(\mathcal{M}_{X_0}) & & \end{array} \tag{7}$$

We want a map $(r_+)_! \circ (r_-)^*$, Then we need to look at SOMETHING THAT GOT ERASED (JESSE - I MISSED THIS TOO)so that we can pullback and pushforward to get maps from $K^\tau(\mathcal{M}_{X_0}) \to K^\tau(\mathcal{M}_{X_1})$. But then we need this to be well-defined for compositions of bordisms, which will require twistings. Then for every 1-manifold S we assign a cocycle, $\tau \in Z^3(\mathcal{M}_S$, and to each surface Σ we want to assign a coboundary $B^3(\mathcal{M}_\Sigma)$ which is some kind of compatibility between the source and target twistings once we pull them back to $\mathcal{M}_\Sigma$.

9. K-Theory of Topological Stacks, Ryan Grady, Notre Dame

Throughout, G is sufficiently "nice:" simple, maybe π_1 is free, or perhaps it's even simply connected. Anyway, there are some assumptions lurking. As motivation, consider the following black-box theorem:

Theorem 9.1.

$$ {}^{k}R(LG) \cong {}^{k+n}K_G^{dim\ G}(G) \cong {}^{k+n}K(G//G) \cong {}^{k+n}K(A/LG) $$

where A is some space of connections (with values in $\mathfrak{g}$) on the trivial principle bundle over S^1, and LG acts by gauge transformations.

So consider the groupoid

$$ A/S^1 \rtimes LG, $$

and since the S^1 action must lift to the bundle, the bundle must be trivial. Then there is a corresponding theorem

Theorem 9.2.

$$ {}^{\tau-\sigma}R(S^1 \rtimes LG) \cong {}^{\tau}K^{dim\ G}(A/S^1 \rtimes LG) $$

So let's begin by describing topological groupoids.

9.1. Topological Groupoids.

Definition 9.3. A topological groupoid is a pair of spaces (X_0, X_1) with source and target morphisms, $s, t : X_1 \to X_0$, and identity section $X_0 \to X_1$, an inverse $inv : X_1 \to X_1$ and a composition, $c : X_1 \times_{X_0} X_1 \to X_1$.

For example, if X is a G-space, $X \times G \overset{\to}{\to} X$ is a topological groupoid, denoted $X//G$.

If X is a space and $\mathcal{U} = \{U_i\}$ is a cover, then define a topolgical groupoid $N_{\mathcal{U}}$ whose objects are pairs (U_i, x_i), $x_i \in U_i$ and a morphism from (U_i, x_i) to (U_j, x_j) is an ω in $U_i \times_X U_j$ such that $\pi_i(\omega) = x_i$ and $\pi_j(\omega) = x_j$.

9.2. Central Extensions.

The heuristic for this is:

$$ \begin{array}{ccccc} U(1) & & \rho_1 & & X_1 \\ \downdownarrows & \to & \downdownarrows & \to & \downdownarrows \\ pt & & \rho_0 & & X_0 \end{array} \tag{8} $$

Definition 9.4. A $U(1)$ central extension of $X = X_1 \to X_0$ is a locally equivalent groupoid ${}^{\tau}X$ and a $U(1)$-bundle over ${}^{\tau}X_1$ together with some compatibility.

It is a fact that a $U(1)$-bundle gerbes over X are in bijection with central extensions of X. Also, to a $\mathbb{P}(H)$-bundle there is an associated $U(1)$-gerbe called the lifting gerbe.

As an example, we define M_T. Consider $S^1 \times S^1 \to S^1 \times S^1$ given by $(z_1, z_2) \mapsto (z_1 z_2, z_2)$. Then we take the mapping torus for $B(S^1 \times S^1)$:

$$ \mathbb{CP}^\infty = BS^1 \hookrightarrow BS^1 \times BS^1 \times [0,1]/\sim \overset{p_2 \times p_3}{\to} BS^1 \times S^1 \overset{DD}{\to} K(\mathbb{Z}, 3) $$

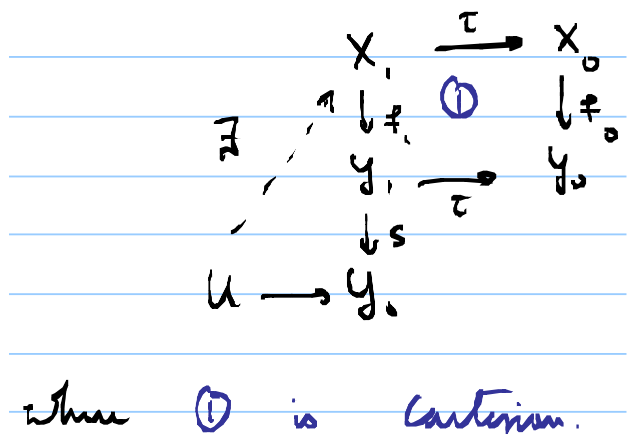


FIGURE 4. A local equivalence.

where DD classifies the Dixmier-Douady class (this class is the obstruction to lifting to a Hilbert bundle). Here, $DD(M_T)$ is not torsion. The above gives a $U(1)$-gerbe.

Definition 9.5. Let X, Y be topological groupoids. Then $F : X \to Y$ is an equivalence if it is essentially surjective and fully faithful. F is a local equivalence if F is an equivalence and for each $y \in Y_0$ there is a neighborhood U such that

Remark: this notion of equivalence is not an equivalence relation. We end up with weird correspondence diagrams to make things work, which is reminiscent of some homotopy category stuff.

The local equivalence basically enforces some notion of local lifting, which we don't get for equivalences because essentially surjective does not imply surjective.

Examples:

(1) For a refinement of covers, $U \to V$ then there is a local equivalence $N_U \to N_V$.
(2) $G \to P \to X$ a principle bundle, then $P//G \to X \overset{\to}{\to} X$ is a local equivalence.
(3) $H < G$, $P//H \to G/H//G$

Definition 9.6. A global quotient qroupoid is one who that is related via a zig-zag of local equivalences to a groupoid of the form $X//G$ for X Hausdorff and G a compact Lie group.

Definition 9.7. A local quotient groupoid is one who admits a cover by open groupoids that are global quotients.

Definition 9.8. For $X_1 \overset{\to}{\to} X_0$ a groupoid, then a fiber bundle is a regular fiber bundle $P \to X_0$ together with a bundle isomorphism $t_f : P_a \to P_b$ for $f : a \to b \in X_1$ satisfying a cocycle condition.

Sections are the naive thing, as maps from X to P in a suitable sense.

Proposition 9.9. *Let $F : X \to Y$ be a local equivalence and $P \to Y$ be a fiber bundle. Then there is a homeomorphism,*

$$\Gamma(X, F^*P) \cong \Gamma(Y, P)$$

Definition 9.10. A Hilbert bundle H on X is a fiber bundle with fiber a $\mathbb{Z}/2$-graded separable Hilbert space. H is universal if it contains any other Hilbert bundle as a summand. H is locally universal if for all open subgroupoids $X_U \hookrightarrow X$, $H|_{X_U}$ is universal.

Proposition 9.11. *Suppose X is a global quotient, $X = S//G$, then*

$$H = S \times L^2(G) \times \mathbb{C}l_1 \times \ell^2$$

is locally universal. Furthermore, if Y is a local quotient groupoid, then there exists a locally universal Hilbert bundle H. The bundles above are unique up to contractible choices.

Notice that on a space, Hilbert bundles are always trivial so that these notions are only interesting if points have automorphisms.

Proposition 9.12. *A local quotient groupoid is a global quotient groupoid if and only if its universal Hilbert bundle splits as a finite sum of finite dimensional bundles.*

Corollary 9.13. *Any gerbe with non-torsion DD-class is not a global quotient.*

In particular, M_T is not a global quotient. Also, $A/S^1 \times LG$ is not a global quotient groupoid, but is a local quotient groupoid.

9.3. **K-Theory.** Let X be a local quotient groupoid, H its locally universal Hilbert bundle. Then define

$$Fred^{(0)}(H) = \{A \in Fred(H) | A^2 + I \text{ is compact}\}.$$

This does half the job, namely this gives us even K-theory. Now we need to get odd K-theory. So let $A \in Fred(\mathbb{C}l_n \otimes H)$ for n odd. Let

$$\omega(A) := \epsilon_1 \cdots \epsilon_n \cdot A \quad n = -1 \bmod 4$$

$$\omega(A) := i^{-1}\epsilon_1 \cdots \epsilon_n \cdot A \quad n = 1 \bmod 4,$$

where the ϵ_i are generators for the Clifford algebra. Then define

$$Fred^{(n)}(H) \subset Fred^{(0)}(\mathbb{C}l_n \otimes H)$$

as odd operators that commute with $\mathbb{C}l_n$ and such that $\omega(A)$ has positive and negative essential spectrum.

Now define

$$\underline{k}(X)_n = \Gamma(X, Fred^{(0)}(H)) \quad n \text{ even}$$

$$\underline{k}(X)_n = \Gamma(X, Fred^{(1)}(H)) \quad n \text{ odd}$$

and

$$K^n(X) = \pi_0(\underline{k}(X)_n).$$

Theorem 9.14. *K^* is functorial, and local equivalent groupoids give isomorphic K-theories, and we have a MV sequence for open subgroupoids, and we have shriek maps for K-oriented maps.*

This shows that local equivalence is the correct notion of weak equivalence for groupoids, and in fact, they are weak equivalences for the model theory on stacks.

Now, what about the twists?

Definition 9.15. A twist of a groupoid X is a central extention, τ_X, which recall is a locally equivalent groupoid equipped with a principle bundle.

$$^{\tau}K(X) := [K(^{\tau}X)]_{deg\ 1}$$

where the degree 1 is with respect to the action of S^1 on the Hilbert spaces.

Remark: $\widetilde{LG} \to LG$ defines a twist of $G//G$.

Theorem 9.16.

$$^{\tau-\sigma}R(S^1 \rtimes LG) \cong {}^{\tau}K(A/S^1 \rtimes LG)$$

From $e \to G$ we get a shriek map

$$ind : {}^{\tau-\sigma}R(S^1 \rtimes G) \to {}^{\tau}K(A/S^1 \rtimes G)$$

and

$$ind^* : {}^{\tau}K(A/S^1 \rtimes LG) \to Hom_{\mathbb{Z}}({}^{\tau-\sigma}R(G); R(S^1))$$

Let H be an irreducible representation of $S^1 \rtimes LG$. Then

$$ind^*[H] = \sum_{\mu} \epsilon(u) q^{||\mu||^2/2} Tr(g) V_{p-\mu}$$

where the right hand side is the Kac numerator and μ ranges over $\lambda + \rho$ where λ is the lowest weight of H and ρ is a positive energy rep.

Remark: If we take a trivial groupoid we get ordinary K-theory back and when we take an action groupoid, we recover G-equivariant K-theory.

10. Loop Groups and Positive Energy Representations, Harold Williams, UC Berkeley

We've already seen pieces of this in previous talks. Now we're going flesh all of it out.

The idea is that LG (or more appropriately $\widetilde{LG}$) behaves very much like a compact group, so we want to study its representation in a similar way.

So first, we want to think of

$$LG := C^{\infty}(S^1, G)$$

as a C^{∞}-manifold. So for $C^{\infty}([a,b])$ we want $f_n \to f$ if and only if $f_n^{(k)} \to f^{(k)}$ uniformly for all k. Then it is a fact that

$$T_e LG = LT_e G = L\mathfrak{g}.$$

It turns out that the Fourier guys are dense in the above, i.e.

$$\oplus_{k \in \mathbb{Z}} z^k \mathfrak{g} \hookrightarrow L\mathfrak{g}$$

is dense.

Definition 10.1. For $f \in Aut(G)$, let

$$L_f G = \{\gamma : \mathbb{R} \to G | \gamma(\theta + 2\pi) = f(\gamma(\theta))$$

This is the twisted loop group.

Definition 10.2. Let G be compact. Then $\widetilde{LG}$ is the universal extension

$$\begin{array}{ccccccccc} 1 & \to & \pi_1 G & \to & \tilde{G} & \to & G & \to & 1 \\ & & \downarrow & & \downarrow = \exists! & & \downarrow & & \\ 1 & \to & A & \to & E & \to & G & \to & 1 \end{array} \tag{9}$$

for any abelian group A.

From now on, assume G is simple, so $\pi_1(G) = 0$. Given an invariant bilinear form, $\langle\ ,\ \rangle : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{R}$, define

$$\omega : \Gamma^2 L\mathfrak{g} \to \mathbb{R}$$

by

$$\omega(\xi, \eta) = \int_{S^1} \langle \xi, d\eta \rangle.$$

Then ω is a 2-cocycle, so it defines a central extension:

$$0 \to \mathbb{R} \to \widehat{L\mathfrak{g}} \to L\mathfrak{g} \to 0 \tag{10}$$

by

$$[(\xi, a), (\eta, b)] = ([\xi, \eta], \omega(\xi, \eta))$$

for $a, b \in \mathbb{R}$ and $\xi, \eta \in L\mathfrak{g}$. We'd like to get a corresponding central extension

$$1 \to T \to \widehat{LG} \to LG \to 1 \tag{11}$$

So need a class in $H^2(LG)$. So we extend ω by left translation to get $\overline{\omega} \in \Omega^2(LG)$. Now if $\overline{\omega}/2\pi$ is integral, we get a S^1 bundle with Chern class $\overline{\omega}/2\pi$, which lifts the group structure.

From now on, normalize $\langle\ ,\rangle$ so that $\overline{\omega}/2\pi$ is minimal among integral classes. The corresponding $\widehat{LG}$ is universal and $\pi_1 \widehat{LG} = 0$.

As a remark, for simply connected groups we have isomorphisms

$$H^4(BG) \overset{\cong}{\to} H^3_G(G) \overset{\cong}{\to} H^3(G) \overset{\cong}{\to} H^2(LG),$$

so our class ω really comes from one on $H^4(BG)$. So $\overline{\omega}/2\pi$ is cohomologous to the transgression of $\overline{\sigma} \in \Omega^3(G)$,

$$\sigma : \Lambda^3 \mathfrak{g} \to \mathbb{R},$$

$$\sigma(X, Y, Z) = \langle [X, Y], Z \rangle,$$

which is G-invariant as in the above sequence of isomorphisms.

Let P be a projective representation of LG. This gives a central extension, LG_p, which gives a multiple $\ell\langle\ ,\ \rangle$ of $\langle\ ,\ \rangle$, where ℓ is the *level* of P.

The universal map $\widehat{LG} \to LG_P$ restricts to the center as $z \mapsto z^\ell$. Also, $Diff^+(S^1)$ acts on LG and lifts to an action of $Diff^+(S^1)$ on $\widehat{LG}$ so we can form

$$\widetilde{LG} = \pi \rtimes \widehat{LG}.$$

We want to study the representation theory of this guy.

Let's quickly review the representation theory of compact groups. Let

$$T \hookrightarrow G$$

be a maximal torus. We start by looking at the representation theory of this, since we know what representations of abelian groups look like. We have the cocharacter lattice,

$$\check{T} := Hom(S^1, T)$$

and the character lattice

$$\hat{T} := Hom(T, S^1).$$

Taking derivatives, we regard

$$\check{T} \hookrightarrow \mathfrak{t}, \quad \hat{T} \hookrightarrow \mathfrak{t}^*.$$

If G now acts on V, we can decompose V as

$$V = \oplus_{\lambda \in \hat{T}} V_\lambda$$

where

$$V_\lambda = \{v | t \cdot v = \lambda(t) v\}.$$

The nonzero weights of $Ad_{\mathfrak{g}} : G \to G$ are the roots of G.

For nonzero $\alpha \in \hat{T}$ a root, $dim(\mathfrak{g}_\alpha) = dim(\mathfrak{g}_{-\alpha}) = 1$. Define the coroot h_α by $h_\alpha \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\alpha(h_\alpha) = 2$. We note $h_\alpha \in \tilde{T}$ comes from $SU(2) \to G$

Now $\hat{T}$ acts on $N(T)$ by

$$(g \cdot \alpha)(t) = \alpha(g^{-1} t g)$$

for $g \in N(T)$.

If $v \in V_\lambda$,

$$tgv = gg^{-1}tgv = (g\lambda) gv,$$

so $gv \in V_{g\lambda}$, and thus the weights of V are invariant under

$$W = N(T)/T,$$

the Weyl group.

Now, to recover the representation theory of G choose positive/negative weights. We say $\lambda \in \hat{T}$ is antidominant if $\lambda \leq \omega\lambda \ \forall \omega \in W$, which holds if and only if

$$\lambda(h_\alpha) \leq 0 \quad \forall \alpha \geq 0.$$

Theorem 10.3. *Irreducible representations are in bijection with antidominant weights, which are in turn in bijection with $\hat{T}/W$.*

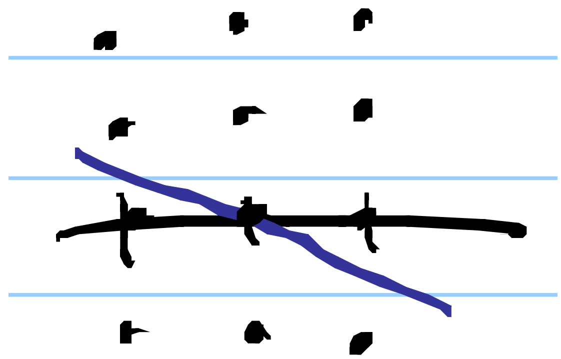


FIGURE 5. Roots of SL_2.

10.1. **The Affine Weyl Group.** Let T_R be the torus which acts by rotations on the base, T_C for the central torus, and T is the maximal torus of G, thought of as living in LG by constant loops. Then

$$T_R \times T \times T_C \hookrightarrow \tilde{LG}.$$

We claim that

$$W_{aff} := \frac{N(T_R \times T \times T_C)}{T_R \times T \times T_C} = \tilde{T} \rtimes W.$$

To see this (at least half way) consider

$$R_\theta \in T_R, \quad f \in LG.$$

Then

$$R_\theta^{-1} f(z) R_\theta = f(e^{i\theta} z)$$

and for $f \in \tilde{T}$

$$f(z) R_\theta f(z)^{-1} = R_\theta f(e^{i\theta} z) f(z)^{-1} = R_\theta f(e^{i\theta}) \in T_R \times T$$

so $\check{T}$ normalizes T_R, and since it takes values in S^1, it normalizes T and T_C as well. This "shows" the claim.

Roots of $\tilde{LG}$ are defined as

$$\{(k, \alpha, 0) | k \in \mathbb{Z}, \ \alpha \text{ a root of } G\}$$

positive roots are

$$\{(0, \alpha, 0) | \alpha > 0\} \bigsqcup \{((k, \alpha, 0) | k > 0\}.$$

We see this by considering

$$\mathbb{R}_0 \oplus (\oplus_{k \in \mathbb{Z}} z^k \mathfrak{g}) \oplus \mathbb{R}_C.$$

The new simple root is $(1, -\theta, 0)$ where θ is the highest root.

Now, the coroots are

$$h_{(k,\alpha,0)} = [z^k e_\alpha, z^{-k} e_{-\alpha} = (0, h_\alpha, \frac{-k}{2} ||h_\alpha||^2).$$

We remark that $\overline{\omega}/2\pi$ is integral if and only if $||h_\alpha||^2$ is even. We can then characterize the normalized form as

$$\langle \theta, \theta \rangle = 2.$$

So in particular, our integral forms take the character lattice to the cocharacter lattice and $h_\theta \mapsto \theta$, under induced map $\mathfrak{t} \to \mathfrak{t}^*$.

We remark further that $\tilde{L\mathfrak{g}} \otimes \mathbb{C}$ is an affine Kac-Moody algebra. Kac-Moody algebras are the right generalization to infinite dimensions of a semisimple algebra.

10.2. **Positive Energy Representations.**

Definition 10.4. A representation V of $\tilde{LG}$ is of positive energy if in the decomposition

$$V = \oplus_{\mathbb{Z} \times \hat{T} \times \mathbb{Z}} V_{n,\lambda,\ell},$$

we find that

$$V_{n,\lambda,\ell} = 0$$

for all $n < 0$.

It's a fact that PERs are completely reducible, unitary and have heighest weights.

Theorem 10.5. *PER of $\tilde{LG}$ are in bijection with antidominant affine weights.*

So if V is irreducible, then T_C acts by a unique weight, and V has a well-defined level.

So let's classify the antidominant weights, (n, λ, ℓ). First note that given a representation of $\tilde{LG}$, we can always twist by a character,

$$T_R \to T,$$

which will shift the energy. So we can always assume that $n = 0$. We can check if a weight is antidominant by evaluating on positive coroots, i.e.

$$\lambda(h_\alpha) \leq 0$$

for $\alpha > 0$. So we want to consider

$$(0, \lambda, \ell) \cdot (0, -h_\theta, -\frac{1}{2}||h_\theta||^2) \leq 0$$

working this out we find

$$-\lambda(h_\theta) - \frac{\ell}{2}||h_\theta||^2 \leq 0,$$

and by our normalization we find

$$\lambda(h_\theta) \geq -\ell.$$

Now let's look at $SU(3)$, finding θ, determining Weyl chambers, where the positive coroots are, where the Weyl alcoves are. The antidominant weights at level ℓ are those in the Weyl alcove.

We claim that

$$\eta \in \tilde{T} \hookrightarrow \mathfrak{t}$$

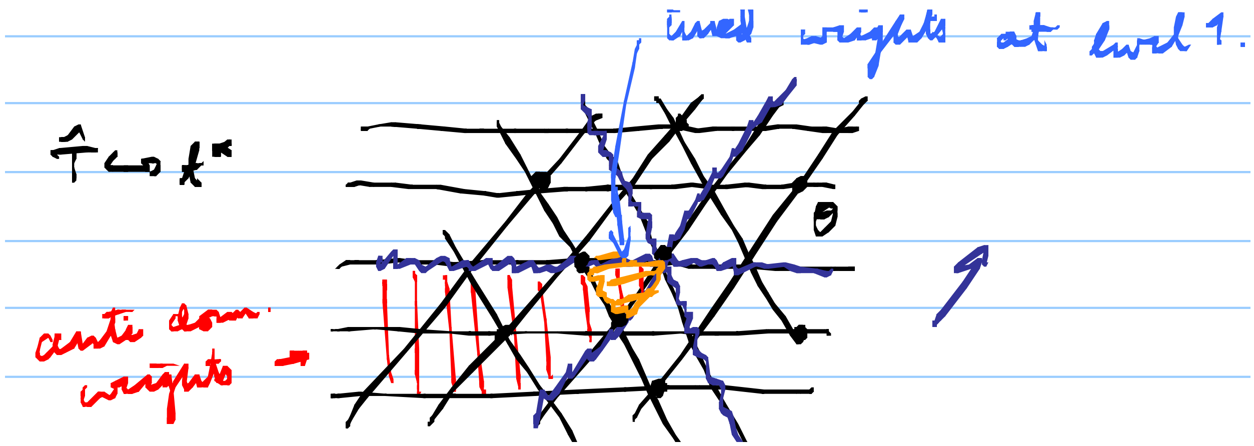
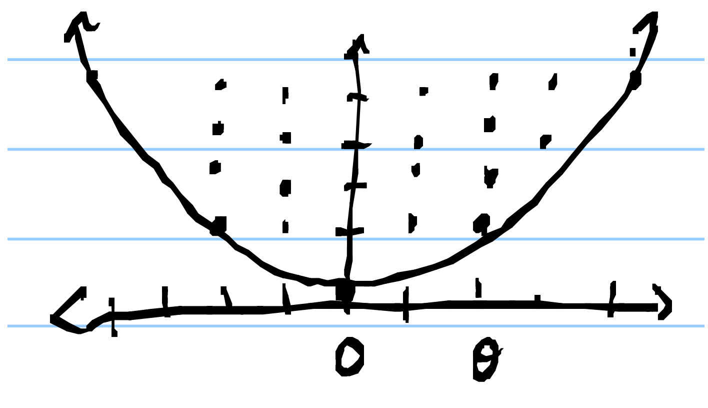


FIGURE 6. Roots for $SU(3)$.



acts on $\mathbb{R}^*_R \oplus \mathfrak{t}^* \oplus \mathbb{R}^*_C$ by

$$\eta \cdot (n, \lambda, \ell) = (n + \lambda(\eta) + \frac{\ell}{2}||\eta||^2, \lambda + \ell\eta^*, \ell).$$

Notice that η fixes the level ℓ. The fact that η is acting by affine transformations is why we have an affine Weyl group and an affine Kac-Moody algebra.

In general we'll always have a representation of the form $(0, 0, 1)$, called the basic representation.

For example, consider $LSU(2)$ at level 1. Then we have $(0, 0, 1)$ and $(0, -\theta/2, 1)$. Then

$$\tilde{T} = mh_\theta \hookrightarrow \mathfrak{t},$$

and

$$mh_\theta \cdot (0, 0, 1) = (\frac{1}{2}||mh_\theta||^2, m\theta,) = (m^2, m\theta, 1)$$

which gives us the Weyl orbit are the lattice points lying on some parabola. For a fixed energy level, we get a finite dimensional representation, see the figure above.

10.3. **Existence of PERs.** One might be tempted to build these representions naively, say $LSU(2)$ acting on on $L^2(S^1;\mathbb{C}^n)$. But this is not of positive energy. So we need to do something more sophisticated.

Now it is a fact that

$$G/T \cong G_{\mathbb{C}}/B^+,$$

the generalized flag variety. From the description on the right hand side above, we have a complex manifold.

$$G_{\mathbb{C}} = \bigsqcup_{w\in W} N^{-1}wB^+.$$

For example for $SU(N)$, B^+ is upper triangular matrices, N^{-1} is lower triangular with ones on the diagonal. We find then that

$$G/T \cong \bigsqcup_{w\in W} N^{-1}w$$

and the $N^{-1}w$'s are affine spaces, one of which is dense.

Any $\lambda \in \hat{T}$, extends to $B^+ = T_{\mathbb{C}} \rtimes N^{01} \to \mathbb{C}^\times$, so we get a representation $\mathbb{C}_\lambda$ of B^+. Then

$$\mathcal{L}_\lambda = G_{\mathbb{C}} \times_{B^+} \mathbb{C}_\lambda$$

is a complex line bundle over the complex manifold G/T.

Theorem 10.6 (Borel-Weil)**.**

$$\Gamma(\mathcal{L}_\lambda) = 0,$$

unless λ is antidominant. Then

$$\Gamma(\mathcal{L}_\lambda) = L_\lambda.$$

We recall that

$$K_G(G/H) \cong K_H(pt) = Rep(H).$$

For example, consider the Hopf bundle $U(1) \to SU(2) \to \mathbb{CP}^1$. So irreps of SU(2) sections of holomorphic line bundles on $\mathbb{CP}^1 = SU(2)/U(1) = G/T$. These line bundles correspond to

$$Sym^k(\mathbb{C}^2),$$

which is the sheaf $\mathcal{O}(k)$.

Theorem 10.7. *Borel-Weil holds for LG/T.*

Unpacking the analogue of this,

$$\tilde{LG}/\tilde{T} \cong LG/T \cong LG_{\mathbb{C}}/L^+B^+$$

where L^+B^+ is bounded values of holomorphic maps γ from the unit disc into $G_{\mathbb{C}}$, where $\gamma(0) \in B^+$. From here, everything works out as one would expect: we get an analogue of the Bruhat decomposition:

$$LG = \bigsqcup_{w\in W_{aff}} L^-N^-wL^+B^+.$$

and

$$LG/T = \bigsqcup_{w \in W_{aff}} L^- N^- w,$$

and all the cells are contractible. We have a "big cell" that is isomorphic to $L^- N^-$. From the exponential map on the Lie algebra, we get that a neighborhood in $L^- N^-$ is diffeomorphic to a neighborhood in $L^- \mathfrak{n}^-$. So if

$$s \in \Gamma(\mathcal{L}_\lambda),$$

we restrict to get a holomorphic function on $L^- \mathfrak{n}^-$. So in particular, a section is determined by its Taylor series at the the basepoint, which is in

$$\Pi_{n \geq 0} Sym^n (L^- \mathfrak{n}^-)^*,$$

and one can see that this is positive energy, so the section we started with was positive energy.

11. Character Formulas, Dario Beraldo, UC Berkeley

Let G be a compact Lie group. And $T \subset G$ the maximal torus. We have some lattices:

$$\text{roots of } \mathfrak{g} \subset Hom(T, S^1) \supset \text{ weight } \mathfrak{g} \subset i\mathfrak{t}^*$$

Then there is a map

$$exp : \mathfrak{t}/L \overset{\cong}{\to} T.$$

$$\Lambda = \left(\frac{L}{2\pi i}\right)^* = \{\phi : \mathfrak{t}^* \to i\mathbb{R} | \phi(L) \subset 2\pi i \mathbb{Z}\}.$$

Then irreducible reps for G are in bijection with Λ_+, where $\lambda \in \Lambda_+$ maps to (π_λ, V_λ). Then we have the Weyl character formula for $X \in \mathfrak{t}$

$$\chi_\lambda(exp(X)) = \frac{\sum_w \epsilon(w) e^{(w \cdot (\lambda + p))(X)}}{e^{\rho(X)} \Pi_{\alpha > 0} (1 - e^{-\alpha(X)})},$$

where

$$\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha = \sum_{i=1}^{\ell} w_i.$$

Note that in general

$$w \cdot (\lambda + \rho), \rho \notin \Lambda^+,$$

but $w(\rho) - \rho$ is a root of $\mathfrak{g}$.

Now we will explain Kirilov's orbit method philosophy. Let G be a non-compact Lie group. Then coadjoint orbits (denoted Ω) in $\mathfrak{g}^*$ are in bijection with unitary irreducible representations of G (denoted π_Ω).

Proposition 11.1. *Each coadjoint orbit is a homogeneous symplectic manifold.*

The map $\Omega \to \pi_\Omega$ is called quantization, and we may think of Ω as some classical phase space.

Unfortunately, this doesn't work for all Lie groups, for example the above is false for a compact Lie group. However this does work if G is connected with nilpotent Lie algebra.

So backing up, any Lie group acts on its Lie algebra by the adjoint representation, ad, which is the deriviative of the adjoint action Ad of the Lie group on itself. We will define another action of G on $\mathfrak{g}$ by

$$K(g) \cdot \phi = \phi \circ Ad(g^{-1}).$$

Taking derivatives, $\mathfrak{g}$ acts on itself by

$$K_*(X)\phi = \phi \circ (-ad(X)).$$

for $X \in \mathfrak{g}$ and $\phi \in \mathfrak{g}^*$.

So now we wish to get a symplectic manifold

$$G/stab(\phi) \to \Omega_\phi$$
$$g \mapsto k(g)\phi.$$

Now we define a form σ (that will turn out to be symplectic). Now,

$$T_\phi \Omega = \mathfrak{g}/s_\phi$$

where

$$s_\phi = \{X \in \mathfrak{g} | K_*(X)\phi = 0\}.$$

Then

$$\sigma : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$$
$$(X, Y) \mapsto \phi([X, Y]).$$

One can show that σ is closed using the Jacobi identity. It is nondegenerate as

$$ker(\sigma) = s_\phi.$$

For example, consider the Heisenberg group of 3×3 upper triangular matices with 1's on the diagonal:

$$H := \{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} | a, b, c \in \mathbb{R}\}.$$

Let g_{abc} denote an element of H. Then we want to look at the irreducible representations of H, denoted π_λ and $\pi_{\mu,\nu}$.

Then $\pi_\lambda : L^2(\mathbb{R}, dx) \to L^2(\mathbb{R}, dx)$ where

$$(\pi_\lambda(g_{abc})f)(x) = e^{2\pi i(bx+c)} f(x_a)$$

which is irreducible by Wiener's theorem, and

$$\pi_{\mu,\nu}(g_{abc} = e^{2\pi i(a\mu_b\nu)}.$$

Define

$$X := g_{100}, \quad Y = g_{010}, \quad Z = g_{001}.$$

and note $[X, Y] = Z$. Define a pairing

$$\langle A, B \rangle = tr(AB).$$

Now for computing K, since $\mathfrak{g}^* = Mat(3 \times 3)/\mathfrak{g}^{\perp}$,

$$K(g_{abc}) \begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix} = g_{abc} \begin{pmatrix} * & * & * \\ x+bz & * & * \\ z & y-az & * \end{pmatrix} g_{abc}^{-1} = \begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix}$$

Now let $z = \lambda$. Then if $\lambda \neq 0$ we get 1 orbit $\{z = \lambda\}$ and if $\lambda = 0$, orbits are points $\{\mu, \nu, 0)\}$.

We have a measure on these orbits, given by the symplectic volume,

$$\mu_\Omega := \frac{\sigma^d}{d!}$$

where $d = \frac{1}{2} dim\Omega$.

Then there is the Kirillov character formula

$$\chi_{\pi_\Omega} = \hat{\mu_\Omega}.$$

The character is the trace of the representation evaluated on the exponential of a Lie algebra element. Given

$$\pi_\lambda(\phi) = (\pi_\lambda(g), \phi) = \int_G \pi_\lambda(g_{abc}\phi(a,b,c)dadbdc.$$

In the above example, $\sigma = \frac{dx \wedge dy}{\lambda}$. Now, we want to verify that $(\chi_{\pi_\Omega}, \varphi) = (\hat{\mu}_\Omega, \varphi) = (\hat{\varphi}, \mu_\Omega)$. We have an operator on $L^2(\mathbb{R})$ given by:

$$\pi_\lambda(\varphi) = (\pi_\lambda(g), \varphi) = \int_G \pi_\lambda(g_{abc})\varphi(a,b,c)dadbdc$$

Now,

$$\begin{aligned}(\pi_\lambda(\varphi)(f))(x) &= \int_{\mathbb{R}^3} \varphi(y,b,c)e^{2\pi i\lambda(bx+c)} f(y)dydbdc \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\phi}(y-x, \lambda x, c)e^{2\pi i\lambda c} f(y)dcdy\end{aligned}$$

So

$$\pi_\lambda(\phi)f(x) = \int_{\mathbb{R}}\int \hat{\phi}(y-x, \lambda x, c)e^{2\pi i\lambda c}dcf(y)dy.$$

Since f was Schwartz, this is trace class,

$$tr\pi_\lambda(\phi) = \int_{\mathbb{R}} K(x,x)dx = \frac{1}{\lambda}\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\phi}(0,x,c)e^{2\pi i\lambda c}dcdx = \int_{\mathbb{R}} \phi(0,0,c)\frac{e^{2\pi i\lambda c}}{\lambda}dc,$$

where

$$K(x,x) = \hat{\phi}(y-x, \lambda x, c)e^{2\pi i\lambda c}.$$

Then

$$\chi_{\pi_\Omega} = \frac{e^{2\pi i\lambda c}}{\lambda}\delta(a)\delta(b).$$

So

$$(\hat{\mu_\Omega}, \phi) \to (\mu_\Omega, \hat{\phi}) = \int_{\Omega_\lambda}\left(\int_{\mathbb{R}^3} \phi(x,y,z)e^{2\pi i(ax+by+cz)}dadbdc\right)\frac{dx \wedge dy}{\lambda}$$

which gives

$$\int \phi(x,y,z)\left(\int e^{2\pi i a x}dx\right)\left(\int e^{2\pi i b y}dy\right)\left(\int e^{2\pi i c z}dz\right)$$

WASN'T ABLE TO COPY THIS ONE IN TIME. (Jesse: NEITHER WAS I.)

For compact groups, this fails roughly because there are too many coadjoint orbits and too few irreducible representations of G. We do have a Killing form on $\mathfrak{g}$, so adjoint orbits can be identified with coadjoint orbits. Adjoint orbits are flag varieties, G/Γ, for $\Gamma \supset T$. It is a fact that each orbit intersects the positive Weyl chamber in exactly one point.

Another problem is something about $exp : \mathfrak{g} \to G$. A third problem is a ρ-shift.

Now lets look at $SU(2)$. So coadjoint orbits are flag varieties, i.e. spheres, which are centered at $0 \in \mathfrak{su}(2)$. So let Ω_r be the sphere of radius r, and $\sigma = r\sin\phi d\theta \wedge d\phi$. Let Z be a diagonal Lie algebra element with $iz, -iz$ on the diagonal. Then

$$\frac{1}{p(Z)}\int e^{2\pi i<Z,F}d\sigma(F) = \frac{1}{p(Z)}\frac{\sin rz}{z} = \frac{\sin rz}{\sin z}$$

where

$$p(Z) = \frac{\sinh(ad\ Z/2)}{ad\ Z/2}$$

If $r = n+1$, then

$$\frac{\sin((n+1)z)}{\sin z} = \chi_n(z).$$

Then the Kirillov character formula:

Theorem 11.2 (Kirillov)**.**

$$\chi_\lambda(exp(X)) = \frac{1}{p(X)}\int_{\Omega_{\lambda+\rho}} e^{2\pi 1<X,F>}d\mu(F)$$

$exp^*\chi_\lambda(X)$.

Now let's move to loop groups. So

$$\tilde{L\mathfrak{g}} \cong L\mathfrak{g} \oplus \mathbb{R}C$$

and

$$\hat{LG} = L\mathfrak{g} \oplus \mathbb{R}C \oplus \mathbb{R}D$$
$$[D, X\otimes z^n] = nX\otimes z^n.$$

We have an invariant bilinear form defined by

$$< D, C >= 1, \quad < X \times z^m, Y \otimes z^n >=< X, Y > \delta_{m,-n}.$$

In principle we want to calculate the action of $\hat{LG}$ acting on $\hat{L\mathfrak{g}}^*$, but for now we'll just do LG acting on $\tilde{L\mathfrak{g}}^*$. We have sequences

$$0 \to \mathbb{R}C \to \tilde{L\mathfrak{g}} \to L\mathfrak{g} \to 0$$

$$0 \to (L\mathfrak{g})^* \to \tilde{L\mathfrak{g}}^* \to \mathbb{R}\delta \to 0$$

where δ is an element determined by $\delta : C \mapsto 1$. Then

$$(\tilde{L\mathfrak{g}})^*_\ell = \{\phi : \tilde{L\mathfrak{g}} \to \mathbb{R} | \phi(C) = \ell\},$$

which are connections on the trivial principle G bundle over S^1. Then we need to calculate this dual:

$$(L\mathfrak{g})^* = \Omega^1(S^1; \mathfrak{d}^*) = \Omega^1(S^1; \mathfrak{d})$$

where the second isomorphism uses the Killing form, which are like infinitessimal gauge transformations.

$$(X \otimes f, Y \otimes \omega) =< X, Y > \int f\omega$$

In $(\tilde{L\mathfrak{g}})^*_\ell$, we have an element $\ell\delta + \omega$.

$$K_*(X)(\ell\delta + \omega) = X \cdot (\ell\delta + \omega)$$

and

$$\begin{aligned} < X \cdot (\ell\delta + \omega), Y + \alpha C > &= - < \ell\delta + \omega, [X, Y + \alpha C] > \\ &= - < \ell\delta + \omega, [X, Y] > \\ &= -\ell < dX, Y > - < \omega, [X, Y] > \\ &= -\ell < dX, Y > + < [X, \omega], Y > \\ &=< -\ell dX + [X, \omega], Y + \alpha C \end{aligned}$$

so we find that the infinitessimal gauge action is

$$x \cdot (\ell\delta + \omega) = [x, \ell d + \omega]$$

where $d = \ell\delta$. So LG acts on $(L\mathfrak{g})^*_\ell$ by $\gamma \cdot (d + \omega) = \gamma(d + \omega)\gamma^{-1}$, and we see

$$(L\mathfrak{g})^*_\ell / LG \cong \mathcal{A}/LG = G//G$$

11.1. Frankel has given a Kirillov character formula for $\hat{LG}$. Let λ be a weight.

$$\chi_\lambda(exp(bD + Y)\frac{1}{p(bD + Y)} e^{b/a(H,H)} \int_{\Omega_{\lambda+\hat{p}}} e^{2\pi i < dB + Y, F >} d\mu^{a/b}(F).$$

The roots of $\hat{LG}$ are

$$\{R_\mathfrak{g} + \mathbb{Z}C\} \bigsqcup \{\mathbb{Z} - 0C\}$$

and the simple roots were $\alpha_1, \ldots, \alpha_\ell$ and $\alpha_0 = C - \theta$. The fundamental weights are $\{\tilde{\omega_0}, \ldots, \tilde{\omega_\ell}\}$. Then

$$\frac{2(\tilde{\omega}_i, \alpha_j)}{(\tilde{\alpha_j}, \tilde{\alpha_j})} = \delta_{ij}.$$

The Killing term is

$$\hat{p} = \sum_{i=0}^{\ell} \tilde{\omega}_i = \frac{D}{2} + p$$

where

$$p = \sum_{i=1}^{\ell} \omega_i.$$

The above character formula is defined if and only if $Im(b) < 0$ and

$$\lambda + \hat{p} = aD + H$$

for $a > 0$.

Constantine: Which elements of the loop group are trace class? You have to do something to force things to be trace class which involves something about energies in the unit disc. DIDN'T CATCH THIS PART (Jesse - NEITHER DID I).

12. Dirac Family Construction of K-Classes, Sander Kupers, Utrecht

This is either FHT II or parts 3 and 5 or FHT III.

There will be 3 analogous constructions giving a map between loop group representations and equivariant K-theory.

So let T be a torus, G a compact Lie group. There are three setups for the different constructions:

(1) Have a finite dimensional complex representation $R(G) \cong {}^{\sigma}K_G^{dim\ G}(\mathfrak{g}^*)_{cpt}$
(2) Positive energy representations at level $\tau - \sigma$ $R^{\tau-\sigma}(LG) \cong {}^{\tau}K_G^{dim\ G}(G)$.
(3) Positive energy representations at level τ, which are in bijection to representations of Γ^τ,

$$R^\tau(LT) \cong R(\Gamma^\tau) \cong {}^{\tau}K_T^{dim\ T}(T)$$

All of these isomorphisms can be implemented by Dirac families on spinor fields. We will also see the twistings σ in these constructions.

First we'll look at a special class of twistings, then look at $Pin^{\mathbb{C}}$ and spinors. Then look at G, LG, LT and Γ^τ.

We'll again be thinking of twists as central extensions of some topological groupoid. So recall a twist $K_G^\tau(X)$ is the following data:

(1) $\tilde{X}//\tilde{G} \to X//G$ locally equivalence of groupoids such that $\tilde{N} \subset \tilde{G}$ is normal, freely acting on $\tilde{X}$, $\tilde{X}/\tilde{N} \cong X$, $\tilde{G}/\tilde{N} \cong G$, $N = \Omega G$.
(2) A central extension $\tilde{G}^\tau$ of $\tilde{G}$
(3) A grading homomorphism $\epsilon : \tilde{G} \to \mathbb{Z}/2$.

So we want to get our hands on classes in $K_G^\tau(X)$. Unwinding a definition from yesterday, we need $\tilde{G}^\tau$-equivariant families of skew-adjoint odd Fredholm operators on a τ-twisted Hilbert bundle, indexed by $\tilde{X}$. We'll construct Dirac operators, and they'll be Fredholm operators of this type.

12.1. $Pin^{\mathbb{C}}$ **and Spinors.** If V is a finite dimensional real vector space with inner product, one can define

$$Pin^{\mathbb{C}}(V) \subset \mathbb{CL}(V)$$

This is a group and therefore it acts on spinor representations S. If the dimension of V is even , we get two irreducible representations, $S^\pm$, and we just fix one. If the dimension of V is odd, we get one irreducible representation S, but we need to remember a $\mathbb{C}l(1)$ action too. There is a central extension

$$1 \to T \to Pin^{\mathbb{C}}(V) \to O(V) \to 1$$

which is split at the Lie algebra level.

We can construct spinors on loop groups, but we'll need to make some choices, like a polarization which roughly allows us to distinguish between positive and negative energy representations.

Now pick a G invariant inner product on $\mathfrak{g}$ with basis e_i and dual basis e_i^*. Then $Ad : G \to O(\mathfrak{g})$.

Definition 12.1. $G^\sigma = G \times_{O(\mathfrak{g})} Pin^{\mathbb{C}}(\mathfrak{g})$.

Pick our spinor representation σ on S, so G^σ acts on S, $\mathbb{C}l(\mathfrak{g}^*)$ acts on S, $\mathfrak{g}^*$ acts on S via γ, from splitting

$$\mathfrak{g} \to \mathfrak{g}^\sigma \to pin^{\mathbb{C}}.$$

Now we want to define spinor fields. Via left translation, identify spinor fields with $C^\infty(G) \otimes S$. Then G acts on $C^\infty(G) \otimes S$ on the left via a map R and $\mathfrak{g}$ and $\mathfrak{g}^*$ act on the right via γ and σ.

Theorem 12.2 (Peter-Weyl). *$L^2(G) \otimes S \cong \widehat{\bigoplus}_{V\ irr} V^* \otimes V \otimes S$ where the first V is right translation, and the second V is left translation.*

Look at a single summand,

$$V^* \otimes V \otimes S.$$

Then

Definition 12.3. $D_\sigma = i\gamma^a R_a + \frac{i}{3}\gamma^a \sigma_a \in End(V \otimes S)$ where $\gamma^a = \gamma(e^a)$, $R_a = R(e_a)$ and $\sigma_a = \sigma(e_a)$.

Note this is not the Levi-Civita Dirac operator! However, its square is a generalized Laplacian.

Now, $D(V) : \mathfrak{g}^* \to End(V \otimes S)$ and $D_\mu = D_\sigma + \mu_a \gamma^a$ where $\mu = \mu_a e^a$.

There are some properties of D:

(1) D_μ is odd, skew-adjoint, if S admits a commuting $\mathbb{C}l(1)$ action, D_μ commutes with this.
(2) $D(V)$ is G^σ-equivariant.

Proposition 12.4. *$ker(D)$ is supported on a coadjoint orbit in $\mathfrak{g}^*$ if and only if $D(V) \in {}^\sigma K_G^{dim\ G}(\mathfrak{g}^*)_{cpt}$ for*

$$\sigma = (\mathfrak{g}^*//G \xrightarrow{id} \mathfrak{g}^*//G, G^\sigma, \epsilon^a).$$

To do calculations, fix for μ a maximal torus T_μ such that $\mu \in \mathfrak{t}_\mu^*$ and a Weyl chamber such that μ is antidominant.

Proposition 12.5. *If V is irreducible of lowest weight λ then $ker(D)$ is $V_{-\lambda} \otimes S_{-\rho}$ when $\mu = -\lambda - \rho$ and 0 otherwise.*

$\rho = \frac{1}{2}\sum_{a\in} a$ is the lowest weight of S as a G^σ-representation.

To prove the above,

(1) introduce energy operator E_μ which acts as multiplication by a constant on weight spaces, with lowest value

$$V_{-\lambda} \otimes S_{-\rho}.$$

(2) $D_\mu^2 + 2E_\mu$ is multiplication by a constant, which implies D_μ^2 is multiplication by a constant and attains its maximum on $V_{-\lambda} \otimes S_{-\rho}$.
(3) $D_\mu = -\gamma(\mu+\lambda+\rho)$ on $V_{-\lambda} \oplus S_{-\rho}$ if and only if ($D_\mu^2 = 0 \Leftrightarrow \mu = -\lambda-\rho$).

From this it follows that $ker(D(V))$ is supported on coadjoint orbit of $-\lambda-\rho$. So we've constructed a map from irreducible representations to coadjoint orbits.

Working from the K-theory side,

$$R(G) = K^0_G(pt) \xrightarrow{D} {}^\sigma K_G^{dim\ G}(\mathfrak{g}^*)_{cpt}$$

where the map is induced by $[V] \mapsto (V \otimes S, 1 \otimes cl)$.

Now for a coadjoint orbit Ω in the case that G is connection (for disconnected G need another twist), $[\tilde{ker}] = [V] \in K_G^{\sigma(\Omega)+dim\Omega}(\Omega)$ and this maps to $[D(V)]$ DIDN'T GET A CHANCE TO GET THE DIAGRAM HERE (JESSE - I MISSED THIS TOO).

Now we wish to do this for loop groups. The major differences are:

(1) Positive energy representations are projective, i.e. a representation of LG^τ. So we use $\mathcal{A}$ instead of $(L\mathfrak{g}^*)$ as an indexing space, which is equivalent to choosing linear splittings of $L\mathfrak{g}^\tau \to L\mathfrak{g}$.
(2) Spinors become a bit harder (this is where the polarization comes in)
(3) Need to do some analysis: work on a dense subspace of finite energy loops, the fourier components described earlier today.

Let τ be an admissable extension (LG^τ extends to $\tilde{LG}^\tau$). And let V be a positive energy representation of level $\tau - \sigma$. Then S is an infinite dimensional spinor representation of $\mathbb{C}l(L\mathfrak{g}^*)$.

To fix notation, let $e_a z^n$ be a fourier basis in $L\mathfrak{g}$, $\gamma^a(n)$ Clifford multiplication of $e^a z^n \in L\mathfrak{g}^*$ on S, $\sigma_a(n)$ the spinor action of $e_a z^n \otimes L\mathfrak{g}$ on S, $(R_a(n))_A$ is the action of $e_a z^n$ on V (which will depend on the splitting).

Remark: S is like the Fock space constructed from the polarization (see Pressley and Segal, Chapter 12).

Definition 12.6. $D(V) : \mathcal{A} \to End(V \otimes S)_{fin})$, $D_A = i\gamma(-n)(R_a(n))_A + \frac{i}{2}\gamma(\Omega)_A$.

If we fix the trivial connection A_0, $\mathcal{A}$ with $L\mathfrak{g}^*$

$$D_\mu = i\gamma^a(-n)R_a(n) + \frac{i}{3}\gamma^a(a)\sigma_a(n) + \gamma(\mu)$$

for $\mu \in L\mathfrak{g}^*$, which is superficially like the Dirac operator we had before, only now it is indexed by $n \in \mathbb{Z}$.

Unfortunately, this operator is unbounded. But we fix this.

Theorem 12.7. *$D(V) : \mathcal{A} \to Fred(V \otimes S)$, $A \mapsto D_A(1 - D_A^2)^{-1/2}$ gives an element of ${}^\tau K_G^{dim\ G}(\mathcal{A} \cong {}^\tau K_G^{dim\ G}(G_1)$.*

Here

$$\tau = (\mathcal{A}//G \to^{hol} G//G, LG^\tau, \epsilon^+)$$

and $D : R^{\tau-\sigma}(LG) \to^{\cong} {}^\tau K_G^{dim(G)}(G_e)$ is an isom.

Now we can do this relatively easily LT, T a torus with cocharacter latice $\Pi \cong \pi_1(T)$ and Π^* the character lattice.

$$LT \cong \Gamma \times exp(V)$$

where

$$\Gamma = \{\gamma \in LT | d(\gamma\gamma^{-1}|_t \text{ is constant}\} \cong \Pi \times T$$

and V s $\mathfrak{t}^\perp \subset L\mathfrak{t}$.

Admissable central extensions of LT are $LT^\tau = \Gamma^\tau \times (exp(V))^\tau$ so $\tau : \Pi \to \Pi^*$ and

$$\tau \in H^1(T) \times H^3_T(T) \cong H^1(T) \otimes H^1(T) \cong \Pi \otimes \Pi.$$

Now $(exp(V))^\tau$ has a single irreducible representation, the Fock space representation $\mathbb{F}$. On the other hand, Γ^τ has irreducible representations

$$V_{[\lambda]} = \oplus_{\lambda \in [\lambda]} V_\lambda,$$

where

$$[\lambda] \in \Pi^*/\tau(\Pi)$$

So irreducible positive energy representations of LT are level τ are all of the form $V_{[\lambda]} \otimes \mathbb{F}$.

Now $\tau : \Pi \to \Pi^*$ extends to a linear map, $\tilde{\tau} : \mathfrak{t} \to \mathfrak{t}^*$, so we can take $\tilde{\tau}^* D$ of the map $D : \mathfrak{t} \to C^\infty(T) \otimes S$ we defined earlier.

Proposition 12.8. *$\tau^* D$ restricts on $V_{[\lambda]} \otimes S$ and then is a Γ^τ-equivariant family of odd skew-adjoint Fredholm operators with commuting $\mathbb{C}l(1)$-action if dim T is odd. So*

$$D(V)_{\tau+?} \in {}^\tau K^{dimT}_{T^\tau}(\mathfrak{t}) \cong {}^\tau K^{dim\ T}_T(T)$$

and

$$\tau = (\mathfrak{t}//T \to T//T, \Gamma^\tau, \epsilon).$$

Theorem 12.9. *${}^\tau R(LT) \cong R(\Gamma^\tau) \cong {}^\tau K^{dim\ T}_T(T)$, and in particular*

$${}^\tau K^{dim\ T}_T(T) \cong Hom_{W^e_{aff}}(\Lambda^\tau, H^n_c(\mathfrak{t}) \otimes \mathbb{Z}[\epsilon]$$

Λ^τ is a Π^*-torsor, and there is a bijection

$$\Lambda^\tau / W^e_{aff} \cong \Pi^*/\tau(H).$$

13. 2-Tier Field Theory and the Verlinde Algebra, AJ Tolland, SUNY Stony Brook

I'd like to explain how the Verlinde algebra first is a Frobenius algebra, and show how it really is an extended field theory too. Then we'll try to see the connection to K-theory.

First, we need a bunch of notation and preliminaries. Afterwards we'll get to the fun stuff.

So let G be a simple 1-connected compact group of dimension d (as usual, for concreteness think $SU(2)$). Let $G_\mathbb{C}$ be its complexification. Let $\ell \in H^4_G(pt, \mathbb{Z})$ be a *level*. As we've seen this week, this group has many different incarnations:

$$\mathbb{Z} \cong H^4_G(pt, \mathbb{Z}) \cong H^3(LBG, \mathbb{Z}) \cong H^3(BLG; \mathbb{Z}) \cong H^3_{LG}(pt, \mathbb{Z}) \cong H^2_{LG}(pt; U(1)),$$

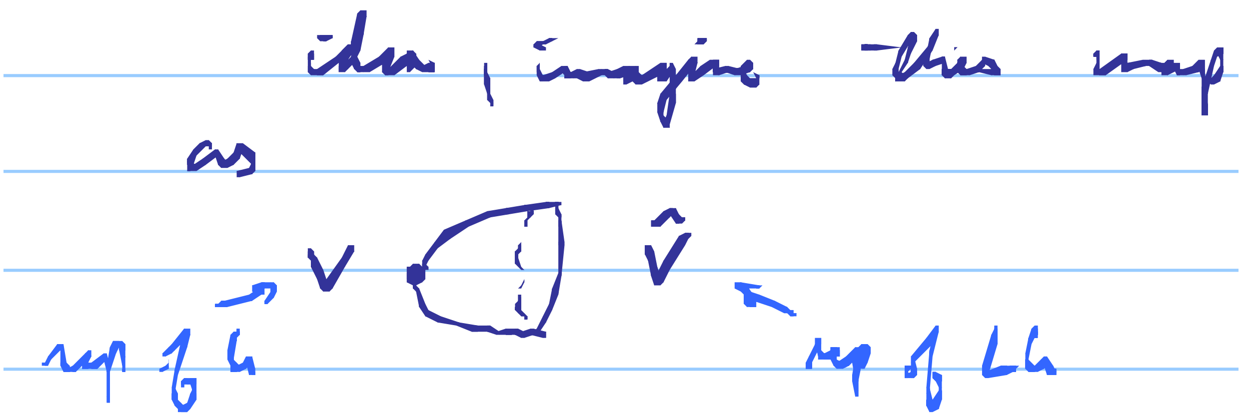


FIGURE 7. Picture of holomorphic induction.

and we can interpret the last guy as an ad-invariant symmetric bilinear form. A level ℓ is positive, written $\ell \geq 0$, if the bilinear form is.

The map $\ell : T \to \tilde{T}$ gives us a trace on the representation ring of G, denoted

$$\theta_\ell : R(G) \to \mathbb{Z},$$

as follows. Let $F = Ker(d\ell)$ thought of as living in $\mathfrak{t}$, the Lie algebra of the maximal torus of G. Then we may define

$$\theta_\ell(V) = \frac{1}{|F|} \sum_{f \in F^{reg}/W} |\Delta(f)|^2 \chi_V(f),$$

where $\Delta(f)$ is the Weyl denominator and $\chi_V(f)$ comes from the character formula.

Then our bilinear form looks like

$$< V, W >_\ell = \theta_\ell(V^* \otimes W)$$

The Twisted K-theory side gives us

$${}^\ell K^d_G(G) = R(G)/ker(< \ , \ >).$$

The Verlinde algebra has been sadly neglected so far. We're going to try to rectify this now. So denote it by

$$R^\ell(LG) = Gr(PER),$$

the Groethendieck group of positive energy representations. This is related to the story of holomorphic induction, which is a map

$$R(G) \to R^\ell(LG).$$

If you're trying to remember what this map actually does, think of a disk with a marked point with a representation V of G attached to the marked point an a representation $\hat{V}$ attached to the boundary of the disk, so we get a map $V \mapsto \hat{V}$; see the picture.

Now define the affine flag variety (or the Grassmannian, if you like) X_D associated to the disk D:

$$X_D := \frac{G_{\mathbb{C}}(D^\times)}{G_{\mathbb{C}}/D}$$

If we were to choose a coordinate, the numerator would be Laurant series in z and the denominator would be power series in z. Recall from Harold's talk that there is a central extension $\hat{LG}$ of LG, which comes from a line bundle $\mathcal{L}$ on $G_{\mathbb{C}}(D^\times)$, which in turn gives a line bundle $\mathcal{L}$ on X_D.

We have an action of $G_{\mathbb{C}}(D)$ on $G_{\mathbb{C}}(D^\times)$, but $G_{\mathbb{C}}(D^\times)$ also acts on V, so we can get an associated bundle $\mathcal{V}$ on X_D.

This ends the preliminaries. Now assume $\ell \geq -\tilde{h}$ (which for $SU(2)$ is $\ell \geq -2$). If $V = V_\lambda$ and $\lambda(\theta) \geq -\ell$ (so that we're in the Weyl alcove). Then we take our line bundle from before and consider the global sections of $\mathcal{L}^{\otimes \ell} \otimes \mathcal{V}$:

$$\hat{V} := H^0(X; \mathcal{L}^{\otimes \ell} \otimes \mathcal{V}),$$

is the positive energy representation of level ℓ with $\hat{V}_0 = V_\lambda$. Now it might seem that something goes wrong for negative ℓ, but really all we get it the zero representation, which is fine.

Just as a comment: LG doesn't honestly sit inside of $G_{\mathbb{C}}(D^\times)$. Rather, there are polynomials which are dense in both spaces, and their representation theories agree.

As a special case, consider

$$\hat{H} = H^0(X; \mathcal{L}^{\otimes \ell})$$

is the vacuum representation.

To think about fusion, rather than looking at a punctured disk, we'll think about the pair of pants (see the picture). We're going to have to make some choices, but this will descend to something well-defined on the Verlinde algebra. So as before we label the boundary circles by representations of LG, denote the incoming by $\hat{V}_1$ and $\hat{V}_2$. We want to construct something for the outgoing boundary, $\hat{V}_1 \star \hat{V}_2$, the fusion product. We do this by capping off both the incoming boundaries with marked points labeled by V_1 and V_2. We also cap off the boundary with a disk D, but don't put a marked point on it. Then if Σ is the resulting surface, we consider

$$X_{\Sigma - D} = \frac{G_{\mathbb{C}}(D^\times)}{G_{\mathbb{C}}(\Sigma - D)}.$$

THERE WERE SOME QUESTIONS ABOUT HOW THIS REALLY WORKS.

So we get a line bundle $\mathcal{L}$ on this flag variety $X_{\Sigma-D}$ and bundles $\mathcal{V}_i$ on $X_{\Sigma-D}$.

Definition 13.1. $\hat{V}_1 \star \hat{V}_2 = H^0(X_{\Sigma-D}; \mathcal{L}^{\otimes \ell} \otimes \mathcal{V}_1 \otimes \mathcal{V}_2)$ which descends to a product on $R^\ell(LG)$ that is independent of the choices since the data we chose is continuous and $R^\ell(LG)$ is discrete.

This is of level ℓ since the extension is classified by the line bundle $\mathcal{L}^{\otimes \ell}$.

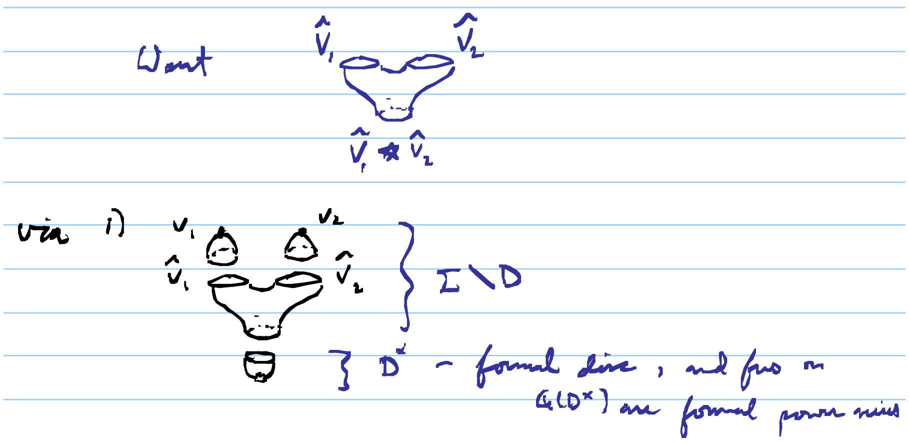


FIGURE 8. The fusion product.

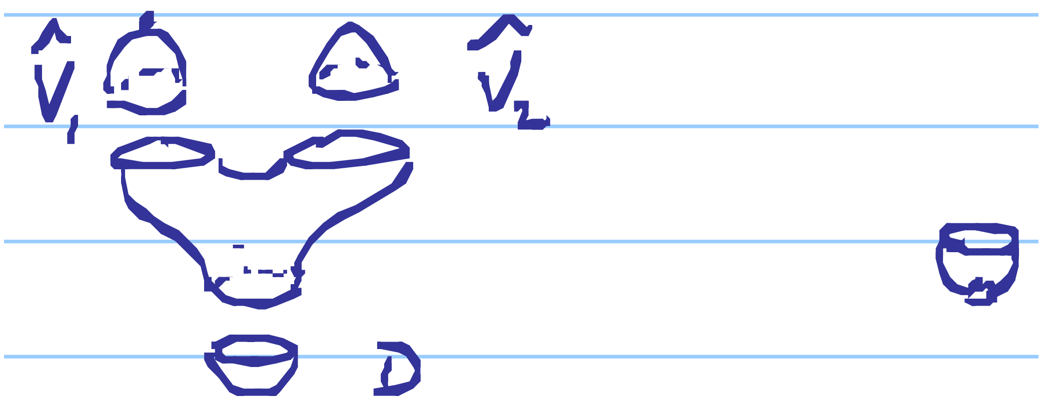


FIGURE 9. Fusion multiplicities.

Theorem 13.2 (FHT).

$$R^\ell(LG) \cong {}^{\ell+\tilde{h}}K_G^d(G)$$

and it is the fusion product in the Verlinde algebra that goes to the Pontryagin product in twisted K-theory.

The Pontryagin product really comes from the multiplication on G. MIGHT BE GOOD TO SPELL THIS OUT A BIT MORE.

AJ MADE A CRYPTIC (AND VERY INTERESTING) REMARK ABOUT HOW THE PRODUCT IS RELATED TO SOME OPERATOR-STATE CORRESPONDENCE BETWEEN THE MARKED POINT ON A DISK AND ITS BOUNDARY.

We want to now see how this is some TQFT in that we're really doing some integral over a space of fields. First we need to think about fusion multiplicities (i.e. the multiplicities of irreps in the fusion product). So consider a pair of pants with incoming labeled by $\hat{V}_1$ and $\hat{V}_2$ and outgoing labeled by the fusion product, which the funny capping off we had before (see the picture). Then, if $\hat{V}$ is an irrep, by definition

$$mult_{\hat{V}}(\hat{V}_1 \star \hat{V}_2) = H^0_{G_{\mathbb{C}}(D)}(\hat{V}_1 \otimes \hat{V}_2 \otimes \hat{V}^*).$$

and then we compute

$$H^0_{G_{\mathbb{C}}(D)}(\hat{V}_1 \otimes \hat{V}_2 \otimes \hat{V}^*) = H^0_{G_{\mathbb{C}}(D)}(H^0(X_{\Sigma-D}, \mathcal{L}^{\otimes \ell} \otimes \mathcal{V}_1 \otimes \mathcal{V}_2) \otimes \hat{V}^*)$$

and this is the E_2 page of some spectral sequence converging to

$$H^0(X_{\Sigma-D}/G_{\mathbb{C}}(D), \mathcal{L}^{\otimes \ell} \otimes \mathcal{V}_1 \otimes \mathcal{V}_2 \otimes \mathcal{V}^*) = Bun_{G_{\mathbb{C}}}(\Sigma)$$

(We only have to worry about H^0 above because Teleman's thesis says the higher cohomology vanishes).

Now $\Sigma = (\Sigma - pt) \bigsqcup D$, and this gives us something like a clutching description.

Corollary 13.3. *$\chi(\mathcal{L})_{Bund_{G_{\mathbb{C}}(\Sigma)}} = mult_{\hat{H}}(\hat{H} \star \hat{H}) = Z_{R^\ell(LG)}(punctured\ sphere)$, where $\hat{H}$ is the vacuum vector.*

This is the tip of the iceberg! We can generalize this wildly by decorating things with complex structures. Also, since we have a TQFT above, we can start throwing in surfaces of higher genus.

Now after giving the Verlinde algebra it's due attention, let's switch back to K-theory. Everything we're going to do should be traced back through the FHT isomorphism.

Theorem 13.4 (Atiyah-Bott)**.**

$$Bund_{G_{\mathbb{C}}}(\Sigma) = \mathcal{A}^G_\Sigma / C^\infty(\Sigma, G)$$

is an isomorphism of stacks.

Now define

$$\mathcal{M}_G(\Sigma, \partial\Sigma) = \mathcal{A}^G_\Sigma / C^\infty((\Sigma, \partial\Sigma), (G, e)),$$

and we have a LG-equivariant projection

$$\pi : \mathcal{M}_G(\Sigma, \partial\Sigma) \to \mathcal{A}_{\partial\Sigma},$$

and

$$\mathcal{A}_{\partial\Sigma}/LG \cong G//G.$$

Now let's talk a bit about twistings.

$$H^4_\sigma(pt) \cong H^3_G(G),$$

and

$$\begin{array}{lcl} S^1 \times LBG & \overset{ev}{\to} & BG \\ \quad p \downarrow & & \\ \quad LBG & & \end{array} \tag{12}$$

and $\ell \in H^4(BG) \to H^3_G(G)$, $\tau = p_* ev^* \ell \in H^3_G(G)$. COULDN'T QUITE COPY ALL THIS IN TIME (JESSE: YOU HAD EVERYTHING I HAVE).

On Σ,

$$\begin{array}{ccc} \mathcal{M}_{G_{\mathbb{C}}}(\Sigma,\partial\Sigma) \cong \Sigma \times Map(\Sigma, BG) & \xrightarrow{ev} & BG \\ p\downarrow & & \\ \mathcal{M}_{G_{\mathbb{C}}(\Sigma,\partial\Sigma)} & & \end{array} \tag{13}$$

Then

$$d\int_\Sigma ev^*\ell = \int_\Sigma d(ev^*\ell) = \int_{\partial\Sigma} ev^*\ell = \tau,$$

and we find

$${}^{\pi^*\tau}K_{LG}(\mathcal{M}_G(\Sigma,\partial\Sigma)) \cong \tau^0 K_{LG}(\mathcal{M}_G(\Sigma,\partial\Sigma)).$$

A difference between two trivializations is a line bundle on $\mathcal{M}_G(\Sigma,\partial\Sigma)$, denoted $\mathcal{O}(\tau)$. Then $\mathcal{O}(\tau) = \mathcal{L}^{\tau-\tilde{h}}$, which we can think of as the first sign of the FHT isomorphism.

We have an index from

$${}^{\pi^*\tau}K_{LG}(\mathcal{M}_G(\Sigma,\partial\Sigma) \xrightarrow{\pi} {}^\tau K_G(G),$$

and

$$index_D(\mathcal{O}(\tau)) = \pi_!({}^\tau 1),$$

and this "Dirac" index agrees with the Doulbealt one

$$index_{\partial}(\mathcal{O}(\tau - \tilde{h})).$$

If we consider a 2-holed torus with genus 1 pieces Σ_1 and Σ_2 and want the index of the whole thing... MAYBE A CAREFUL EXPLAINATION HERE?

Remark: Now, if G is not simply connected, and you happen to be working with twistings that are transgressed you still get a Verlinde algebra fusion product, but in general this may not be.

14. Survey 2: Known and unfinished business, Constantin

Let's take stock of where we are and highlight a few things.

The following two things are closely related:

(1) Character Formulas: Kac-Weyl and Kirillov (connected by Fourier transform)
(2) Chern character and Dirac Family give computational constructions of ${}^\tau K_G(G)$ (Chern does this after tensoring with $\mathbb{C}$).

The Fourier transform is a shadow of a higher structure connection Chern character and Dirac family, a structure called modularity, as Kac-Weyl is related to Chern and Kirillov is connected to Dirac family.

Now modularity is related to 3-d TQFT via an S-transformation in Chern-Simons field theory, and is also related to Elliptic cohomology. However, these two are somewhat disjoint as elliptic cohomology needs a circle action and Chern-Simons only makes sense if we forget the circle action.

So let's review all the ingredients in this picture.

So Let G be a compact group and $T \subset G$ and $W = N_G(T)/T$. For example, $G = U(n)$ T is diagonal matrices and $W = S_n$. Then

$$\rho = \frac{1}{2}\sum_{\alpha>0}\alpha$$

which in the semisimple case is $\sum w_i$. Now for (anti)-dominant λ and V_λ a representation of heighest weight λ,

$$\chi_\lambda = Tr_{V_\lambda}|_T = \frac{\sum_{w \in W} e^{w(\lambda+\rho)}}{e^\rho \Pi_{\alpha>0}(1-e^{-\alpha}},$$

Let Δ be the denominator above, called the Weyl denominator; it is a character of the spin representation $S(\mathfrak{g}/\mathfrak{t})$ which as a graded space is $det^{-1/2} n \otimes \Lambda^*(\Pi^*)$.

Now for loop groups $\gamma \in LG$ and H a PER, we might want to look at $Tr_H(\gamma)$, but this need not be trace class. We can use $E = -i\frac{d}{dt}$ acts on H with positive spectrum, as some kind of damping term. It turns out that for any $q \in \mathbb{C}$ with $|q| < 1$, $q^E\gamma$ is trace class on H, and $Tr(q^E e^\gamma)$ is holomorphic in q. Then we have for a fixed level ℓ

$$\chi_\lambda = \frac{\sum_{\mu \in W_{aff}^{c+\ell}(\lambda+\rho)} \epsilon(\mu) e^\mu q^{(||\mu||^2 - ||\lambda+\rho||^2)/2}}{\Delta(L\mathfrak{g}/\mathfrak{t})}$$

where

$$W_{aff} = W \ltimes \pi_1(T),$$

and $\ell + c$ defines a map $\pi_1 T \to \pi_1(T)^*$ (the weight) and Δ is the character of the spin representation of $L\mathfrak{g}/\mathfrak{t}$. THERE WAS A THOM CLASS COMMENT HERE I DIDN'T CATCH THAT HAD TO DO WITH $\mathfrak{t}$ OR SOMETHING (JESSE - ME NEITHER).

As an aside, we have the transgression map:

$$H^4(BG;\mathbb{Z}) \to H^3_G(G),$$

and from

$$Sym^2((\pi_1 T)^*) \to H^1(T) \otimes H^2(BT) \cong (\pi_1 T)^* \otimes (\pi_1 T)^*$$

THERE IS SOME PROBLEM IF OUR LEVEL IS NOT TRANSGRESSED IN DEFINING A CIRCLE ACTION (JESSE - DIDN'T CATCH THIS), i.e. there is no q in the game.

Recall from the Dirac construction

$$D : H_\lambda \otimes S^\pm(L\mathfrak{g})$$

and then ${}^\tau K_G(G)$ classifies the PER of ${}^\tau LG \ltimes Cliff(L\mathfrak{g})$, so we're really classifying PER of spinors. The character we actually see is just the Kac *numerator*. Think: $S(L\mathfrak{g})$ =Kac denominator $\times$ Thom class along T.

So for example for $U(1)$ at level 2, we get

$$\sum u^{2n} q^{n^2} \quad \sum u^{2n+1} q^{n(n+1)}$$

for $u \in U(1)$. Now let's get rid of this circle action and focus on Chern-Simons at $q = 1$.

Say $\pi_1 G$ is torsion free. Then the Kac numerator at $q = 1$ becomes a linear combination of δ-functions supported at

$$\Gamma \hookrightarrow T \overset{d(\ell+c)}{\to} T^*.$$

Proposition 14.1. *This inclusion is a $\mathbb{C}$-linear isomorphism between $\mathbb{C} - Rep^{\ell+c}(LG)$ and the span of such δ functions. Moreover, this is in fact the R_G-linear under the multiplication action of the character of G and the natural action of $R_G = K_G(pt)$ on ${}^\tau K_G(G)$, which is the same as the action of $Rep(G)$ on $Rep^{\ell+c}(LG)$.*

So after a normalization, the Kac numerator diagonalizes the multiplication in the complexified $K_G^{\ell+c}(G)$.

So recall that the twisted global Chern character

$$ {}^\tau K_G(X) \to \Gamma(G_{\mathbb{C}}/G_{\mathbb{C}}; sheaf\ {}^\tau\mathcal{K}) $$

and recall $G_{\mathbb{C}}/G_{\mathbb{C}} = T_\sigma/W$. In the case that $X = G$ and τ nondegenerate the sheaf is a sum of skyscraper sheaves of rank 1 at the regular points in $F/W \hookrightarrow T/W$, where by regular we mean not fixed by an Weyl group element. Now the stake is

$$ {}^\tau H_T^*(T; {}^\tau\mathbb{L}(f)) \overset{\cong}{integrationinT}\ \mathbb{C}. $$

Theorem 14.2. *The Kac numerator agrees with the Chern character image:*

$$ \sum_{f \in F} c_f \delta_f = (f \mapsto c_f). $$

(after an overall normalization).

Recall the ordinary Chern character can be described as

$$ K_G^*(pt) \xrightarrow{ch} H^*(BG; \mathbb{C} = (\mathbb{C}[[\mathfrak{g}]])^G, $$

and given $V \in K_G^*(pt)$, χ_V is a function on the group, and

$$ ch[V] = exp^* \chi_V. $$

Now the loop group version is basically the same, but we have the spinors in the game.

Now let's recall the Kirilov versus the Dirac family construction. Kirillov takes a H_λ and identifies it with a coadjoint orbit $\Omega_\lambda \in \mathcal{A}$. The Dirac family construction takes a $C_\lambda \subset G$ and a K-theory class on it, $\mathcal{L}$-family up to stabilization is the Thom pushforward of $[\mathcal{L}_\lambda]$ along inclusion $C_\lambda \hookrightarrow G$.

Theorem 14.3. *The following match:*

$$ \begin{array}{ccc} G/G & \cong & \mathcal{A}/LG \\ f_\lambda & \mapsto & \Omega_\lambda/LG \end{array} \tag{14} $$

where note $G/G \cong T/W$.

Where is this f_λ? $f_\lambda \in F^* \hookrightarrow T \to T^*$. A word of caution: the map here is the transpose of the map

$$ H^1(T) \otimes H^2(B)) = \pi_1^* \otimes \pi_1^*. $$

FELL A LITTLE BEHIND ABOVE, LOOK OUT FOR MISTAKES.

Proposition 14.4. *The bilinear form $d(\ell + c)$ gives a duality pairing between F and F^*.*

Note that in the interesting case (symmetric levels) $F \cong F^*$.

Theorem 14.5. *The two constructions are related by the Fourier transform. The first generalizes the product and the 2nd sees the integral forms.*

Conjecture 14.1. *The product in the Kirillov or Dirac family picture is given by "Dirac convolution of conjugacy classes."*

Where Dirac convolution is something like $C_\lambda \star C_\mu \xrightarrow{m} G$. There is a baby (or decategorified version) of this. The big version is secretly a 3d-TQFT statement.

If we take

$$\delta_\lambda \star \delta_\mu = \sum N^\nu_{\lambda\mu} \delta_\nu \star \delta_{\rho/\ell+c}$$

where the N are structure constants in the Verlinde algebra and the $\delta_{\rho/\ell+c}$ plays the role of a unit.

Now for transgressed levels, $F = F^*$, so the Fourier transform happens on the same space. Then

$$S : {}^\tau K_G(G) \otimes \mathbb{C} \to {}^\tau K_G(G) \otimes \mathbb{C}$$

diagonalizes the product. It is known that $K_G(G)$ is the "reduction along ρ" of Chern-Simons theory for G at level ℓ. Take

$${}^\tau K_G(G) \otimes \mathbb{C} = CS(S^1 \times S^1)$$

which is also associated to S^1 in the 2-d theory, and there is an $S \in SL_2(\mathbb{Z})$, $S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Theorem 14.6. $S = S$

Then

$$CS(? \times S^1) = HH_*(CS(?)).$$

Let's make everything precise and let G be a finite group. Then in the 2-d theory, what we assign to a circle is what we assign to a torus in Chern-Simons. So a circle in the 2-d theory gets $K_G(G)$, and the torus in CS theory gets a vector space

$$\Gamma(space\ of\ G - bundles\ on\ T; \mathbb{C})$$

which are functions on G-bundles on $\mathbb{C}$, which is $\mathbb{C}[commuting\ pairs]^G$. Thinking about $K_G(G)$ as G-equivariant vector bundles on G, each conjugacy class in G is a representation of $Z_G(repofg)$ which gives character class function on $Z_G(g)$. Claim: if you repeat the previous story for finite G, you see that s^* corresponds to switching commuting pairs in $\mathbb{C}[commuting\ pairs]^G$, i.e. $(g, h) \mapsto (h, g^{-1})$.

Have said that $K_G(G) \otimes \mathbb{C} = HH_0$((semi-simple) braided tensor category), which is Chern-Simons of a circle. A canidate category here is $Vect_G[G]$ with convolution along G because we want something like

$$S^1 \mapsto G//G,$$

and so we should assign $\Gamma(G//G; Vect)$, which is exactly $Vect_G[G]$.

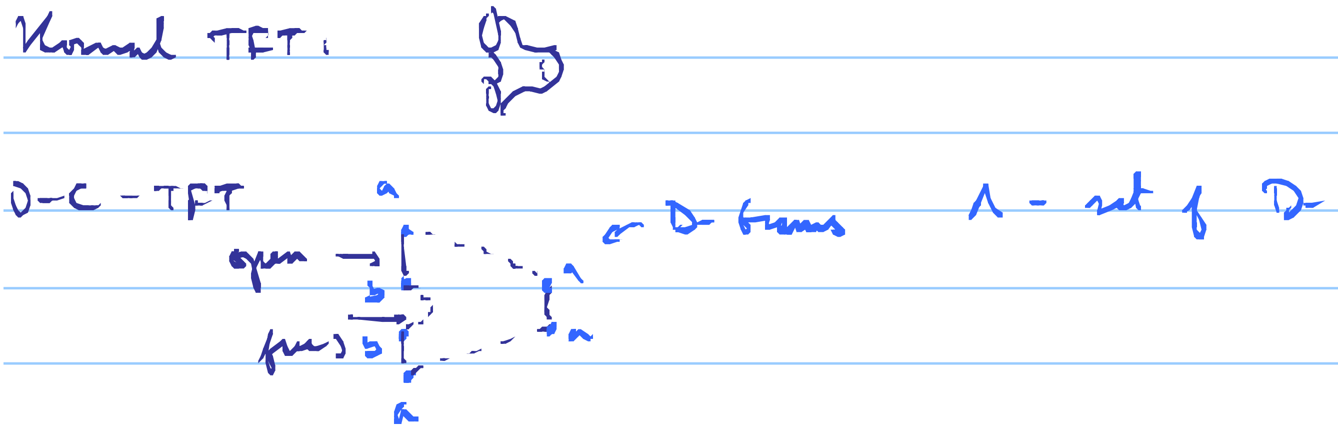


FIGURE 10. Open-Closed field theories.

Proposition 14.7. *We can "see" the braided structure.*

We want $V \star W \cong W \star V$. This is untwisted, as otherwise we would need some braiding.

Now

$$\begin{aligned} V \star W(g) &= \oplus_h V(h) \otimes W(h^{-1}g) \cong \oplus_h V(gh) \otimes W(h^{-1}) \cong \oplus_h W(h^{-1}) \otimes V(gh) \\ &\cong \oplus_h W(h) \otimes V(gh^{-1}) \cong \oplus_h W(h) \otimes V(h^{-1}g) \cong W \star V(g) \end{aligned}$$

where we used equivariance of the conjugation in the second to last isomorphism. This gives a braiding.

15. Open-Closed Field Theories, Matt Young, SUNY Stony Brook

We'll go through open closed things in increasingly complex situations, starting with Atiyah's notion and working up to Costello's theorem about open-closed TCFTs. Everything here will be a 2-1 theory.

An open-closed TFT is roughly about cobordisms between manifolds with boundary. What this means practically is that we have two types of boundaries to our bordisms: open boundaries (which are intervals) and free boundaries. This is coming from string theory where the "usual" notion of field theory and cobordism is for closed strings and open-closed stuff should describe open strings. Now, if you take the physics seriously, open strings should be attached to some D-brane, a set denoted by Λ.

So let's define a category M_Λ whose objects are closed circles or intervals and morphisms are as usual, with free boundaries labeled by D-branes. This is easily seen to be a symmetric monoidal category. So an open closed TFT is a symmetric monoidal functor Z from this category to $Vect$:

$$Z \in Fun^{\otimes}(M_\Lambda, Vect_{\mathbb{C}}).$$

Then $Z(S^1) = H$ is a commutiative Frobenius algebra. Let $Z(I_{ab}) = \mathcal{O}_{ab}$ where a, b are labels for D-branes. Then

$$\mathcal{O}_{ab} \otimes \mathcal{O}_{ba} \to \mathcal{O}_{aa} \to \mathbb{C}$$

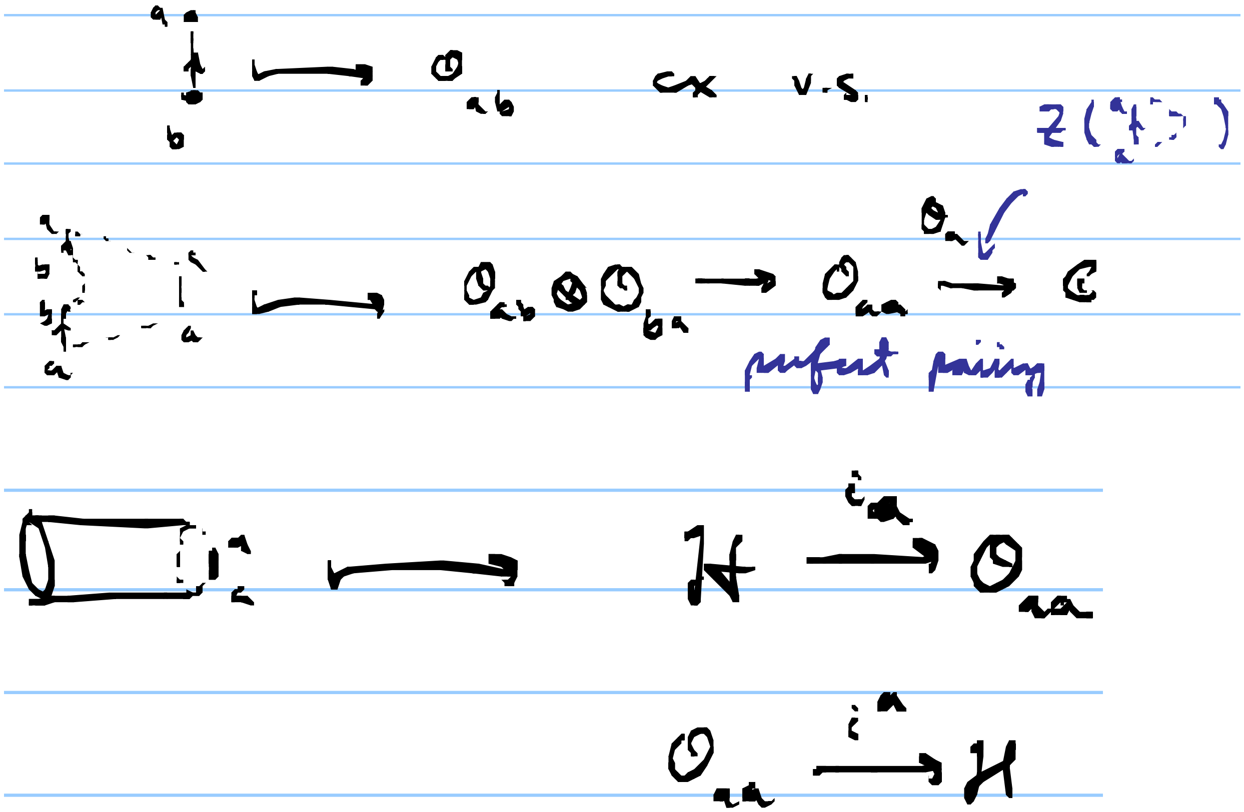


FIGURE 11. Data of an open-closed field theory.

is a perfect pairing, so $\mathcal{O}_{aa}$ is also a Frobenius algebra, but it need not be commutative!

A whistle is an example of a morphism from an open string to a closed one. Notice that we need some free boundary to make this possible. Then

$$Z(whistle) : H \xrightarrow{i_a} \mathcal{O}_{aa},$$

and we also have the dual, i^a. The claim is that i_a is central. We also have the Cardy condition, which we will explain now. Let

$$\pi^a_b = i_b \circ i^a$$

So fix a basis ψ_μ of $\mathcal{O}_{ab}$ and ψ^μ for the dual, $\mathcal{O}_{ba}$. Then

$$\pi^a_b(\psi) = \sum \psi_\mu \psi^\mu,$$

see the picture.

Theorem 15.1 (Sewing)**.** *This data is the same as an open-closed TFT.*

There is a D-brane category B, whose objects are element of Λ and

$$Hom_B(a, b) = \mathcal{O}_{ab}.$$

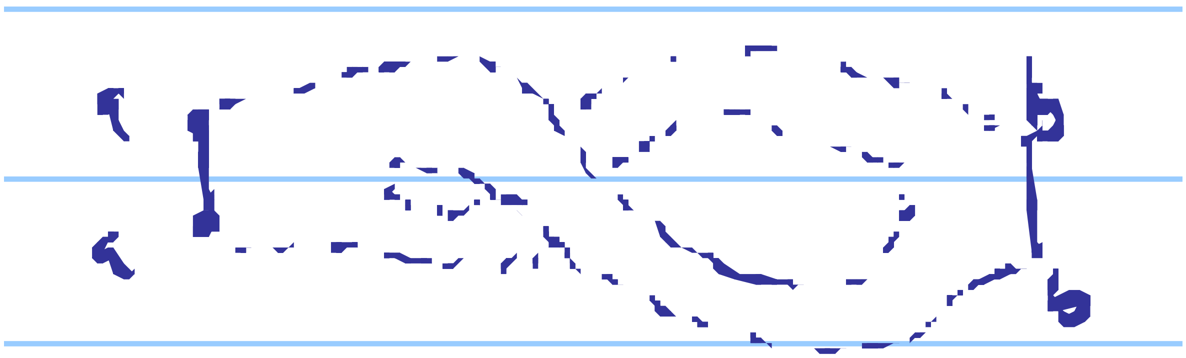


FIGURE 12. Cardy condition.

Corollary 15.2. *B is a CY category, which just means there is a non-degenerate pairing coming from the Frobenius pairing:*

$$Hom(a, b) \otimes Hom(b, a) \to \mathbb{C}.$$

(Maybe this should also be called a Frobenius category.)

We can think of this as the usual theorem involving commutative Frobenius algebras. So, can we classify/construct these things? For example, is there a way of getting an open-closed field theory from a closed one? A good reference for this is Moore-Segal or Lazarvin (the latter being the physics perspective).

Theorem 15.3. *Given H a semisimple commutative Frobenius algebra, we can reconstruct $B \simeq Vect(X)$ where $X = Spec(H)$, which is unique up to tensoring with a line bundle on X. We can also recover the Frobenius structure on B.*

Conversely, given a semisimple CY category B (it's hom-spaces are semisimple), then we can reconstruct H as the ring of endomorphisms,

$$H = End_B(id).$$

In the above, if B is not semisimple, we may run into issues with defining a trace, giving uniqueness problems. Notice when H is semisimple, $spec(H)$ is a finite space. If it weren't finite we would have some twistings, which is more interesting but also more difficult. I THINK THERE ARE SOME ISSUES HERE TO BE DISCUSSED.

Then given $\xi \in H$, $\xi_a = L_a(\xi) \in \mathcal{O}_{aa}$ gives

$$\xi_a \circ \eta = \eta \circ \xi_b,$$

for $\eta \in \mathcal{O}_{aa}$.

Now let's look at some examples. Let G be a finite group, and

$$B = Rep_{\mathbb{C}}(G).$$

Now if Σ is a closed $2-manifold$, then

$$Z(\Sigma)=\sum_{P\to\Sigma/\sim}\frac{1}{|Aut(P)|}$$

where $P\to\Sigma$ is a principle G-bundle. We take

$$H=\mathbb{C}[G]^G.$$

Then

$$Z(\Sigma,\partial\Sigma)=\sum_{\substack{bundles\ with\\ prescribed\ holonomy}}\frac{1}{|Aut(P)|}.$$

Also let

$$\mathcal{O}_{AA}=End_G A.$$

Consider the pair of pants where the boundaries are free boundaries and we've shrunk the open strings to points.Then

$$Z(\Sigma, decorations)=\sum_{P\to\Sigma/\sim}\frac{\Pi\chi_A(hol_P(C_i))}{|Aut(P)|}.$$

Now let's look at open-closed TCFTs. To do this, we replace $Vect_{\mathbb{C}}$ by $Chain_{\mathbb{C}}$. This is supposed to record something about ghost numbers and BRST differentials.

Now,

$$\mathcal{OC}_\Lambda := C_\bullet(\mathcal{M}_\Lambda^{conf})$$

Definition 15.4. An open-closed TCFT is an h-split symmetric monoidal functor

$$Z:\mathcal{OC}_\Lambda\to Chain_{\mathbb{C}}$$

where

$$\phi(a\otimes b)\mapsto\phi(a)\otimes\phi(b).$$

There are restrictions of these to functor to $\mathcal{O}_\Lambda$ and $\mathcal{C}$. It turns out by pullback, we also get field theories from these restrictions (the inclusions of the categories make things h-split still) but there is no obvious way to pushforward open field theries, unlike the topological case.

Theorem 15.5 (Costello)**.**
(1) *The category of open-closed TCFTs is equivalent to the category of unital extended CY A_∞-categories.*
(2) *Also, there is a way to make the pushforward above exact so that we get a functor*

$$\text{Open } TCFT\to\text{Open}-\text{Closed } TCFT.$$

where the map is given by

$$Z\mapsto j^*\mathbb{L}i_!Z.$$

(3) *Finally*

$$HH_*(Z)=H_*(j^*\mathbb{L}i_!Z)$$

where i is the inclusion of open stuff into open-closed.

The unital part comes from a strip that gives a map $\mathcal{O}_{aa} \to \mathcal{O}_{aa}$. THERE WAS SOME CONFUSION ABOUT HOW THIS MIGHT ALSO BE THE DIFFERENTIAL SINCE NOW WE ARE IN CHAIN COMPLEXES, NOT VECTOR SPACES.

A is an A_∞ algebra if

$$A = \oplus_{p\in\mathbb{Z}} A^p$$

and degree $2d$ maps $m_d : A^{\otimes d} \to A$, where m_1 is a differential, m_2 measure how much if fails to satisfy Leibniz, m_3 measures how much it fails to be associate, etc.

Definition 15.6. $\mathcal{A}$ is an A_∞-category if

$$m_d : Hom_{\mathcal{A}}(A_0, A_1)\otimes Hom_{\mathcal{A}}(A_1, A_2)\otimes\cdots\otimes Hom_{\mathcal{A}}(A_{n-1}, A_n) \to Hom_{\mathcal{A}}(A_0, A_n)$$

satisfying the analogous properties to an A_∞-algebra.

For example and A_∞ category with a single object is just an A_∞-algebra.

Definition 15.7. A CY A_∞-category has a pairing

$$< , >_{A,B}: Hom(A, B) \otimes Hom(B, A) \to \mathbb{C}$$

and cyclic symmetric, i.e. $< m_d(\varphi_0, \dots \varphi_{d-1}), varphi_d >= \pm < \varphi_1, \dots, \varphi_d), \varphi_0 >$.

So in particular, the CY category in the Costello theorem is the category of D-branes.

Now we need to talk about $HH_\bullet$ of an A_∞-category.

Recall the cobar construction for an associative $\mathbb{C}$-algebra. We can do this exactly the same way for a dg-category. To compare with Moore-Segal, we just need to check that HH_* is just the endomorphisms of the identity functor. We can choose a quasi-isomorphic dg-category to our A_∞-category, or there is some A_∞-category construction to calculate HH_*.

A motivation for this increase in abstraction is to understand string topology. IT WAS UNCLEAR FROM THE DISCUSSION HOW CLOSE WE ARE. THERE WERE SOME INTERESTING AND INTRICATE COMMENTS HERE. JACOB MAY OR MAY NOT HAVE CONSTRUCTED THIS, CLAIMS SOMETHING LIKE IT IN HIS SURVEY?

Let's talk about the B-model as an example. Take X to be a compact Calabi-Yau. Define $Perf(X)$ or $P(X)$ as having objects complexes of holomorphic vector bundles on X and morphisms

$$Hom_{P(X)}(E, F) = \Omega^{0,\bullet}(E^* \otimes F)$$

This is naturally a dg-category and you can get a nondegenerate pairing on the homology of this category.

Theorem 15.8. *For A and A_∞-algebra $H_\bullet(A)$ can be made into A_∞ such that $m_1 = 0$, $m_2 = m_2^A$, with a map,*

$$H_\bullet(A) \to A.$$

So then we use the theorem to get a CY A_∞ structure on

$$H_\bullet(P(X)) = D^b_\infty(X)$$

So then we take this, apply Costello's theorem to get an open-closed TCFT, and then use the other part of the theorem to a closed TCFT. This is the B-model, though it is still unclear exactly how this relates to the physicists B-model.

16. Landau-Ginzburg B-Models, Kevin Lin, UC Berkeley

We'll start with some rambling on homological mirror symmetry for compact Calabi-Yau (after Kontsevich '94). Recall that a manifold is CY if it is compact Kähler with trivial canonical bundle, or equivalently if it has vanishing $c_1(TX)$ (this equivalence being Yau's theorem). So if X is a compact CY, there is a mirror Y, which is also a compact CY, satisfying some relations:

$$D^b Coh(X) \cong D^\pi Fuk(Y), \quad D^b Fuk(X) \cong D^b Coh(X)$$

relating the derived category of coherent sheaves on one manifold ("B-Model") with the Fukaya category on the other ("A-Model").

We can extend mirror symmetry beyond the CY case. For example, Let X be Fano (compact Kähler with ample anticanonical bundle). The mirror model Landau-Ginzburg model is a noncompact Kähler manifold Y with a holomorphic function $W : Y \to \mathbb{C}$ called the superpotential.

Now the A-model on X is $D^\pi Fuk(X)$ and the B-Model on Y is $D^b Coh(X)$. The B-model on (Y, W) is $D^b Sing(W)$, $MF(W)$ (the latter being "matrix factorization"), and the A-model on (Y, W) is the Fukaya-Seidel category. And there are relations among these models.

There is something called the C-Y/L-G correspondence which says that

$$D^b Coh(X) \cong MF(W)$$

Let X be a variety over $\mathbb{C}$ (not necessarily smooth or compact, and more generally X can be a nice scheme). Now

$$D^b(X) = D^b(Coh(X)) = \text{localization of } K^b(X) \text{ at quasi} - \text{isomorphisms}$$

where $K^b(X)$ has as objects bounded complexes of coherent sheaves on X and morphisms are morphisms of complexes up to homotopy. Then the universal property of localization says that for every functor $K^b(X) \to \mathcal{C}$ that preserves quasi-isomorphisms, it factors uniquely through $D^b(X)$.

Now inside of $D^b(X)$ we have $Perf(X)$ which is the full subcategory of objects quasi-isomorphic to locally free sheaves, i.e. vector bundles. So we think of it as just complexes of vector bundles. Then we take

$$D^b Sing(X) := D^b(X)/Perf(X)$$

that has universal property that any functor from $D^b(X) \to \mathcal{C}$ that sends $D^b Sing(X)$ to stuff quasi-isomorphic to zero factors uniquely through $D^b Sing(X)$.

Proposition 16.1. *If X is nonsingular, then*

$$D^b Sing(X) = 0$$

This is just some result from algebraic geometry saying that coherent sheaves can be resolved by locally free ones.

So now let's think about the L-G model, $(Y, W : Y \to \mathbb{C})$.

Definition 16.2. The cateogry of B-branes in the L-G model are

$$\Pi_{\lambda \in critical\ values\ of\ W} D^b Sing(Y_\lambda)$$

where $Y_\lambda = W^{-1}(\lambda)$.

Definition 16.3. The category of B-Branes for matrix factorization models is

$$MF(W) = \Pi_{\lambda \in Crit(W)} MF_\lambda(w).$$

Now assume Y an affine variety, $Y = Spec(A)$, as this will make $MF_\lambda(W)$ easier to define.

So MF_λ has as objects P which consists of P_0, P_1, finitely generated projective A-modules with maps $d_P : P_0 \to P_1$, $d_P : P_1 \to P_0$, where $d_P \circ d_P = (W - \lambda)$ which we think of as "curvature;" and MF_λ has as morphisms for $P = (P_0, P_1, d_P)$, $Q = (Q_0, Q_1, d_P)$:

$$\oplus_{i,j} Hom_A(P_i, Q_j)$$

with obvious $\mathbb{Z}/2$ grading and differential

$$d\phi = d_q \circ \phi - (-1)^{|\phi|} \phi \circ d_P.$$

So this is a differential $\mathbb{Z}/2$-graded category.

There is an obvious triangulated structure on both of these categories, and an equivalence of triangulated categories:

$$MF_\lambda W \to D^b Sing(Y_\lambda)$$

where

$$(P_0, P_1, d_P) \mapsto (coker(P_0 \to P_1))|_{Y_\lambda}.$$

Theorem 16.4 (Orlov)**.** *This is an exact equivalence of triangulated categories.*

THERE WAS SOME CONFUSION ABOUT WHETHER THE SINGULARITIES SHOULD BE ISOLATED OR NOT. FOR NOW WE'LL ASSUME THEY'RE ISOLATED. CONSTANTINE SAYS THE RESULT IS KNOWN FOR NONISOLATED, BUT NONE OF US KNEW IT.

For example, let's look at the mirror L-G model to $\mathbb{CP}^1$. Then

$$W : \mathbb{C}^\times \to \mathbb{C}.$$

Now, $\mathbb{C}^\times = Spec(\mathbb{C}[x^{\pm 1}]$, and we define

$$W = x + \frac{1}{x},$$

and so to find the critical points of W,

$$\partial W / \partial x = 1 - \frac{1}{x^2} = 0$$

so we get $x = \pm 1$, and thus the critical values are $\lambda = \pm 2$. Then

$$W - 2 = x + \frac{1}{x} - 2 = \frac{1}{x}(x^2 + 1 - 2x) = \frac{1}{x}(x-1)^2$$

and so we get maps $\mathbb{C}[x^{\pm 1}] \to \mathbb{C}[x^{\pm 1}]$ given by multiplication by $(x-1)$ and $\frac{1}{x}(x-1)$, so we clearly satisfy the relation for an object of $MF_\lambda(M)$, so we have a matrix factorization.

As an exercise one can check that

$$H^0(Hom_{MF}(P, P) \cong \mathbb{C}[t]/(t^2 - 1)$$

which has complex dimension 2. Notice that

$$H^0(\oplus_{i,j} Hom_A(P_i, Q_j), d) = \frac{\text{chain maps}}{\text{homotopy}}$$

THEN THERE WAS SOMETHING ABOUT GREAT CIRCLES AND MIRROR STUFF I DIDN'T CATCH (JESSE - ME NEITHER).

We can do noncommutative and graded cases. So say B is a nice (possibly noncommutative) graded algebra. Then we can define

$$D^b Sing^{gr}(B) = D^b(gr(B))/D^b(gr\ Proj(B))$$

where $D^b(gr(B))$ is the category of finitely generated graded right modules, and $D^b(gr\ Proj(B))$ is the category of finitely generated projective right modules.

We again want to define $MF_\lambda^{gr}\lambda(W)$, and so say $W \in B_n$, the nth grade of B. Now we say $d_P : P_1 \to P_0$ is a degree n map and W in the center of B.

Theorem 16.5 (Orlov)**.** *Say W is in the center of B. Then*

$$D^B Sing^{gr}(A) \cong MF_0^{gr}(W)$$

where $A = B/WB$

Here the quotient comes about because we're thinking of $Spec(B)$ ... MISSED IT (JESSE - ME TOO).

Theorem 16.6 (Orlov, C-Y/L-G correspondence)**.** *If W is a homogeneous polynomial in $k[x_1, \ldots, x_n]$ then consider*

$$W : \mathbb{A}^n \to \mathbb{A}^1, \quad (\text{or } \mathbb{C}^n \to \mathbb{C}),$$

Then we have an equivalence of categories $MF_?(W) \cong D^b Coh(X)$ where X is the hypersurface defined by W in $\mathbb{P}^{n-1}$.

THE QUESTION MARK MAY OR MAY NOT BE A ZERO. SEEMS LIKELY. (JESSE - I DIDN'T HAVE A SUBSCRIPT ON THE MF(W). I DON'T KNOW IF IT WAS A STRAY MARK ON THE WHITEBOARD OR IF I JUST MISSED IT).

A remark: these matrix factorization should be thought of as curved modules over A.

Let's return to the commutative case and compute the Hochschild homology of this category.

$$C_*(A, W) := \Pi_k A \otimes A^{\otimes k}$$

we have a differential

$$\partial_1 : A \otimes A^{\otimes k} \to A \otimes A^{\otimes k-1}$$

which is the usual bar construction differential. But we have a second differential:

$$\partial_W : A \otimes A^{\otimes k} \to A \otimes A^{\otimes k+1}$$

where we plug in a W in various spots of the tensor product with signs, but only at either the beginning or the end (kind of like thinking of the list as living on a circle and plugging between all gaps, so the beginning and the end are the same). Now let $\partial = \partial_1 + \partial_W$. WE COULDN'T QUITE TELL IF THIS THING SQUARES TO ZERO, BUT IT SHOULD COME DOWN TO SIGNS. We can also define

$$C_\bullet(MF(W)) := \Pi_k \oplus_{\substack{\alpha_0,\ldots,\alpha_k \\ \alpha_i \in Ob}} Hom(\alpha_k, \alpha_0) \otimes \cdots \otimes Hom(\alpha_{k-1}, \alpha_k).$$

Proposition 16.7. *There is a quasi isomorphism*

$$C_*(MF(W)) \to C_*(A, W).$$

Now assume Y is nonsingular.

Then

Proposition 16.8. *There is a quasi isomorphism*

$$C_*(A, W) \to (\Omega^*_Y, \wedge dW).$$

Sketch of Proof.

$$\begin{array}{ccccc} C_3 & \overset{\partial_W}{\leftarrow} & C_2 & \leftarrow & C_1 \\ \partial_1 \downarrow & & \downarrow & & \downarrow \\ C_2 & \overset{\partial_W}{\leftarrow} & C_1 & \leftarrow & C_0 \\ \partial_1 \downarrow & & \downarrow & & \downarrow \\ C_1 & \overset{\partial_W}{\leftarrow} & C_0 & \leftarrow & 0 \\ \downarrow & & \downarrow & & \\ C_0 & \overset{\partial_W}{\leftarrow} & 0 & & \\ \downarrow & & & & \\ 0 & & & & \end{array} \tag{15}$$

and there is another double complex, together with a map $\phi : C_k \to \Omega$ of double complexes,

$$a_0 \otimes \cdots \otimes a_k \mapsto \frac{1}{k!} a_0 da_1 \wedge \cdots \wedge da_k.$$

It is a theorem of Hochschild, Konstant and Rosenburg that this map is a quasi-isomorphism. Then we do some things with spectral sequences of these double complexes to prove our theorem. Say $dim(Y) = n$. Then isolated singularities implies that

$$H^n(\Omega^*, \wedge dW) \cong \Omega^n / dW \wedge \Omega^{n-1}$$

and 0 otherwise. This shows that both spectral sequences collapses in the E_2 plane. Now we'd like to compute

$$coker(\Omega^{n-1} \overset{\wedge dW}{\to} \Omega^n)$$

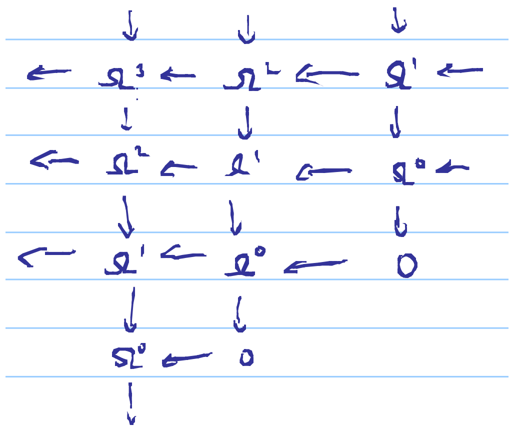


FIGURE 13. The other double complex.

and if we assume Y has a volume form then HH_* is isomorphic to HH^* and we compute HH^* as

$$coker(T \overset{i_{dW}}{\to} \mathcal{O})$$

which when $Y = \mathbb{A}^n$ is

$$\mathcal{O}/(\partial_{x_1} W, \dots, \partial_{x_n} W).$$

The CY structure comes from the pairing on $\Omega^* \times \Omega^* \to \mathbb{C}$,

$$(\omega, \eta) \mapsto \int \frac{\omega \wedge \eta}{\partial_{x_1} W \dots \partial_{x_n} W}.$$

THEN THERE WERE SOME QUANTUM COHOMOLOGY COMMENTS.

17. Twisted $K_G(G)$ as open closed theory, Constantin

Say X is a CY manifold, but not necessarily compact. Then $DCoh(X)$ is not quite a CY category, but we can get a partial TQFT in 2-d (partial in the sense of string topology). One fix is to add a superpotential W with proper critical set. Then we get a CY category and a full B-model TQFT.

So how is this related to equivariant K-theory? Well $K_G(G)$ is a version of $K(LBG)$. Then this is a "string-topology-like" structure from

$$S^1 \mapsto K_G(G)$$

resulting in differentials on $\mathbb{Z}[x_1, \dots, x_\ell]$ (HOW?) (JESSE - I HAVE IN MY NOTES THAT WE GET THIS BECAUSE WE CAN INTEPRET $K_G(G)$ AS KAHLER DIFFERENTIALS ON $\mathbb{Z}[x_1, \dots, x_\ell]$. I'M NOT SURE HOW THIS HELPS OR IF I MISHEARD).

As soon as we turn on a twisting $\tau \in H^3_G(G)$, then we get an honest TQFT from a Frobenius algebra on ${}^\tau K_G(G)$. Now we want to see how such TQFTs are related to the B-model. Some of these "results" have not been precisely proved or even precisely stated! One of the things we'll find is that the L-G model story is a bit of a lie: W shouldn't really be a function, but instead an element of the Brauer group.

Let's back up and talk about the difficulty here. First we need to talk a little about string topology. So say X is a compact, oriented, simply connected manifold (though for string topology we want $X = BG$). Now consider the following algebras:

$$C^*(X), \quad C_*(\Omega), \text{ with convolution.}$$

These algebras are related by Koszul duality (or bar/cobar duality):

$$\mathbb{R}Hom_{C^*X}(k,k) \cong C_*(\Omega X), \quad C^*(X) \cong \mathbb{R}Hom_{C_*\Omega X}(k,k).$$

We have an equivalence of categories:

$$\text{Perfect } C_*\Omega X - \text{Modules} \to k\text{−finite } C^*X\text{−Modules.}$$

and an equivalence of subcategories of these

$$k - \text{finite } C_*\Omega X\text{−Modules} \to \text{Perfect } C^*X - -\text{Modules.}$$

Morally this equivalence comes from taking a fiber at base point and taking the action of holonomy, which is equivalent to choosing a local system:

$$\mathbb{R}Hom_{C^*X}(k, M) \leftarrow M.$$

Theorem 17.1. *The subcategory, $\mathcal{C}$ above is obviously a CY category (using Poincare duality pairing on X). Furthermore*

$$HH_*(\mathcal{C}) \cong HH_*(C^*X) \sim C^*(LX)$$

From this we should get something about string topology, since now we're looking at cochains on loop space.

Theorem 17.2. *The "larger" category, $\mathcal{C}'$ above can also be made into a CY object in linear categories. Furthermore*

$$HH_*(\mathcal{C}') \sim C^*(LX)$$

and

$$HH_*(C_*\Omega X) \cong C_*(LX)$$

Now we want to twist these guys to try to get some string topology stuff, hoping

$$HH_*({}^\tau C_*\Omega X) \cong {}^\tau C_*(LX).$$

So let $\tau \in H^1(\Omega X; GL(h^*)$ for h^* a cohomology theory. Then we can form ${}^\tau C_*\Omega X$. So can we define a Pontryagin product and hence a TQFT?

$$\Omega X \times \Omega X \xrightarrow{m} \Omega X.$$

Need $m^*\tau = \tau \otimes 1 + 1 \otimes \tau$ (so that τ is primitive), and it suffices to get a TQFT if τ is transgressed from X.

Now, if X is a ringed space with a sheaf of rings $U \mapsto h^*(U)$, then we have a map $Br(X; h^*) = H^2(X; GL_1(h^*)) \to H^1(\Omega X; GL_1(h^*))$ so for example (JESSE: I THINK THIS IS WHAT WAS MEANT) $\mathbb{Z}/2 \times K(\mathbb{Z}, 2) \hookrightarrow GL_1(K^*)$ and $H^2(X; K(\mathbb{Z}, 2)) = H^4(X; \mathbb{Z})$.

So candidates for twisted string topology: start with $\beta \in Br(X; h^*)$, and form

$${}^\beta h_*(\Omega X)$$

with its Pontryagin product.

Theorem 17.3. *$HH_*({}^\beta h_*(\Omega X)) = {}^\beta h_*(LX)$ has a string product and a partial TQFT algebra.*

Now let's try this with $K_G(G)$. Now X becomes BG, and the Brauer group is $H^4(BG)$ and $\Omega X = G$. Get "twisted group ring of G over K." Also ${}^\tau K_*(G)$ has a Pontryagin product. We would hope

$$HH_*({}^\tau K_*(G) \cong {}^\tau K_*^G(G),$$

but this isn't true. We get the completion at the augmentation ideal. However, if we had a twisting there would not be a completion problem.

Theorem 17.4 (Brylinski,?)**.** *For G simple, simply connected,*

$$K_G(G) \cong HH_*(K_G(pt))$$

and notice that the right hand side is differentials on $Spec(R_G)$.

So we need to twist $h^*(X)$ by a class $\beta \in H^2(X; Gl_1(h^*))$. So if we have K^*, then we would get stuff in $H^4(G)$ but this is not where twistings come from, though this somehow gets fixed by curved modules.

So what are curved modules over $K^*(X), \beta$? So think by analogy to vector bundles, but now rather than the sheaf of functions we have some sheaf with values in K-theory.

So cover X by opens $\{U\}$ where $\beta = \delta\alpha_U$. Then on $U \bigcap V$ we get

$$\alpha_U - \alpha_V \in C^1(X; Gl_1(K)) = C^3(X; \mathbb{Z}),$$

and $K(U \bigcap V)$ is a module over $K(U)$ and $K(V)$. Then a curved module is a $K * U$-module M_U for each U and an "isomorphism"

$$M_U \otimes_{K(U)} K(U \bigcap V) \cong M_V \otimes_{K(V)} K(U \bigcap V)$$

together with some Cech compatibility conditions.

Conjecture 17.1. *This will be a CY category and give the same TQFT as ${}^\beta K_*(\Omega X)$.*

Also note that Koszul duality is a bit funny here, as it uses a bimodule k which is neither a module over either of the things that act on it, but the failures cancel.

So now let's comment about what happens for BG. This definition for H^4-curved modules over $K_G(pt)$ fails. The completion business doesn't go away. If we tensor with $\mathbb{C}$, this seems to work with a bit of creativity.

$$K_G(pt) \otimes \mathbb{C} = \mathbb{C}[G]^G,$$

class function on $G_{\mathbb{C}}$. Now what's the potential? Well, $\beta \in H^4(BG; \mathbb{Z})$ are quadratic invariant function on the Lie algebra,

$$\beta \in H^4(BG; \mathbb{Z}) \cong (Sym^2 \mathfrak{g})^G.$$

and we get these badly multivalued functions

$$g \mapsto \beta(\log g, \log g).$$

In physics, we're used to multivalued functions because all one cares about are the critical points, that is the derivative. Then we get a map $dW : T \to \mathfrak{t}^*$ given by

$$t \mapsto i_{\log t}\beta.$$

(JESSE - I HAVE $t \mapsto logt\Delta\beta$ HERE).

If we postcompose with the map $\mathfrak{t}^* \to T^*$ we get a well-defined map. The critical points are the kernel of the deriviative F, and locally it has Morse singularities. Then

$$MF(Rep(G); \beta_)) \cong \text{sheaves on } F$$

where $F \hookrightarrow T$ is the critical set. Then we consider F^{reg}/W. The twisted $K_G(G)$ has an analogue with higher twistings (can only define over $\mathbb{C}$) that also gives a TQFT related to the index over moduli space of flat connections of virtual vector bundles of virtual dimension 1. From the Hessian determinants, we get the correct trace for computing the index above.

How do we interpret twistings of string topology geometrically? Take $\beta \in H^4(X)$ and $\beta_L \in H^3(LX)$ and $\beta' \in H^3(\Omega X)$. Assume β pairs nontrivially with $\pi_4 X$. Then β' is not trivial. Then look at ${}^\tau C_*(\Omega X)$ with differential $\partial \beta' \cap$.

If β vanishes on $\pi_4 X$, e.g. is decomposable then if given a quadratic form on $H_2 = \pi_2$, we get an extension of the group ΩX and a deformation of $C_*(\Omega X)$.

We'll try quickly to explain the connection to Chern-Simons. Let G be finite, $\tau \in H^4(BG; \mathbb{Z})$. Chern-Simon gives us a 3-d TQFT/$\mathbb{C}$ in dimensions 3-2-1-0. We dimensionally reduce to 2-d TQFT/$\mathbb{C}$, we assign ${}^\tau K_G(G) \otimes \mathbb{C}$ to the circle. But we have something slightly better here, a 2-d TQFT/$K^*(pt)$, which aspires to be an extended 3-d TQFT, but isn't. This is also related to the completion problem.

Let's spell out CS a bit more. To a point we want to assign a tensor category, to 1-manifolds we want to assign the Drinfeld center (a braided tensor category with ...). So our braided tensor category is ${}^\tau Vect[G]$, vector bundles on G with τ-twisted convolution. For $\tau = 0$, the center ends up being $Vect_G[G]$ that we saw earlier. Before we tried to get ${}^\tau K_G(G)$ from ${}^\tau K_*(G)$, which failed; instead we got the completion.

Turn to Koszul duality: $K_G(G) \to (Vect[G], \star)$, $K_G^*(pt) \to (Rep(G), \otimes)$.

$$Vect[G] = Hom_{(Rep(G),\otimes)}(Vect, Vect)$$

We'd like to get the curved version of this tensor category.

The take-away point is that the 2-d TQFT over $K^*(pt)$ above is some intermediate between Chern-Simons and its dimensionally reduced theory.

18. Chern-Simons as a 3-2-1 Theory, Hiro Tanaka, Northwestern

The goal here is to relate Chern-Simons to the 2-1 theory coming from $K_G(G)$.

In 1989, Ed Witten told a fairytale. The characters were a 3-manifold M, and a simple, simply connected group G (think $SU(2)$, as usual), and a trivial

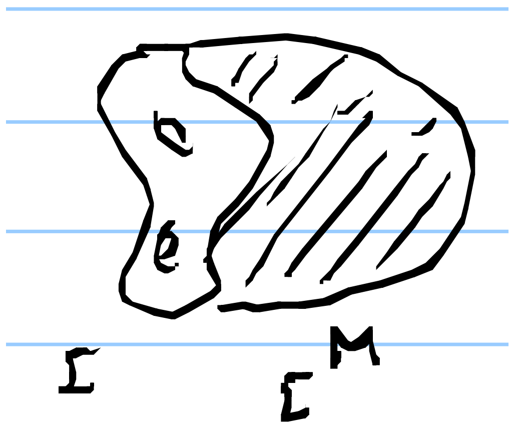


FIGURE 14. Cutting the 3-manifold along a surface Σ.

principle G-bundle over M. Then take a connection $A \in \Omega^1(M; \mathfrak{g})$ and find a number

$$CS(A) = \frac{1}{8\pi^2} \int_M tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A).$$

So far this isn't such a fantastic tale, but now let's do something crazy: let's integrate over the space of all connections

$$Z_k(M) = \int_{\mathcal{A}_M/G^M} e^{2\pi i k CS(A)}$$

where $\mathcal{A}_M$ is the space of connections and G^M denotes $C^\infty(M, G)$, the space of gauge transformations. An element of the gauge group $g \in C^\infty(M, G)$ acts on a connection by

$$g^*A = g^{-1}Ag + g^{-1}dg.$$

Then it is an exercise to show that

$$CS(g^*A) = CS(A) + n, \quad n \in \mathbb{Z}.$$

It is another exercise to show that for $G = SU(2)$, $n = deg(g)$. The reason this is a fairtale is that there is no measure on the space of connections mod gauge transformations. However, it still appears that we can define some 3-dimensional TFT from this data. So let's think heuristically for now. Here is a table that will say what we assign in each dimension for this extended TFT.

$$\begin{array}{cc} 3-\text{manifold } M & \text{partition function} \\ 2-\text{manifold} & ? \\ 1-\text{manifold} & Rep(\tilde{LG}),\ \text{a linear category} \end{array} \tag{16}$$

Now let's figure out what to assign to a surface. So say we cut our 3-manifold along M, leaving a 3-manifold with boundary $_\Sigma M$, Now let $\mathcal{A}_\Sigma$ be connections on $_\Sigma M$ restricted to Σ. Now fix one of these $a \in \mathcal{A}_\Sigma$. Now define

$$\mathcal{A}_a := \{A \in \Omega^1(M; \mathfrak{g}) | A|_\Sigma = a\}$$

Then we can assign to this 2-manifold:

$$Z_k({}_\Sigma M)(a) = \int_{\mathcal{A}_a/ker(G^M \to G^\Sigma)} e^{2\pi ikCS(A)}.$$

As another excercise, show that

$$\forall g \in G^\Sigma, a \in \mathcal{A}_\Sigma,$$

$$Z_k(M)(g^*a) = e^{2\pi ikf(g,a)} Z_k(M_\Sigma)(a)$$

where

$$f : \mathcal{A}_\Sigma \times G^\Sigma \to \mathbb{R}/\mathbb{Z}$$

$$(a, g) \mapsto \frac{1}{8\pi^2} \int_\Sigma tr(g^{-1}ag \wedge g^{-1}dg) - \int_M \tilde{g}^*\sigma$$

where $\tilde{g} \in G^M$ such that $\tilde{g}|_\Sigma = g$, and $\sigma \in H^3(M;\mathbb{Z})$, where $\sigma[M] = n$ in the n above in the indeterminacy of the CS action when we change by a gauge transformation.

The punchline of this is G^Σ acts on $\mathbb{C} \times \mathcal{A}_\Sigma$,

$$g(z, a) = (e^{2\pi if(g,a)}z, g^*a)$$

and then

$$\mathbb{C}^{\otimes k} \times \mathcal{A}_\Sigma \to \mathcal{A}_\Sigma$$

is a nice line bundle and $Z_k(M_\Sigma)$ is a G^Σ-equivariant section. So maybe it is the vector space of sections of this line bundle that we should be assigning to Σ.

18.1. **Geometric Quantization.** Say $L \to X$ is a line bundle over a complex manifold X with a G action, and $c_1(L) = \omega$ for ω a Kähler form. If G preserves ω, we may define a moment map,

$$\mu : X \to \mathfrak{g}^*$$

where $\xi \in \mathfrak{g}$ and $v \in T_xX$.

$$< d\mu_X(Y), \xi >= \omega_X(v, \xi_x)$$

where μ is equivariant (so that we still have an action of $\mu^{-1}(0)$).

Definition 18.1. $\mu^{-1}(0)/G$ is called a symplectic quotient of X.

So just to give a sense of where we're headed, $\mathcal{A}_\Sigma$ turns out to be a symplectic manifold and μ evaluates the curvature of a connection, so $\mu^{-1}(0)/G$ is the space of flat connections mod gauge transformation.

As a remark, notice that it might seem that μ is only defined up to a constant, but for simple groups this ambiguity goes away.

Now we want to complexify some stuff. We need the following:

Theorem 18.2. *G-equivariant sections of L are in bijection with sections of $L/G_\mathbb{C} \to X/G_\mathbb{C}$.*

So then we see that to a surface in Chern-Simons theory we associated to a surface, Γ_{hol}(flat connections on Σ/Gauge;$\mathcal{L}$), where the sections are holomorphic.

It is another theorem that if we do the symplectic reduction or the quotient by the complexification, we get the same manifold.

So now let's explain what we assign to the circle. Let's consider a bordism from the circle to the empty set, Σ. Then $S^1 \xrightarrow{\Sigma} \emptyset$ gives

$$Z_k(S^1) \xrightarrow{Z_k(\Sigma)} Z_k(\emptyset).$$

We find that $Z_k(\emptyset) = Vect$ since if $Z_k(S^1)$ assigns an object of a linear category $\mathcal{C}$, we get $Z_k(\emptyset) \in Hom_{\mathcal{C}}(X, Y) \in Vect$. What is the monoidal structure? $\mathcal{C} \otimes \mathcal{D}$ has as objects pairs (C, D) and

$$Hom((C, D), (C', D')) = Hom(C, C') \otimes Hom(D, D')$$

and so the unit for this tensor product is the category of vector spaces. PROBABLY NEED TO REWRITE THIS MORE CLEARLY. THERE WAS ALSO SOME ISSUE ABOUT THE CATEGORY NEEDING ENOUGH COLIMITS, SAY FINITE COLIMITS.

So again, consider $\mathcal{A}_\Sigma$ as the space of connections on the trivial bundle over $G \times \Sigma \to \Sigma$, and ${}_G\Sigma$ acts on $\mathcal{A}_\Sigma$ as before. We have a moment map

$$\omega(\alpha, \beta) = \frac{1}{8\pi^2} \int_\Sigma tr(\alpha \wedge \beta)$$

for $\alpha, \beta \in \Omega^1(\Sigma; \mathfrak{g})$, and moment map

$$\mu : \mathcal{A}_\Sigma \to (Lie(G^\Sigma))^*$$

is

$$\mu(A) = Curv(A) - \psi(A)$$

where ψ is the composition

$$\mathcal{A}_\Sigma \to \mathcal{A}_{S^1} \overset{A}{\hookrightarrow} (\tilde{L\mathfrak{g}})^* \xrightarrow{B} (Lie(G^\Sigma))^*.$$

where B is the dual of the map of lie algebras obtained from lifting a map $G^\Sigma \to LG$ to $G^\Sigma \to \tilde{LG}$ over LG.

Theorem 18.3. *Fix an irreducible representation $\hat{R}$ of $\tilde{LG}$. Then*

$$\mu^{-1}(W)/G^\Sigma = \{\text{cong classes of } \pi_1(\Sigma) \to G \text{ s.t. } \pi_1(\partial\Sigma) \to C\}$$

where W is the image of the coadjoint orbit in $(\tilde{LG})^*$ associated to $\hat{R}$ and C is a conjugacy class in G given by the holonomy of the LG-orbit in $\mathcal{A}_{S^1}$ that come from the $\tilde{LG}$-orbit $(\tilde{LG})^*$.

Remark: this is closely related to the Kirillov story, but it is "holomorphic, not Dirac," so there are some minor differences.

Now let's define

$$\mathcal{M}_{\Sigma;C} = \mu^{-1}/G^\Sigma.$$

This giant theorem continues:

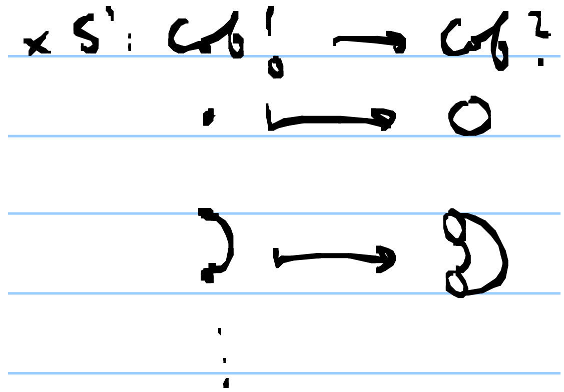


Figure 15. Dimensional reduction.

Theorem 18.4. *$\mathcal{M}_{\Sigma;C}$ is Kähler and we pick a line bundle $\mathcal{L}$ with $c_1(\mathcal{L}) = \omega$. Then the space of sections of $\mathcal{L}^{\otimes k} \to \mathcal{M}_{\Sigma;C}$ gives a functor*

$$Z_k(S^1) \to Vect$$

$$\hat{R} \mapsto \Gamma(\mathcal{M}_{\Sigma;C}, \mathcal{L}^{\otimes k})$$

which is the one we want.

Theorem 18.5. *$Z_k(S^1 \times S^1) =$Verlinde algebra for $\widetilde{LG}$.*

The way to see this is to look at the characters of the loop group representation which can be interpreted as sections of some line bundle... CONSTANTINE MADE A VERY INTERESTING COMMENT ABOUT THIS, INVOLVING ELLIPTIC CURVES, ETC.

The simple objects of the Verlinde algebra are the irreducible representations of the loop group at level k and form a basis. Note that the tensor product does not preserve the level, whereas the fusion product does, so fusion is the right multiplication to take.

The relationship between CS assigning the Verlinde algebra to the torus and ${}^{\tau}K_G(G)$ assigning the Verlinde algebra to a circle is not a conicindence: we can see it from dimensional reduction.

So let's talk about dimension reduction by example. If we have

$$Z : Cob_1^2 \to Vect,$$

we want to get

$$Cob_0^1 \to Vect.$$

This comes about from a functor

$$\times S^1 : Cob_0^1 \to Cob_1^2,$$

and then we just compose.

Proposition 18.6. *The 2-1 TFT ${}^{\tau}K_G(G) \otimes \mathbb{C}$ is dimensional reduction by S^1 of the 3-2-1 TFT given by CS.*

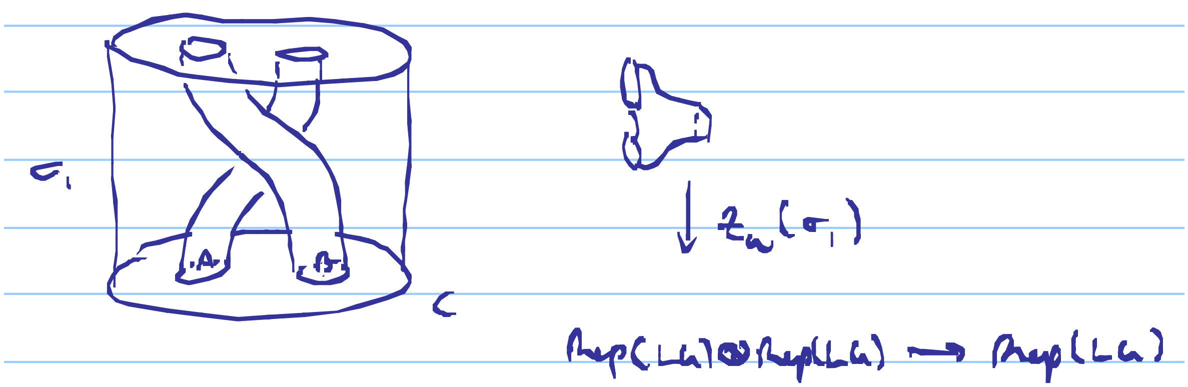


FIGURE 16. Braiding coming from 3-d geometry.

The conjecture is that this extends down to points.

Now let's talk about braided monoidal categories. Consider the squashed pair of pants (a disk with two holes removed). Then consider a bordism that "swaps the legs," see the picture. This is going to lead to a braiding. Now a pair of pants gives a functor

$$Rep(\tilde{LG}) \otimes \tilde{LG} \to \tilde{LG},$$

and this braiding gives a natural transformation, which we will call σ_1. The appropriate tensor product here is again the fusion product, so that we preserve the level of the representation.

More abstractly, say we have a symmetric monoidal category $(\mathcal{C}, \otimes)$. Then there is the natural transformation given by switching the factors

$$\mathcal{C} \times \mathcal{C} \stackrel{swap}{\to} \mathcal{C} \times \mathcal{C} \stackrel{\otimes}{\to} \mathcal{C}.$$

However, in a braided monoidal category, *swap* and its inverse need not compose to the identity (we can see this via looking at actual braid diagrams up to isotopies).

So now we see the open question from yesterday: Is there a "Dirac convolution" on ${}^{\tau}Vect_G(G)$ that corresponds to the fusion product $Rep(\tilde{LG})$?

Remark: for negative energy representations at negative levels, we get some dual picture, in Chern-Simons, but it doesn't give anything new.

Another remark: ${}^{\tau}Vect_G(G)$ is a categorification of ${}^{\tau}K_G(G)$. This is the extending-down problem, roughly stated this trying to find a product structure that works on the vector bundles themselves, not their image under K-theory.

19. Something About Local Field Theory, Chris Douglas

So as in the last talk, we're looking at Chern-Simons theory, and Witten told us about making a 3-2 theory, then Resh/Turaev gave a precise way of making it a 3-2-1 theory. So the question is: Is CS a 3-2-1-0 theory? If so, what do we assign to a point.

First, let's talk about delooping. Recall the definition of TQFT as a symmetric monoidal functor from the bordism category to Vect. Extending means we start with some symmetric monoidal n-category of bordisms, but we also need an appropriate target category. We find that we want to assign to a closed $(n-1)$-manifold a vector space, so we postulate the existence of a different category $\mathcal{C}$ where

$$\Omega\mathcal{C} := Hom_{\mathcal{C}}(1,1) = Vect$$

and we say $\mathcal{C} = BVect$.

Definition 19.1. A local field theory is

$$Bord_0^n \to \mathcal{C}$$

where again $\mathcal{C} = B^{n-1}Vect$.

A priori there are many deloopings, so constructing these is more of an art than a science.

There are two ways of delooping, in general:

$$Vect \overset{categorification}{\mapsto} 2-Vect, \quad Vect \overset{algebraification}{\to} Alg$$

The objects of $2-Vect$ are linear categories, 1-morphisms are functors, and 2-morphisms natural transformations, and for $3-Vect$ the story is similar, but we also get modifications. For Alg we have algebras as objects, bimodules as morphisms and bimodule maps as 2-morphisms.

We can fill in a rectangle:

$$\begin{array}{ccccc} Vect & \to & 2-Vect & \to & 3-Vect \\ \downarrow & & \downarrow & & \downarrow \\ Alg & \to & TensorCat & & \\ \downarrow & & \downarrow & & \\ 2-Alg & \to & Braided\ Tensor\ Cat & & \end{array} \tag{17}$$

Then $2-Alg$ has 2-algebras, h-bimod v-alg, and h-bimod v-bimod maps. One of the issues here is classifying 2-field theories with target $\mathcal{C}$, and the other is finding a good $\mathcal{C}$.

Let's think about local field theories in dimension 1. We draw our favorite 1-manifold and try to figure out what morphism in $Vect$ to choose. We chop up our 1-manifold first into elementary pieces and work with these. So the classification goes as follows: the positively/negatively oriented point gets a vector space, then the unit and counit get some morphisms satisfying Zoro's lemma, and then we get local field theory. One way to encapsulate all this data is the the positively oriented point is dualizable:

$$pt^+ \in Vect^{dualizable}.$$

Now for $2-d$ local field theories we try to do something similar. So again draw your favorite 2-manifold, say the torus, chop it into elementary pieces (say via a Morse decomposition), giving us some notion of a 2-1 field theory. Then we need to chop these elementary pieces further to get a 2-1-0 theory.

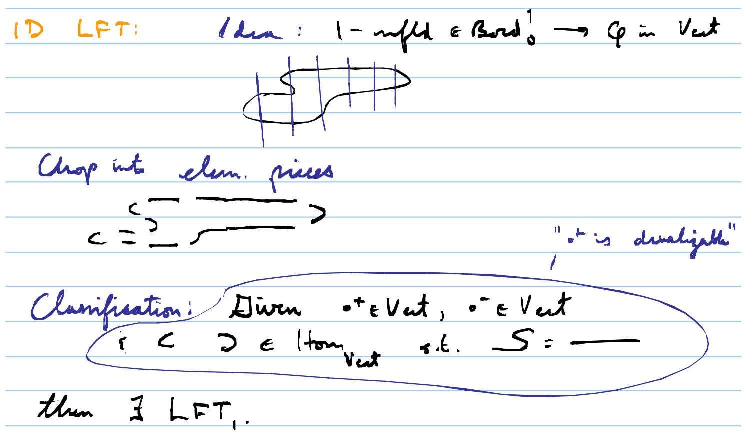


FIGURE 17. One dimensional local field theories.

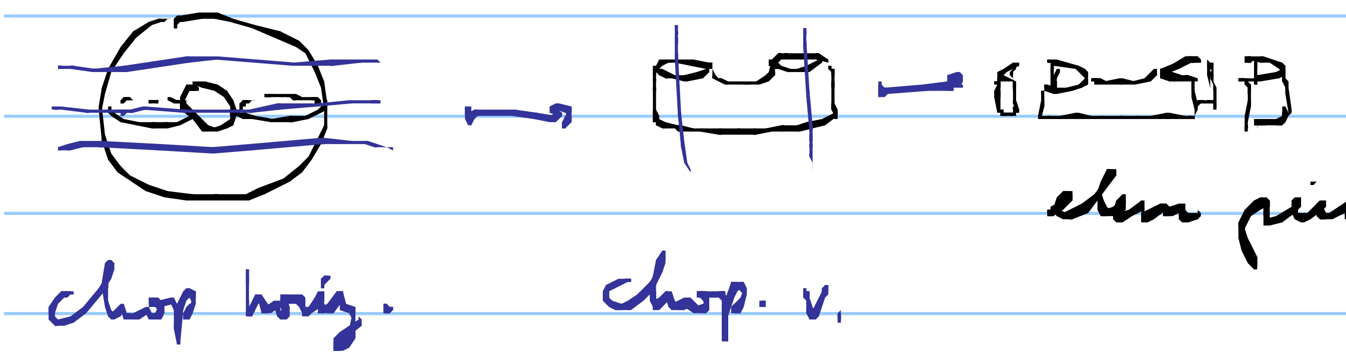


FIGURE 18. Two dimensional local field theories.

THERE IS THE POSTER OF ALL THE ELEMENTARY PIECES IN DIMENSION 2. CAN PROBABLY GET THIS PDF FROM SOMEWHERE. (JESSE - I DIDN'T GET THESE DOWN)

As an aside: this story is a bit of a fudge: we're classifying *framed* field theories, not oriented ones, where the pictures all get framings from the blackboard framing. But let's ignore the finesse for now.

So now what does our classification look like? The statement is that we get a 2-dimensional local field theory provided we get something in $\mathcal{C}$ for pt^+, satisfying a bunch of relations.

Now, this is all painstaking and complicated, and was done in low dimensions ($d \leq 3$) before Jacob Lurie came along and transformed the field. So let's talk about his result: the cobordism hypothesis.

Theorem 19.2 (Hopkins-Lurie)**.**

$$Fun^{\otimes}(FrBord_0^n, \mathcal{C}) \cong \mathcal{C}^{fd}$$

where $\mathcal{C}^{fd}$ *are the fully dualizable objects of* $\mathcal{C}$*.*

Recall: For $\mathcal{C}$ a 2-category, $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ are adjoint if there exist

$$u : 1_{\mathcal{C}} \to gf, \quad v : fg \to 1_{\mathcal{D}}$$

such that

$$f \xrightarrow{fu} fgf \xrightarrow{vf} f = 1_f$$
$$g \xrightarrow{vg} gfg \xrightarrow{gv} g = 1_g.$$

The picture is a left and right elbow satisfying Zoro.

Definition 19.3. (1) For $\mathcal{C}$ a symmetric monoidal n-category, $\mathcal{C}$ has adjoints for 1-morphisms if all 1-morphisms have left and right adjoints in $Ho_2\mathcal{C}$ (the homotopy 2-category)
(2) $\mathcal{C}$ has adjoints for k-morphisms if for $1 < k < n$, $Hom_{\mathcal{C}}(a, b)$ has adjoints for $(k-1)$ morphisms.
(3) $\mathcal{C}$ has duals if
   (a) every object of $\mathcal{C}$ has a dual in $ho(\mathcal{C})$
   (b) $\mathcal{C}$ has adjoints for k-morphisms, $0 < k < n$.
(4) $c \in \mathcal{C}$ is dualizable if it is int he maximal symmetric monoidal sub-n-category with duals.

As a remark: I'm on a crusade to remove the word "fully" from the typical notion of "fully dualizable," replacing dualizable with k-dualizable, and fully dualizable with simply dualizable.

For example, dualizable objects in vector spaces are finite dimensional vector spaces.

In algebras, dualizable objects are finite dimensional semi-simple algebras.

In tensor categories, we would like $G \mapsto S(G) \in TC$ and $S(G)$ dualizable, giving rise to 3-dimensional local field theories. It isn't exactly clear how to do this. Here is one result:

Theorem 19.4 (D-Schommer-Pries-Snyder). *If $\mathcal{C} \in TC$ is fusion, then $\mathcal{C}$ is dualizable.*

Here fusion roughly means finite dimensional hom sets, semi-simple, finitely many simple, and duals.

We are reasonably sure that if $\mathcal{C}$ is dualizable, then $\mathcal{C}$ is fusion.

Corollary 19.5. *$Rep(LG, k) \in TC$ is dualizable.*

Corollary 19.6. *There exists a 3-dimensional local field theory that sends a point to $Rep(LG, k)$*

But, this is not the same theory as Chern-Simons! In fact, calling this ψ and Chern-Simons ϕ

$$\psi(M^3) = \phi(M)\overline{\phi(M)}.$$

THIS THEORY GOES BY SOME OTHER NAME, BUT I DIDN'T CATCH IT. (JESSE - ME NEITHER)

THERE WAS SOME CONFUSION IN THE AUDIENCE ABOUT THIS. Explaining how TC is a three category seems to resolve all the issues.

What happens:

We have a 4-category BTC with $\Omega BTC = TC$ and there is an $A \in BTC$ dualizable, giving a 4-dimensional field theory. This is supposed to help us with the CS anomaly. Then there is a morphisms

$$A \overset{S(T)}{\to} 1$$

in BTC, that is dualizable, and $S(T)$ should lead to the localization of CS.

We can ask for the value of this theory on a circle, and in general we'd get $\mathcal{C} \otimes_{\mathcal{C} \otimes \mathcal{C}^{op}} \mathcal{C}$, which we can define as some kind of Drinfeld center of $\mathcal{C}$, $Z(\mathcal{C})$. THERE IS SOME ISSUE HERE ABOUT DUALIZABILITY, SOMETHING ABOUT HH^* AND HH_*. SEEMS FINE TO IGNORE FOR NOW.

As a remark, often it's easier to guess than answer and compute a field theory rather than construction one from the cobordism hypothesis. Really, computing the invariant of (say) a closed manifold can by quite a chore.

Now let's think about 2-dimensional algebras (everything from here on is joint work with Bartels and Henriques). We'd like to go from 0-algebras to 1-algebras by adding some horizontal multiplication, and then to 2-algebras by adding a vertical multiplication.

Then we confront the Eckman-Hilton problem, which says that if our two multiplications commute then they must agree. To get around this you have to give up something. The solution here is to take horizontal multiplication as associative and unital, and take vertical multiplication a coh. associative, while keeping the fact that horizontal and vertical multiplication commute. In particular here, we get an algebraic incaranation of the pentagon axiom, which is pretty awesome. THERE IS SOME PICTURE SHOWING THIS. (JESSE - I DON'T HAVE THIS PICTURE IN MY NOTES)

One might also want to a category where multiplication is encoded by bimodules rather than maps, and this is a weaker notion. But the reason we don't take this notion is that there is a lot in conformal field theory that related to our choice.

There is a functor from 2-Algebras to tensor categories.

So now we define the category of conformal nets.

Definition 19.7. A conformal net is a cosheaf of von Neuman algebras on the category of intervals.

What on earth is this thing? Well, a cosheaf is just a functor:

$$N : Int \to vNalg, \quad I \mapsto N(I)$$

such that certain gluing maps work.

THERE WERE SOME NICE DESCRIPTIONS OF VERTICAL MULTIPLICATION USING SOME PICTURES. (JESSE - I DON'T HAVE THESE PICTURES EITHER)

Proposition 19.8. *A conformal net is a* $2 - Alg$*.*

Theorem 19.9. *Nets for a symmetric monoidal 3-category. There exists a conformal net associated to a loop group at level* k*,* $N_{LG,k} \in Net$ *and it is dualizable.*

Corollary 19.10. *There exists a local field theory whose target is nets.*

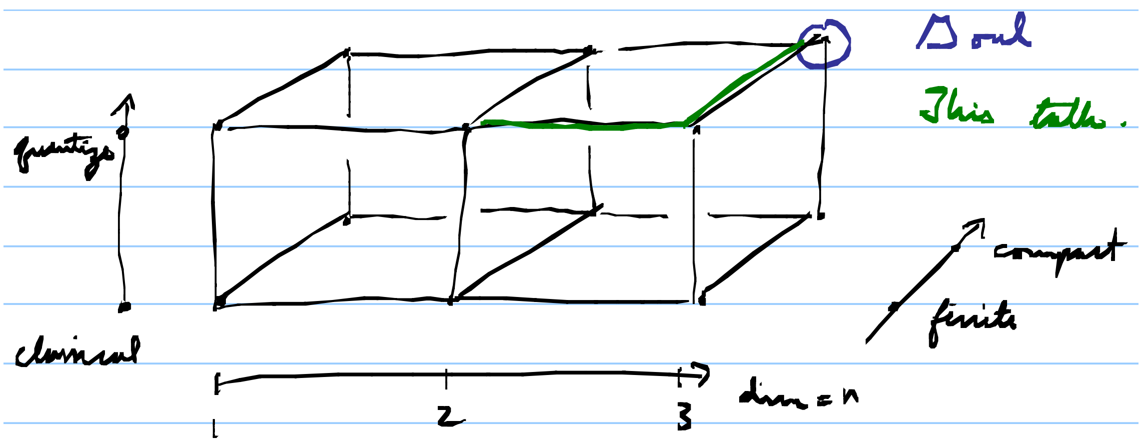


FIGURE 19. Picture of where this talk fits in the Chern-Simons story.

We note that under the functor from 2-algebras to tensor categories $N_{LG,k}$ goes to $Rep(LG,k)$. (Though there are still some technical issues in proving this is a 3-functor, though it appears to preserve dualizable objects.)

Now that we know the condition for dualizability we can build all kinds of 3-dimensional field theories, at least in principle. Stuff is out there related to the monster group.

20. Chern-Simons Theory and the Categorified Group Ring, Konrad Waldorf, UC Berkeley

Paper by FHT and L.

So as we've been hearing today, we would like to extend Chern-Simons theory to the point. For all this there will be a parameter in the theory,

$$\tau \in H^4(BG;\mathbb{Z}).$$

There appear to be two problems to take care of to implement this:

(1) Lurie: Whatever we assign to a point determines the whole theory, but we already know Chern-Simons theory at higher levels, so we must be very careful with our choice.
(2) The anomaly.

The above mentioned paper says we should look at:

(1) The case of a finite group (also called Dijkgraaf-Witten theory)
(2) Look at classical CS-theory and quantize
(3) We should decategorify and look for similar cases of 1- and 2-dimensional TQFTs

There is a picture to keep in mind: have a horizontal axis labeling the dimension of the theory, a vertical axis of going from classical to quantum, and a third axis from from finite to compact groups. We want to end up at the point the furthest from the origin. In this talk we start with a 2-dimensional quantum theory on finite groups, categorify this to a 3-dimensional theory on finite groups, and then hint about how to go to general compact groups.

The method we use is geometric realizations of twisting parameter, τ, by geometrically realizing $H^{n+1}(BG;\mathbb{Z})$. We have a table:

n	Group Theoretical Realization	Geometrical Realization
1	characters of G	multiplicative map $G \to S^1$
2	central extentions of G by S^1	multiplicative S^1-bundle over G, (K,θ)
3	*String* extension of G by BS^1	multiplicative S^1-gerbe over G

So let's think about how we construct these: a line bundle is a cover with transition functions, and a gerbe is a cover with "transition functions" on 2-fold intersections, and we can iterate this idea as we wish.

Definition 20.1. A multiplicative gerbe is

(1) a gerbe $\mathcal{G}$ over a group G (like a transgressed twist, if you like)
(2) a 1-isomorphism over $G \times G$ "fiberwise:"

$$K_{x,y} : \mathcal{G}_x \otimes \mathcal{G}_y \to \mathcal{G}_{xy}$$

(3) a 2-isomorphism over $G \times G \times G$:

$$\begin{array}{ccc} \mathcal{G}_x \otimes \mathcal{G}_y \mathcal{G}_z & \overset{K_{x,y}\otimes id}{\to} & \mathcal{G}_{xy} \otimes \mathcal{G}_z \\ id \otimes K_{y,z} \downarrow & \swarrow \theta_{x,y,z} & \downarrow K_{xy,z} \\ \mathcal{G}_x \otimes \mathcal{G}_{yz} & \overset{K}{\to} & \mathcal{G}_{xyz} \end{array}$$

(4) and a pentagon axiom for θ over $G \times G \times G \times G$.

As a remark, all the models we've taking for twists in K-theory are gerbes. Or if you like, gerbes are geometric realizations of $H^3(G;\mathbb{Z})$.

Now, the underlying gerbe may trivial even though the multiplicative gerbe is not. So what is the multiplicative structure on a trivial gerbe? A multiplicative structure on a trivial gerbe is a pair (K,θ) with K an S^1-bundle over $G \times G$ and θ an isomorphism:

$$\theta_{x,y,z} : K_{xy,z} \otimes K_{x,y} \to X_{x,yz} \otimes K_{y,z}$$

satisfying the pentagon. Note composition of morphisms becomes tensor product of line bundles.

The G-equivariant structure on the trivial multiplicative gerbe (K,θ) is an S^1-line bundle$L$ over $G \times G$ is given by

$$L_{g,x} = K^*_{g\times g^{-1},g} \otimes K_{g,x}$$

where we write (g,x) to emphasize that although both are group elements, g is acting on x.

20.1. **2-Dimensional TQFTs for Finite Groups.**

Theorem 20.2. *There is an equivalence*

$$\begin{aligned} & \{0-1-2 \text{ TQFTs for oriented manifolds with target } Alg\} \\ \overset{\text{Lurie}}{\cong} & \{\text{Dualizable objects in } Alg \text{ homotopy fixed under } SO(2)\} \\ \cong & \{\text{semi}-\text{simple Frobenius algebra}\} \end{aligned}$$

Above Alg is the 2-category of algebras, bimodules, and bimodule maps. We're eventually going to categorify this picture to get to the 3-dimensional theory.

Now choose a finite group G and

$$A = \mathbb{C}^\tau[G] = \Gamma(G; K_\mathbb{C})$$

where for the K we got from our gerbe (and hence our twist τ) we define

$$K_\mathbb{C} := K \times_{S^1} \times \mathbb{C}.$$

We get a multiplicative structure:

$$(\gamma_1 \star \gamma_2)(g) = \sum_{h \in G} \theta(\gamma_1(gh^{-1}) \otimes \gamma_2(h).$$

This is the "twisted group ring." We get a trace,

$$tr(\gamma) = \frac{1}{|G|}\gamma(e) \in K_\mathbb{C}|_e \overset{\theta}{\cong} \mathbb{C}.$$

From the theorem we get an associated 2-d TQFT: to a point we assign $A = \mathbb{C}^\tau[G]$; the circle gets sent to $A \otimes_{A \otimes A^{op}} A \cong Z(A)$ where we use the trace to identify A with its dual; and Σ is sent to

$$\sum_{[p]} \frac{1}{|Aut(P)|} e^{2\pi i < \Sigma, \xi_p \tau >}$$

where $\xi_p : \Sigma \to BG$ classifies a principle bundle p and we regard $\tau \in H^2(BG; S^1)$.

20.2. **Categorification: From 2-d to 3-d.** The theme here is to replace elements of $\mathbb{C}$ to objects of $Vect$. So we take as our target category the category of 2-algebras (here we're using tensor categories as a model for 2-algebras). Now we need to find a (fully) dualizable object. We want to go from the twisted group ring to the category Constantine talked about, $Vect^\tau[G]$. Now,

$$Vect^\tau[G] := VBun(G)$$

where we take $VBun(G)$ as a category. This is very bad definition when G is not finite, and we'll need some sheafy thing. For now let's not worry so much. We need a monoidal structure on $Vect^\tau[G]$:

$$(W \star W')_g := \oplus_{hh'=g} K^\tau_\mathbb{C}|_{h,h'} \otimes W_h \otimes W'.$$

You can see better motivation for all this coming from the quantization of a classical theory (via the picture mentioned in the introduction).

Now we want to prove that $Vect^\tau[G]$ is dualizable and does in fact lead to Chern-Simons theory.

Theorem 20.3. (1) *The 3d TQFT defined by $Vect^\tau[G]$ assigns to the circle the Drinfeld center*

$$S^1 \mapsto Z(Vect^\tau[G]).$$

(2) *The Drinfeld center Z consists of vector bundles W over G together with isomorphisms*

$$L_{x,y} \otimes W_x \to W_{yxy^{-1}}.$$

Corollary 20.4. $K(Z(Vect^{\tau}[G])) \cong {}^{\tau}K^0_G(G)$

Corollary 20.5. *"$Z(Vect^{\tau}[G]) \cong PER^{\tau}(LG)$"*

We're being a bit imprecise here with equivalences of categories versus isos on K-theory. Need to do some sum over conjugacy classes and then prove product structures are the same, or something. Also there are issues with G being disconnected in discussing $PER^{\tau}(LG)$.

Corollary 20.6. *The 3d TQFT we get is Chern-Simons theory.*

Let's try to sketch the proof of the above theorem. The 1st part Chris has already discussed, so let's look at the 2nd part. A Drinfeld center is the natural generalization of the center of an algebra to the center of a monoidal category. Then for $(W, \epsilon) \in Z(Vect^{\tau}[G])$:

$$\epsilon_x : K_{xy^{-1},y} \otimes W_{xy^{-1}} \to K_{y,y^{-1}x} \otimes W_{y^{-1}x}$$

and for $W' = \mathbb{C}_y$,

$$\epsilon_{xy} : K_{yxy^{-1},y} \otimes W_{yxy^{-1}} \to K_{y,x} \otimes W.$$

20.3. **A Quick Tour of the 3d Theory in the Case of a Torus.** So now, let's consider what happens when $G = T = S^1 \times S^1$. We want vector bundles supported over only finitely many points on the torus, and these are skyscraper sheaves. So define the category $Sky^{\tau}[G]$ as the category of skyscraper sheaves with values in finite dimensional vector spaces supported over finitely many points.

For example, for $y \in T$, $\mathbb{C}_y \in Sky^{\tau}[T]$. Furthermore, these guys generate the category from the monoidal structure, which we define to be:

$$\mathbb{C}_x \star \mathbb{C}_y := K_{x,y} \otimes \mathbb{C}_{xy}.$$

Now we run into some problems:

(1) $Sky^{\tau}[T]$ is not dualizable in $2 - Alg$, so does not define a field theory, This leads us to an anomaly.
(2) $Z(Sky^{\tau}[T]) \cong Sky^{\tau}[\mathfrak{t}] \otimes Sky^{\tau}[F]$ where F is a character lattice of the torus. But, $Sky^{\tau}[F]$ is the modular tensor category we want to assign to a cicle, so $Sky^{\tau}[T]$ is somehow too big.

Let's explain this anomaly. The solution we take is to view the above "theory" as a morphism from the trivial theory to some anomaly theory:

$$1 \overset{CS^{\tau}}{\to} \mathcal{A}_{\tau}.$$

Now, a transformation between 4-functors is a 3-functor, so the dimensions are correct. Thus, we need to describe the 4-dimensional TQFT $\mathcal{A}_{\tau}$. It assigns to the point our "unwanted factor:"

$$pt \overset{\mathcal{A}_{\tau}}{\mapsto} Sky^{\tau}[\mathfrak{t}],$$

and so we would need to show that $Sky^{\tau}[\mathfrak{t}]$ is dualizable in 3-algebras, fixed under $SO(4)$.

So let's get a feeling for what the 4-dimensional theory is by evaluating on a 4-manifold. So let M be a oriented closed 4-manifold. Then

$$\mathcal{A}_{\tau}(M) = \sqrt{|F|}^{\chi(M)} \mu^{sign\tau \cdot sgn(M)}$$

where μ is an 8th root of unity.

Under some circumstances, $\mathcal{A}_{\tau}$ is trivial on 4-manifolds:

(1) if $8|sign(\tau)$ things are good
(2) if M is spin, $8|sgn(M)$, and we similarly get a trivial invariant

SOMEHOW WE CAN GET THE EULER CHARACTERISTIC PART TO GO AWAY. (JESSE - I DIDN'T CATCH THIS)

In the above cases we can choose a trivialization

$$\mathcal{A}_{\tau} \xrightarrow{T} 1$$

and compose

$$1 \xrightarrow{CS^{\tau}} \mathcal{A}_{\tau} \xrightarrow{T} 1$$

can be regarded as a 3-functor

$$Bord^3 \xrightarrow{T \circ CS^{\tau}} 2 - Alg.$$

21. Elliptic Cohomology, Nick Rozenblyum, MIT

First we'll figure out how the title fits in with the other things we've discussed this week. We'll start be reviewing orientations of cohomology theories.

21.1. **Orientations of Cohomology Theories.**

We've already seen that for complex vector bundles there is a canonical orientation for K-theory. In what follows we won't be too picky about the difference between oriented and orientable cohomology theories.

Definition 21.1. A multiplicative cohomology theory E is complex oriented if for every complex vector bundle $V^k \to X$ we have a Thom class $U_V \in \tilde{E}^*(X^V)$ (where X^V is the Thom space) such that

(1) there are natural maps

$$\hat{E}^{2n}(X^V) \to \tilde{E}^{2n}(S^{2n}) \cong E^0(pt)$$

such the $U_V \mapsto 1$

(2) naturality with respect to pullbacks
(3) $U_{V \oplus W} = U_V \cdot U_W$

We've seen that K theory is complex orientable. It is also true that $H\mathbb{Z}$ is complex orientable. It is an interesting exercise to show that KO is not complex orientable.

There is a universal complex oriented cohomology theory MU (called complex cobordism), i.e. if E is a complex oriented cohomology theory, then there exists a canonical map $MU \to E$.

What can we do with complex oriented cohomology theories? For one, we can define Chern classes. So recall that $\mathbb{CP}^\infty \simeq (\mathbb{CP}^\infty)^\xi$ where ξ is the tautological line bundle. To see this notice that

$$(\mathbb{CP}^\infty)^\xi \cong D(\xi)/S(\xi)$$

because $D(\xi)$ deformation retracts onto the zero section $mathbbCP^\infty$, and $S(\xi)$ is the universal principle $U(1)$ bundle, so it is contractible.

So if E is complex oriented then there is $x \in \mathbb{CP}^\infty$ such that under the map

$$\tilde{E}^2(\mathbb{CP} \to \tilde{E}^2(\mathbb{CP}^1) \cong E^0(pt),$$

we have that $x \mapsto 1$. From this we have that $E(pt)[x] \to E(\mathbb{CP}^n)$ for all n and that $x^{n+1} \mapsto 0$. Furthermore, from the Atiyah-Hirzebruch spectral sequence,

$$E(pt)[x]/x^{n+1} \overset{\cong}{\to} E(\mathbb{CP}^\infty),$$

and in particular

$$E(\mathbb{CP}^\infty) \cong E(pt)[[x]].$$

This might seem a bit unusual at first sight, but really we ought to take direct limits, not direct sums.

A complex orientation gives a Thom isomorphism for $V^n \to X$:

$$U_V : E^*(X) \overset{\cong}{\to} \tilde{E}^{*+2n}(X^V).$$

We can also define Chern classes, $c_i(V) \in \tilde{E}^{2i}(X)$ such that

(1) Naturality
(2) $c_n(V \oplus W) = \oplus_{i+j=n} c_i(V) c_j(W)$
(3) $c_1(\xi) = x \in \tilde{E}^2(\mathbb{CP}^\infty)$

Naturality together with the splitting principle defines this for all bundles.

Recall that for line bundles $L_1, L_2 \to X$,

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

for $E = H\mathbb{Z}$, the additive formal group law, and for K theory we get the multiplicative formal group law. Now what happens for other E? Let's consider the universal example, the map

$$S : BU(1) \times BU(1) \to BU(1)$$

which classifies the tensor product of line bundles. The multiplication is induced the the multiplication on $U(1)$, the we apply the functor B. So the what does this do on E? We get

$$S^* : E(pt)[[x]] \to E(pt)[[x, y]],$$

and let $x \mapsto F(x, y)$ be some power series in two variables determining S^*. We see that

$$c_1(L_1 \otimes L_2) = F(c_1(L_1), c_1(L_2)).$$

We claim that F is a formal group law.

21.2. **Formal Groups, Formal Group Laws and Cohomology Theories.**

Definition 21.2. A formal group law $+_F$ over R is a power series in two variables over R such that

(1) $x +_F y = y +_F x$
(2) $0 +_F x = x +_F 0$
(3) $x +_F (y +_F z) = (x +_F y) +_F z$

where $x +_F y := F(x, y)$

This behaves like an abelian group, but there are no elements. We can also take $Spf(R[[x]])$ and get a formal group scheme from this.

Suppose that $F(x, y)$ is a formal group law:

$$F(x, y) = \sum a_{ij} x^i y^j.$$

Notice that we can express conditions 1-3 above as polynomial relations for the coefficients, a_{ij}.

The functor

$$Rings \to Sets$$

sending

$$R \mapsto \{\text{formal group laws over } R\},$$

is corepresented by $L = \mathbb{Z}[a_{ij}]/I$ where I is the ideal of relations. That is,

$$Hom(L, R) \cong \{\text{formal group laws over } R\}.$$

Theorem 21.3 (Lazard)**.** *L is a polynomial ring.*

Now we get to put some of this picture together: we have a universal oriented cohomology theory and a universal formal group law, so one might guess the following:

Theorem 21.4. $\pi_* MU \simeq L$

So, an obvious question is can we go the other way, i.e. is there a functor

$$\{\text{formal group laws}\} \to \{\text{cohomology theories}\}.$$

A first guess might be that for a formal group law classified by

$$L = MU(pt) \xrightarrow{F} R$$

define a new "cohomology" theory

$$h_F^*(X) = MU^*(X) \otimes_{MU^*(pt)} R.$$

(One has to be a little careful, and at first only define this finite complexes, and then extend by colimits.) However, in general this doesn't work because we don't preserve long exact sequences. It *does* work if R is flat over L. However, L is such a huge ring that this rarely happens.

Theorem 21.5 (Conner-Floyd)**.**

$$K^*(X) \cong MU^*(X) \otimes_{MU^*(pt)} K^*(pt)$$

where $MU^*(pt) \to K^*(pt) = \mathbb{Z}[\beta, \beta^{-1}]$ *classifies the multiplicative formal group law,*

$$x + y + \beta xy.$$

There are cohomology operators for MU are given by

$$MU_*MU = \pi_*(MU \wedge MU).$$

We can think about $Spec(\pi_*(MU))$ as classifying formal group laws and $Spec(\pi_*(MU \wedge MU)$ classifies two formal group laws and an isomorphism between them. So

$$Spec(\pi_*MU \wedge MU) \quad \downdownarrows \quad Spec(\pi_*MU) \tag{18}$$

is a presentation for the moduli stack $\mathcal{M}_{FG}$ of formal group laws. Now if we have a cohomology theory, choosing an orientation is equivalent to choosing a coordinate on the corresponding formal group, giving a formal group law. But the more fundamental thing to work with is the formal group itself.

Now let X be a sheaf. Then $\pi_*(X \wedge MU)$ is a quasicoherent sheaf on $Spec(L) = Spec(\pi_*MU)$ and $\pi_*(X \wedge MU \wedge MU)$ is a quasicoherent sheaf on $Spec(MU_*MU)$, and these are compatible, and we can descend to get a quasicoherent sheaf on $\mathcal{M}_{FG}$. So we find that

$$MU_*(X) \simeq p^*(\mathcal{F}_x).$$

The upshot of this is that for $MU^*(X) \otimes_{MU^*(pt)} R$ to be a cohomology theory we only need that R is flat over $\mathcal{M}_{FG}$, i.e.

$$Spec(R) \to Spec(L) \to \mathcal{M}_{FG}$$

is flat. There is a theorem due to Landweber that gives a criteria for this.

For convenience, let's pass to even periodic theories, $E^*(pt) = 0$ for $* = 1 \mod 2$. Then there is a class $\beta \in E^{-2}(pt)$ such that β is invertible. Examples of this are K,

$$Hp = \Pi_{k \in \mathbb{Z}} \Sigma^{2k} H\mathbb{Z},$$

and MP where $\pi_0 PMP = \pi_*MU$.

So how to we construct formal group laws? One approach is to start with an algebraic group and complete at the identity. There aren't that many algebraic groups of dimension one around: the additive group $\mathbb{G}_a$ (leading to ordinary cohomology), the multiplicative group $\mathbb{G}_m$ (leading to K-theory), and elliptic curves (leading to elliptic cohomology). So now lets work over $\mathbb{C}$, and then every elliptic curve is of the form

$$E_\tau = \mathbb{C}/\mathbb{Z} < 1, \tau >$$

for $\tau \in \mathbb{H}$, the upper-half plane, and $E_\tau \cong E'_\tau$ if and only if $\tau' = g\tau$ for $g \in SL_2(\mathbb{Z})$. We can consider the stack

$$\mathcal{M}_{Ell} \text{``} \cong \text{''} \mathbb{H}/SL_2(\mathbb{Z})).$$

Note that the j-invariant, $j : \mathcal{M}_{Ell} \to \mathbb{A}^1$ realizes $\mathbb{A}^1$ as the course moduli space. The noncompactness here can be a problem, so we might want to complete it. That aside, there is a canonical map, given by completing the group law of an elliptic curve at the identity

$$\mathcal{M}_{Ell} \to \mathcal{M}_{FG},$$

and this map is flat. This doesn't give us a cohomology theory yet because we've only done this for rings, and the moduli stack of elliptic curves is not affine. There isn't an obvious way to proceed. It turns out you need to work with E_∞ ring spectra.

Theorem 21.6 (Hopkins-Miller, Lurie)**.** *There is a sheaf $\mathcal{O}^{top}$ of E_∞-ring spectra on $\mathcal{M}_{Ell}$ such that*

$$\pi_0(\mathcal{O}^{top}) = \mathcal{O}_{Ell}$$

and $\Gamma(\mathcal{M}_{Ell}, \mathcal{O}^{top}) = tmf[\Delta^{-1}]$.

First we need to construct $\overline{\mathcal{M}_{Ell}}$. Let's think complex analytically first. Then we have

$$D = \{q \in \mathbb{C} | 0 < |q| < 1\}.$$

Then there is a family of elliptic curves over D given by

$$E_q = \mathbb{C}^\times / q^{\mathbb{Z}},$$

where here we are using the exponential map, $\mathbb{C} \to \mathbb{C}^\times$, taking $\tau \mapsto e^{2\pi i \tau}$. This family has a natural extension over

$$D = \{q \in \mathbb{C} |\ |q| < 1\}$$

where E_0 is the nodal rational curve. This is the usual "fishtail" family of elliptic curves. In fact, it is possible to construct this family T algebraically over the formal disk, $Spec(\mathbb{Z}[[q]])$. The fiber at $q = 0$ is the ration nodal curve. This is the guy we want to glue into our moduli stack. We can consider the restriction of this family to $Spec(\mathbb{Z}((q)))$. This gives the Tate curve and "is" $\mathbb{G}_m / q^{\mathbb{Z}}$.

Now we will call Ell the fiber of $\mathcal{O}^{top}$ at $Spec(\mathbb{Z}((q)))$. This is the Tate elliptic cohomology. The formal group law this corresponds to is still $\mathbb{G}_m$, and in fact

$$Ell = K((q)).$$

There is a G-equivariant version of Ell that differs from the usual G equivariant K-theory, and this is what will be related to loop groups.