An Introduction to Complex K-Theory

May 23, 2010

Jesse Wolfson

Abstract

Complex K-Theory is an extraordinary cohomology theory defined from the complex vector bundles on a space. This essay aims to provide a quick and accessible introduction to K-theory, including how to calculate with it, and some of its additional features such as characteristic classes, the Thom isomorphism and Gysin maps.

Contents

1 Generalized Cohomology Theories

We begin with the definition of ordinary cohomology due to Eilenberg and Steenrod:

Def. 1. An *ordinary cohomology theory* is a collection $\{H^i\}_{i \in \mathbb{Z}}$ such that:

- For each $n \in \mathbb{Z}$, H^n is a contravariant functor from the category of pairs of spaces to abelian groups.[1]
- (Homotopy Invariance) If $f \simeq g$ through maps of pairs, then $H^n f = H^n g$ for all n.
- (Preserves Products) $H^n(\coprod X_\alpha) = \prod H^n(X_\alpha)$ for all n.
- (LES of the Pair) For each pair (X, A), there exists a long exact sequence

  $$\cdots \to H^i(X, A) \to H^i(X) \to H^i(A) \to^\delta H^{i+1}(X, A) \to \cdots$$

  such that the boundary map δ is natural.
- (Excision) If $Z \subset A \subset X$ and $\overline{Z} \subset Int(A)$ then the induced map

  $$H^i(X, A) \to H^i(X - Z, A - Z)$$

  is an isomorphism for each i.
- (Dimension) $H^i(*) = 0$ for $i \neq 0$.

An *extraordinary cohomology theory* satifies all of the above except the dimension axiom. Complex K-theory was one of the first extraordinary cohomology theories to be discovered and studied in depth. My aim here is to present it as such and develop some of the key structures of K-theory as a cohomology theory. Whenever going through the gory details would obscure this development, I'll refrain and refer interested readers to other sources instead.

1.1 K-Theory Take 1

As a disclaimer, assume all spaces X are compact and Hausdorff.

Def. 2. Given a space X, let $Vect_{\mathbb{C}}(X)$ denote the semiring of isomorphism classes of (finite dimensional) complex vector bundles over X with addition given by $\oplus$ and multiplication by $\otimes$.

Def. 3. We define $K^0(X)$ to be the group completion of $Vect_{\mathbb{C}}(X)$.

[1] The assignment $X \mapsto (X, \emptyset)$ makes this into a functor on spaces as well, and this is what is meant by $H^i(X)$.

Example 1. All vector bundles over a point are trivial, so $Vect_{\mathbb{C}}(*) = \mathbb{N}$ and $K^0(*) = \mathbb{Z}$.

Let X_* denote a space with basepoint $*$. For any space X, let X^+ denote the union of X with a disjoint basepoint. Let $S^n(X_*)$ denote the n-fold reduced suspension of X_*. With this notation, we define the negative K-groups as follows:

Def. 4. Letting $i : * \to X_*$ denote the inclusion of basepoint, we define

$$\widetilde{K}^0(X_*) := ker(i^* : K(X_*) \to K(*))$$

This is called the *reduced K-group.* Observe that $K^0(X) = \widetilde{K}^0(X^+)$. Now, for $n \in \mathbb{N}$, let,

$$\begin{aligned} \widetilde{K}^{-n}(X_*) &:= \widetilde{K}^0(S^n(X_*)) \\ K^{-n}(X) &:= \widetilde{K}^0(S^n(X^+)) \\ K^{-n}(X,Y) &:= \widetilde{K}^0(S^n(X/Y)) \end{aligned}$$

To extend our definition of the K-groups to the positive integers, we use Bott Periodicity.

Theorem 1. *(Bott Periodicity v. 1) Let $[H]$ denote the class of the canonical bundle in $K^0(\mathbb{CP}^1)$. Then, identifying $\mathbb{CP}^1$ with S^2, and letting $*$ denote the reduced exterior product, the map*

$$\begin{aligned} \widetilde{K}^0(X_*) &\to \widetilde{K}^0(S^2(X_*)) \\ [E] &\mapsto ([H]-1) * [E] \end{aligned}$$

is an isomorphism for all compact, Hausdorff spaces X. We call $[H]-1$ the Bott class.

Periodicity allows us to define the positive K-groups inductively, setting $K^n(-) := K^{n-2}(-)$, and similarly for the reduced groups.

In order to verify that K-theory gives a cohomology theory, we need two last facts:

Prop. 2. *If X is compact and Hausdorff, and E is any vector bundle on Y, then a homotopy of maps $f \simeq g : X \to Y$ induces an isomorphism of bundles $f^*E \cong g^*E$.*[2]

Prop. 3. *To every pair of compact, Hausdorff spaces (X,Y), there exists an infinite exact sequence*

$$\cdots \to K^n(X,Y) \longrightarrow K^n(X) \longrightarrow K^n(Y) \longrightarrow K^{n+1}(X,Y) \to \cdots$$

[2]The proof is an application of the Tietze extension theorem formulated for vector bundles. c.f. Atiyah [2] L.1.4.3.

which is natural in the usual sense.[3]

Now, checking our definitions against the axioms:

- Pullback of bundles makes the K-groups into contravariant functors so Axiom 1 is satisfied.
- Homotopy invariance follows from the proposition we just stated.
- For products, a quick check shows that the map

$$Vect_{\mathbb{C}}(\coprod X_\alpha) \to \prod Vect_{\mathbb{C}}(X_\alpha)$$

  given by pullback along the inclusions is an isomorphism, and that this isomorphism is preserved under group completion.
- The LES of the pair was given above.
- Excision is satisfied because $X/Y \cong (X-Z)/(Y-Z)$ for any $Z \subset Y \subset X$.

so we see that K-theory is indeed a cohomology theory.

1.2 A Quick Note on K-classes

From the definitions we've given, every K-class is an element of $K^0(X)$ for some compact space X. We can say more than this:

1. Two vector bundles E and F define the same K-class if there exists a trivial bundle ϵ^n such that $E \oplus \epsilon^n \cong F \oplus \epsilon^n$. This is known as stable isomorphism, so we see $K^0(X)$ is the group completion of the semiring of vector bundles modulo stable isomorphism.
2. Every K-class can be written as $[H] - [\epsilon^n]$ for some vector bundle H over X.
3. A vector bundle E is in the kernel of $K^0(X) \to \widetilde{K}^0(X)$ if and only if it is stably isomorphic to a trivial bundle.

The upshot of this is that when we want to make arguments in K-theory, we can actually make arguments using vector bundles and then check that these arguments behave well when we pass to K-classes. This is one of the main techniques for making constructions in K-theory.

These conclusions follow from two facts:

[3] The proof of this requires the most work, after Bott periodicity, in setting up K-theory as a cohomology theory. Both Atiyah [2] (P.2.4.4) and Hatcher[7] provide a detailed construction, but I recommend just taking this as a given when first getting a handle on K-theory, and coming back to the details later.

Prop. 4. *Every vector bundle on a compact space is a direct summand of a trivial bundle.*

This follows from a partition of unity argument, and the finiteness of the cover; in particular, this can fail for paracompact spaces. See Hatcher [7] P.1.4.

Prop. 5. *Given a commutative monoid A, with group completion $K(A)$, $K(A) \cong A \times A/\Delta(A)$ and $x \mapsto (x, 0)$ gives the canonical map $A \to K(A)$.*

Since $K(A)$ is defined by a universal property (that it's a left adjoint to the forgetful functor from groups to monoids), it's sufficient (and straightforward) to check that $A \to A \times A/\Delta(A)$ satisfies the universal property.

Putting these together, we see every K-class is of the form $[E] - [F]$ for two bundles E and F. 1 follows because $[E] = [F] \Leftrightarrow \exists\, G$ s.t. $E \oplus G \cong F \oplus G$. Given such a G, let G' be a bundle such that $G \oplus G' \cong \epsilon^n$ for some n. Then

$$[E] = [F] \Leftrightarrow E \oplus G \oplus G' \cong F \oplus G \oplus G'$$

i.e. $E \oplus \epsilon^n \cong F \oplus \epsilon^n$. The proofs of 2 and 3 are similarly straightforward applications of the two propositions above.

1.3 K-Theory Take 2

We can give another characterization of K-theory that is frequently useful, and which illuminates the definitions. Recall that a *spectrum* is a sequence of spaces (CW-complexes) $\{E(n)\}$ and connecting maps $f_n : E(n) \to \Omega E(n+1)$. A *loop spectrum* is one where the connecting maps are homotopy equivalences.

Theorem 6. *(Brown Representability) Every reduced cohomology theory $\tilde{h}$ on the category of pointed CW complexes has a representing loop spectrum $\{H(n)\}$, unique up to homotopy, such that $\tilde{h}^n(X) = [X, H(n)]_*$ (where $[-,-]_*$ denotes based maps up to homotopy).*[4]

Since we recover an unreduced theory h by adding in a disjoint basepoint, i.e. $h^*(X) := \tilde{h}^*(X^+)$, we see Brown also says that unreduced cohomology theories correspond to unbased maps up to homotopy.

We can start to identify the spectrum KU of complex K-theory using the following:

Prop. 7. *For any compact, Hausdorff space X_*,*

$$\widetilde{K}^0(S^n(X_*) \cong [X_*, \mathbf{U}]$$

where the unitary group $\mathbf{U} := \varinjlim U(n)$.[5]

[4] For a fuller discussion and proof of Brown Representability, see Hatcher [6] § 4.E.

[5] The proof follows from considering a clutching argument and that if X is compact, [X,-] preserves filtered colimits (e.g. direct limits).

This together with periodicity and the suspension-loop adjunction shows that

$$\begin{aligned}\widetilde{K}^0(X_*) &= \widetilde{K}^{-2}(X_*)\\ &= \widetilde{K}^0(S^2(X_*))\\ &= [S(X_*), \mathbf{U}]\\ &= [X_*, \Omega\mathbf{U}]\end{aligned}$$

Thus, we can give a homotopy theoretic definition of complex K-theory as

$$\widetilde{K}^{-n}(X_*) = [X_*, \Omega^{n+1}\mathbf{U}]$$

and Brown Representability, plus periodicity, again shows that this gives a cohomology theory.

In particular, periodicity can be restated as

Theorem 8. *(Bott Periodicity v.2)* $\Omega\mathbf{U} \simeq B\mathbf{U} \times \mathbb{Z}$*, and since* $G \simeq \Omega BG$ *for any topological group* G*, we see* $\Omega^2\mathbf{U} \simeq \mathbf{U}$*. Moreover, For all* $n \in \mathbb{N}$*,*

$$\begin{aligned}\pi_{2n+1}(\mathbf{U}) &= \mathbb{Z}\\ \pi_{2n}(\mathbf{U}) &= 0\end{aligned}$$

This is in fact the original form in which Bott proved it.[6]

We can view this version as a calculation of the reduced K-groups of spheres. Passing to unreduced, we see that the values of K-theory for spheres are:

$$\begin{aligned}K^0(S^{2n}) &= \mathbb{Z} \oplus \mathbb{Z}\\ K^1(S^{2n}) &= 0\end{aligned}$$

and

$$\begin{aligned}K^0(S^{2n+1}) &= \mathbb{Z}\\ K^1(S^{2n+1}) &= \mathbb{Z}\end{aligned}$$

Since the spectrum of a cohomology theory is only specified up to homotopy, it's possible to give several equivalent spectra, each of which can shed light on the theory.

For example we can interpret K-theory in terms of operators on a separable infinite dimensional Hilbert space H. Recall that a bounded operator

[6] Both homotopy equivalences have concrete implementations which Bott was able to formulate and study using Morse theory. The interested reader should see Milnor [11] for the full proof. Alternatively, both Hatcher [7] and Atiyah [2] provide proofs of version 1 in terms of bundle constructions.

on a Hilbert space is *Fredholm* if it has a closed image, and its kernel and cokernel are finite. To each operator T, we can assign an index

$$Index(T) := dim\ ker(T) - dim\ coker(T)$$

It turns out that this is the restriction to a point of an isomorphism

$$index : [X, Fred(H)] \to K^0(X)$$

Appendix A to Atiyah [2] spells this out in detail. Note that this interpretation of the spectrum K provides a link between K-theory and index theory of elliptic operators. Other representing spectra also exist and they illuminate deep connections between complex K-theory and areas of interest to mathematical physics and analysis,[7] but this is beyond my scope right now.

2 Computational Tools

As with any cohomology theory, we have the usual computational tools of Mayer-Vietoris sequences, and the LES of the pair. However, these are often not very useful in K-theory, because periodicity means we rarely have enough zero entries to reduce the long exact sequences to a series of isomorphisms. However, K-theory, and in fact any extraordinary cohomology theory, comes with two additional tools which relate its values to those of ordinary cohomology. These are the Atiyah-Hirzebruch spectral sequence, which is a special instance of a generalized Serre spectral sequence, and the Chern character, which relates K-theory to rational cohomology.

2.1 A Quick Recap of Spectral Sequences

Spectral sequences can seem quite daunting at first, at least they did to me.[8] However, once you get comfortable using them, they open up an incredible number of calculations which look nearly impossible without them. Recall that a spectral sequence is an infinite sequence of "pages" consisting of a grid of groups and of differentials between them. We write $E_r^{p,q}$ for the (p,q)th entry of the rth page, and $d^r : E_r^{p,q} \to E_r^{p+r,q-r+1}$ for the differential. You pass from one page to the next by taking the homology with respect to the differentials, and cooking up a new set of differentials from the data of the previous ones. A spectral sequence *converges* if for for each (p,q), $E_r^{p,q} = E_{r+1}^{p,q}$ for $r \gg 0$; we write $E_\infty^{p,q}$ for this stable group.

[7]For example, see Atiyah-Bott-Shapiro [3].

[8]If you haven't worked with them before, I highly recommend Ch.1 of Hatcher [8], available online. I recommend taking the construction of spectral sequences as a given, and focusing first on how use them. Hatcher has a great set of examples, sample computations and exercises and I really enjoyed this.

In a wonderful variety of situations, we can cook up a spectral sequence such that E_2 page starts with something that we know, and the E_∞ page is closely related to something we want to calculate, e.g. the cohomology ring of a space X. As a shorthand, we say the spectral sequence converges to the thing we want, and we write something like

$$E_2^{p,q} \Rightarrow h^{p+q}(X)$$

However, this notation is shorthand and should not be taken literally. Spectral sequences **do not** converge to the groups written on the right hand side of the arrow, they converge instead to the associated graded objects of a filtraton of these groups. Whether or not we can recover the groups we care about depends on an extension problem, and is often nontrivial.

Alright, with these disclaimers in place, let's lay out the main tools:

2.2 The Atiyah-Hirzebruch Spectral Sequence

Recall that given a fibration $F \to E \to B$, we have the Serre spectral sequence:

$$H^p(B, H^q(F)) \Rightarrow H^{p+q}(E)$$

In fact, the proof and construction carry over to any cohomology theory h giving us a generalized Serre spectral sequence:

$$H^p(B, h^q(F)) \Rightarrow h^{p+q}(E)$$

Taking the trivial fibration $id : X \to X$, we get the Atiyah-Hirzebruch spectral sequence

$$H^p(X, h^q(*)) \Rightarrow h^{p+q}(X)$$

This spectral sequence should be seen as reiterating for generalized cohomology what we already know from ordinary cohomology: namely, that cohomology theories are largely determined by their values on the point.[9] As I'll show in a moment, this spectral sequence, along with the generalized Serre SS for K-theory, provides one of the main tools for computing $K^*(X)$.

Note also, that the generalized Serre SS for K-theory allows us to prove a K-theory version of Kunneth (as the same proof in ordinary cohomology using the Serre SS carries over here), and its formulation is precisely the one we're used to.

[9] More precisely, two cohomology theories h and h' are equivalent if there exists a natural transformation $\alpha : h \to h'$ such that α induces an isomorphism $h(*) \cong h'(*)$. c.f. Adams [1].

2.3 The Chern Character

The Chern Character is a ring homomorphism

$$ch : K^*(-) \to H^*(-;\mathbb{Q}) \otimes K^*(*)$$

induces an isomorphism $K^*(-) \otimes \mathbb{Q} \cong H^*(-;\mathbb{Q}) \otimes K^*(*)$.[10] In other words, for any space X,

$$\begin{aligned} K^0(X) \otimes \mathbb{Q} &\cong \bigoplus_{n\in\mathbb{N}} H^n(X;\mathbb{Q}) \otimes K^{-n}(*) \\ &\cong \bigoplus_{n\in\mathbb{N}} H^{2n}(X;\mathbb{Q}) \end{aligned}$$

and similarly,

$$K^1(X) \otimes \mathbb{Q} \cong \bigoplus_{n\in\mathbb{N}} H^{2n+1}(X;\mathbb{Q})$$

3 Sample Computations

The following set of spaces are suggestions for good examples to apply these tools to calculate the K-theory of spaces.

- Riemann Surfaces
- CP^n
- $SO(3)$
- $O(4)$

The first two can be computed immediately from the Atiyah-Hirzebruch SS. For $SO(3)$, you'll need to combine the isomorphism from the Chern character with the Atiyah-Hirzebruch SS. You can use this to calculate the K-theory of $O(3)$ and then use this plus Kunneth to calculate the K-theory of $O(4)$.

4 Theoretical Tools

As we observed above, since every K-class can be represented as formal difference of actual bundles, we can usually make our arguments for K-theory in terms of actual bundles, and then observe that these arguments behave well when we pass to K-classes. Frequently these constructions involve reducing the structure group, for example, in decomposing a bundle as sum of

[10] Such an isomorphism is actually a general property of cohomology theories, and the map $h \to H^*(-;\mathbb{Q}) \otimes K^*(*)$ is called the *character* of the theory.

line bundles, or reducing its dimension by one. I sketch the general process in the next section; we will use it frequently. Following this, the theoretical tools discussed are:

- Adams operations
- The Thom Isomorphism and Applications
- A construction of the Chern character

These are largely independent, and you should feel free to tackle them in any order.

4.1 Splitting Principles

In the most general case, a splitting principle refers to an operation in which a G-bundle $E \to X$ is pulled-back along a map $f : F \to X$, such that f^*E has a reduced structure group, and the map induced by f on cohomology is injective.

The generic splitting principle arises from a fiber sequence

$$H \hookrightarrow G \to G/H$$

where H and G are topological groups.

Give a G bundle $E \to X$, the splitting principle is the action of pulling back the G-bundle along the projection of the associated bundle

$$E \times_G G/H \to X$$

This pullback gives us a bundle with structure group H, and for nice enough fiber sequences, the projection induces the desired injective map on cohomology.[11]

Some common examples of splitting principles for complex bundles are:

- $SU(n) \to U(n) \to S^1$ which corresponds to orienting the vector bundle.

[11]The cleanest way to see that this construction does reduce the structure group is to pass to universal bundles over classifying spaces. Recall that over paracompact spaces X, isomorphism classes of bundles E correspond to homotopy classes of maps $f_E : X \to BG$, where BG is the classifying space of G-bundles. We recover the bundle E on X by pulling back the universal G-bundle $EG \to BG$ along f_E. Up to homotopy, EG is defined as a contractible space with a free G-action, and BG is defined as EG/G. Now, given an injection $H \to G$, EG also has a free H-action, so $BH \simeq EG/H$. But then $BH \to BG$ is the universal G-bundle with fibre G/H. Forming an associated bundle $E \times_G G/H$ over X corresponds to pulling back the universal bundle BH along the map f_E, and pulling back EG along $f_E^*(BH) \to BH \to BG$ gives the "splitting" of E. See Segal [12] for the basics of classifying spaces.

- $U(n-1) \to U(n) \to S^{2n-1}$ which allows us to make arguments/constructions by inducting on the dimension of the bundle.
- $\mathbb{T}^n \to U(n) \to Fl_n$ which corresponds to reducing a vector bundle to a direct sum of line bundles.[12]

This last principle is the one most commonly referred to as the splitting principle, and we will use it repeatedly going forward. In fact, the moral for basic K-theory constructions seems to be:

- Define the construction for (direct sums of) line bundles.
- Extend to arbitrary bundles.
- Use the splitting principle to show that properties which hold for line bundles hold in general.

4.2 Adams Operations

The Adams operations $\{\psi^k\}$ are cohomology operations on complex K-theory, analogous to Steenrod Squares in mod 2 cohomology.

Their basic properties are:

Prop. 9. *For each compact, Hausdorff space X, and each $k \in \mathbb{N}$, there exists a ring homomorphism*

$$\psi^k : K^0(X) \to K^0(X)$$

satisfying:

1. *Naturality, i.e. $\psi^k f^* = f^* \psi^k$ for all maps $f : X \to Y$.*
2. *For any line bundle L, $\psi^k([L]) = [L]^k$.*
3. *$\psi^k \psi^l = \psi^{kl}$.*
4. *$\psi^p(x) \equiv x^p$ $(modp)$.*

Property 2 characterizes the Adams operations, and we can use this, plus the splitting principle, to give a general construction. Observe that if $E = \bigoplus_{i=0}^n L_i$, then property 2 and being a homomorphism says

$$\psi^k(E) = \sum_{i=0}^{n} L_i^k$$

We can extend this to a general definition using exterior powers. Given any bundle E, let $\lambda^i E$ denote the ith exterior power of E. From linear algebra, we know

[12] Fl_n denotes the complete flag variety over $\mathbb{C}^n$.

- $\lambda^k(E \oplus E') = \bigoplus_{i=0}^k \lambda^i(E) \otimes \lambda^{k-i}(E')$
- $\lambda^0(E) = 1$, the trivial line bundle.
- $\lambda^1(E) = E$.
- $\lambda^k(E) = 0$ for $k > dimE$.

There exists a useful class of integral polynomials called the Newton polynomials, denoted s_k (for $k \in \mathbb{N}$). With a little work,[13] one can show that if $E = \bigoplus_{i=0}^n L_i$ as above, then $\sum_{i=0}^n L_i^k = s_k(\lambda^1(E), \ldots, \lambda^k(E))$. We can then take

$$\psi^k(E) := s_k(\lambda^1(E), \ldots, \lambda^k(E)$$

as the general definition. By applying the splitting principle associated with $\mathbb{T}^n \to U(n) \to Fl_n$, it's enough to verify for (direct sums of) line bundles that the Adams operations satisfy the specified properties, and this is a straightforward check from our definitions.[14]

4.3 The Thom Isomorphism and Applications

Given the dependence of K-theory on vector bundles, we might expect that those features of ordinary cohomology related to vector bundles also arise in K-theory (e.g. the Thom Isomorphism, characteristic classes, and Gysin maps). All of these rely on the orientability of the vector bundles in ordinary cohomology, and formulating these for K-theory will similarly require a suitable notion of K-orientable bundle and manifold.

4.3.1 Orientability in K-theory

Orientations for vector bundles or manifolds are defined formally in any cohomology theory analogously to how they are defined in ordinary cohomology. First, some notation: Given a bundle V on X, let $B(V)$ denote the unit ball bundle, and $S(V)$ the unit sphere bundle (under any metric).

Def. 5. *A bundle V is orientable in the cohomology theory E^* if there exists a class $\omega \in E^*(B(V), S(V))$ such that $\omega|_p$ is a generator of $E^*(B(V)_p, S(V)_p)$ as a module over $E^*(*)$ for all $p \in X$.*

Despite the formal similarity between ordinary orientability and orientability in a general theory, we should not expect these to be closely related. First, while orientability in ordinary integral cohomology is equivalent to orientability in a (differential) geometric sense, there is no guarantee that this is the case in general, or that integral orientability is in any way

[13] See Hatcher [7] §2.3 for the details of this construction.
[14] For the proof of this splitting principle, see Hatcher [7] §2.3.

related to E^* orientability. In general, the question of whether an integrally orientable bundle or manifold is E^*-orientable, for some theory E, is a question of whether the orientation class survives to the E_∞ page of the Atiyah-Hirzebruch SS and then whether we can recover the cohomology from the associated graded. In general, this is not trivial, and it still leaves us with the question of interpreting the meaning of the E-orientability of a space.

In complex K-theory, Atiyah-Bott-Shapiro [3] shows that a vector bundle is K-orientable if and only if it admits a $spin^c$ structure. This has applications for mathematical physics, but is beyond the scope of my introduction. On the other hand, every complex bundle admits a $spin^c$ structure, and is thus K-orientable. More directly, we can construct the orientation class of the bundle from its exterior algebra. Thus, for the remainder of this section, I will assume that any bundle or manifold is almost complex, and avoid worrying about the more general case.

4.3.2 The Isomorphism

In preparation for the Thom isomorphism, we reformulate our test space for orientations of bundles as follows:

Def. 6. Given a vector bundle E on X, we define its *Thom space*, X^E, to be the one point compactification of E. [15]

Notice that $(X^E/\infty) \cong B(E)/S(E)$, so we can replace $K^*(B(E), S(E))$ with $K^*(X^E$ in our definition of orientability above.

Prop. 10. *Every complex vector bundle E is K-orientable, with a canonical orientation class $\lambda_E \in \widetilde{K}^0(X^E)$ satisfying*

1. *Naturality, i.e. $\lambda_{f^*E} = f^*\lambda_E$*
2. *Sums, i.e. $\lambda_{E\oplus E'} = (\pi_1^*\lambda_E) \cup (\pi_2^*\lambda_{E'})$*

For the construction of λ_E from the exterior algebra of the bundle E, and a verification that it satisfies the desired properties, see Atiyah [2], §2.6, p. 98-99. λ_E is known as the *Thom class* of the bundle. We can now state:

Theorem 11. *(Thom Isomorphism) If E is a complex vector bundle over X, then $\widetilde{K}^*(X^E)$ is a free module of rank 1 over $K^*(X)$ with generator ω_E. In other words, the map*

$$\Phi_K : K^*(X) \to \widetilde{K}^*(X^E)$$

given by $\Phi_K(x) = \lambda_E \cdot x$ is an isomorphism.

[15] Equivalently, if E is a complex vector bundle, we can define $X^E := P(E \oplus 1)/P(E)$. A quick check will show these are the same. Namely, $P(E \oplus 1)$ amounts to compactifying the fibers of E by gluing in projective hyperplanes at ∞, and modding out by $P(E)$ sends all these hyperplanes to a point.

The theorem can be proven via showing it in the case for (direct sums of) line bundles and then using the splitting principle associated to $\mathbb{T}^n \to U(n) \to Fl_n$ as usual. Though this proof, as Atiyah [2] presents it, relies on Bott periodicity, Bott periodicity is in fact equivalent to the Thom isomorphism theorem. Indeed, the Thom space of the trivial bundle $X \times \mathbb{C}$ is the reduced suspension $S^2(X^+)$. The Thom isomorphism then gives

$$K^0(X) \cong \widetilde{K}^0(S^2(X^+))$$

i.e. Bott periodicity holds.

4.3.3 Gysin Maps

Gysin, or wrong-way, maps, are a useful tool in cohomology theories. Using either Duality or the Thom Isomorphism, we can associate covariant maps to orientation preserving maps between manifolds, in addition to the contravariant mappings guaranteed by the theory. As a matter of notation, given such a map $M \to M'$, let N_M denote the normal bundle of M in M', and ϵ_M denote a tubular neighborhood of M in M' isomorphic to $B(N_M)$ with boundary $\partial(\epsilon_M)$ isomorphic to $S(N_M)$.[16] Let ϵ_M° denote the interior of ϵ_M, i.e. $\epsilon_M^\circ = \epsilon_M - \partial(\epsilon_M)$.

Def. 7. (Gysin Maps) Given an immersion $f : M \to M'$, of almost complex manifolds, we define $f_! : K^*(M) \to K^*(M')$ as the composite:

$$K^*(M) \to \widetilde{K}^*(M^{N_M}) \to K^*(\epsilon_M, \partial(\epsilon_M)) \to K^*(M', M' - \epsilon_M^\circ) \to K^*(M')$$

where the first map is the Thom isom., the second is the isom. $(B(N_M), S(N_M)) \cong (\epsilon_M, \partial(\epsilon_M))$ given by the tubular neighborhood theorem, the third is the isom. due to excision, and the last map corresponds to the canonical map $(M', \emptyset) \to (M', M' \setminus \epsilon_M^\circ)$.

Note that in ordinary cohomology, the Gysin map raises the degree by $dim(M') - dim(M)$. However, if M and M' are almost complex, they are of even dimension, so the difference is even as well. The periodicity of K-theory then ensures that Gysin maps between almost complex manifolds preserve degree.

4.3.4 Thom Classes and Characteristic Classes

We use the Thom class to construct characteristic classes in K-theory. There are two equivalent constructions for these classes, one using a splitting principle and the other using the Adams operations. I will sketch both and the interested reader can check that they are equivalent by doing the calculations with the universal bundle on BU.

[16] The existence of such a neighborhood is an immediate consequence of the usual tubular neighborhood theorem.

Def. 8. (The Euler Class) Given a complex vector bundle E on X, let ζ denote the 0-section. Then the Euler class, $e(E) \in K^0(X)$, is given by $e(E) = \zeta^* \lambda_E$.

Def. 9. (Chern Classes) Given a complex vector bundle E on X, we define the Chern classes classes $c_1(E), \ldots, c_n(E) \in K^0(X)$ inductively as follows:

1. $c_n(E) = 0$ if $n > rk(E)$
2. $c_n(E) := e(E)$
3. $c_i(E) := \pi_*^{-1}(c_i(\widehat{E}))$ where $\widehat{E}$ is the vector bundle corresponding to the splitting principle given by $U(n-1) \to U(n) \to S^{2n-1}$.[17]

We can also define $c_i(E)$ by $c_i(E) := \Phi_K^{-1} \circ \psi^i(\lambda_E)$ where ψ^i is the i^{th} Adams operation.

An obvious question to ask is how the Chern classes in K-theory behave relative to those in ordinary integral cohomology. Recall that in integral cohomology, if L and L' are line bundles, then $c_1(L \otimes L') = c_1(L) + c_1(L')$.[18] In K-theory, a similar calculation shows that $c_1(L \otimes L') = c_1(L) + c_1(L'_) + c_1(L)c_1(L')$. Jacob Lurie includes a nice discussion of this in [10]. In particular, he introduces the notion of formal group laws, and observes that in this language, ordinary chern classes are governed by the formal additive group, whereas K-theoretic chern classes are governed by the formal multiplicative group. We can exploit this to motivate both the definition and the existence of the Chern character.

4.4 Constructing the Chern Character

I asserted the existence of the Chern character above, and this section outlines a concrete construction of it. Morally, the Chern character for K-theory arises from the observation that, over $\mathbb{Q}$, exponentiation gives an isomorphism between the formal additive group and the formal multiplicative group. We can make this concrete as follows:

Given a K-class $[L]$ represented by a line bundle $L \to X$, define

$$ch([L]) := exp(c_1(L)) = 1 + c_1(L) + \frac{c_1(L)^2}{2} + \ldots + \frac{c_1(L)^k}{k!} + \ldots$$

where $c_1(L)$ is the first Chern class of L in ordinary cohomology. Given a K-class $[E]$ represented by $E = \bigoplus_{i=0}^n L_i$, define

$$ch([E]) := \sum_{i=0}^{n} ch(L_i) = n + (t_1 + \ldots t_n) + \ldots + \frac{t_1^k + \ldots t_n^k}{k!} + \ldots$$

[17] We show this to be a splitting principle via the Gysin sequence coming from the Thom isomorphism.

[18] We show this by doing the computations with the chern class of the universal line bundle over $\mathbb{CP}^n$).

where $t_i = c_1(L_i)$. By the same algebra that we used in the construction of the Adams operations, we can convert this expression into one purely in terms of the ordinary chern classes of E.[19] If c_j denotes the j^{th} Chern class $c_j(E)$ of E, and s_k denotes the k^{th} Newton polynomial, then we can rewrite this as

$$ch(E) = dim(E) + \sum_{k>0} \frac{s_k(c_1, \ldots, c_k)}{k!}$$

and take this as a general definition for arbitrary bundles. The interested reader can check that this is well defined and that it lands in the correct dimensions in $H^*(X;\mathbb{Q})$. Moreover, a straightforward check from our definitions shows that for any line bundles L and L', $ch(L \otimes L') = ch(L)ch(L')$ and $ch(\sum_i L_i) = \sum_i ch(L_i)$, so ch gives a ring homomorphism as desired.

Now apply the splitting principle associated to $\mathbb{T}^n \to U(n) \to Fl_n$ to conclude that any cohomology relation which holds for line bundles under this reduction also holds for arbitrary bundles.[20] This completes the construction of the Chern character. For the proof that it induces the isomorphism claimed above, see Hatcher [7], Ch.4, P.4.3 and 4.5.

5 Hirzebruch-Riemann-Roch

The Hirzebruch-Riemann-Roch Theorem was the first in a series of generalizations of the classical Riemann-Roch theorem, which eventually culminated in the Grothendieck-Hirzebruch-Riemann-Roch Theorem in algebraic geometry, and in the Atiyah-Singer Index Theorem in the differential case. I include it here, both for historical purposes, and because some of the deeper applications of K-theory occur in its implications for index theory, and Hirzebruch-Riemann-Roch is a first sign of these results.

Before we can state the theorem, we need to define another characteristic class $Td^*(-)$ called the *Todd class*. We can think of the Todd class as a formal reciprocal of the Chern character, and its construction is similar.

[19] From the axioms of characteristic classes, the total chern class

$$c(E) = (1 + t_1) \ldots (1 + t_n) = 1 + \sigma_1 + \ldots + \sigma_n$$

where σ_j denotes the jth symmetric polynomial in the t_i, and considering cohomological degrees, we see $c_j(E) = \sigma_j$. As noted in the construction of the Adams operations, there exists a class of polynomials s_k called the *Newton polynomials* with the property that $t_1^k + \ldots t_n^k = s_k(\sigma_1, \ldots, \sigma_k)$. This allows us to rewrite the formula above as $ch([E]) = dim(E) + \sum_{k>0} \frac{s_k(\sigma_1,\ldots,\sigma_k)}{k!}$. See Hatcher [7] §.2.3 for more details on the Newton polynomials.

[20] This is why we need the injectivity of the map $p : Fl(E) \to X$ on cohomology. For a proof of this splitting principle, see Hatcher [7], §3.

5.1 The Todd Class

In the usual manner, we define the Todd class for direct sums of line bundles, use some algebra to massage this into a formula for general bundles, and then prove it has the desired properties by using the splitting principle. We can characterize the Todd class as follows:

Prop. 12. *Given a vector bundle $E \to X$, there exists a unique class $Td^*(E) \in H^*(X;\mathbb{Q})$ satisfying:*

1. *Naturality, i.e. $f^*Td^*(E) = Td^*(f^*E)$.*
2. *$Td^*(E \oplus F) = Td^*(E)Td^*(F)$*
3. *$Td^*(L) = \frac{c_1(L)}{1-e^{-c_1(L)}}$ for all line bundles L.*

Using the Bernoulli numbers $\{B_{2i}\}_{i=1}^{\infty}$, we can expand the righthand side of number 3 as a formal power series. Explicitly:

$$Q(x) := \frac{x}{1-e^{-x}} = 1 + \frac{x}{2} + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}x^{2i}$$

For any finite dimensional basespace X, number 3 describes a polynomial in the chern classes with rational coefficients, and thus it does specify an element of $H^*(X;\mathbb{Q})$. This, together with number 2 then dictate the formula for direct sums of line bundles, and though we are unable to massage out a general expression using the Newton polynomials, there exists another class which does the job and allows us to get get a general expression for $Td^*(E)$ as a power series in the Chern classes of E. The first few terms of this expansion are

$$Td^*(E) = 1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2-2c_2)+\frac{1}{24}c_1c_2-\frac{1}{720}(c_1^4-4c_1^2c_2-3c_2^2-c_1c_3+c_4)+\ldots$$

where the c_i are the Chern classes of E. With this definition, naturality is a consequence of the naturality of the chern classes, and the splitting principle associated to $\mathbb{T}^n \to U(n) \to Fl_n$ shows that 2 is satisfied in general.

5.2 The Theorem

Recall that the euler characteristic of a coherent sheaf F on a space X is the alternating sum of its Betti numbers, i.e.

$$\chi(F) := \sum (-1)^i rk(H^i(X,F))$$

Theorem 13. *(Hirzebruch-Riemann-Roch) Given a vector bundle E on a compact complex manifold M, let $\mathscr{E}$ denote the sheaf of holomorphic sections of E. Then*

$$\chi(\mathscr{E}) = \int_M ch(E) \cdot Td^*(TM)$$

I don't give the proof here, but if you're interested, see Hirzebruch [9], or for a nice motivational discussion, see Griffiths and Harris [4] §3.4.

As an illustration of the theorem, we can quickly derive the classical Riemann-Roch formula for curves, which I go through below. There's a similarly nice derivation of Noether's Theorem for surfaces, and the interested reader should Hartshorne's [5], Appendix A.

5.3 Riemann-Roch for Curves

Let X be a curve of genus g and let $L(D)$ be a line bundle X corresponding to divisor D. We need to quickly recall several facts from algebraic geometry:

- $\chi(\mathscr{O}_X) = 1 - g$.
- The first Chern class gives the isomorphism between divisors and line bundles, so $c_1(L) = D$ and $ch(L) = 1 + D$.
- The tangent bundle TX is the dual of the canonical bundle of X, and so if K is the canonical divisor of X, TX corresponds to $-K$, and $Td^*(TX) = 1 - \frac{1}{2}K$.
- Since D and K define elements of $H^2(X; \mathbb{Z})$, $DK = 0$ for dimension reasons.
- For a divisor D, $\int_M D = degD$ (this is tautological from the definitions, but worth recalling).

. Putting these together, we see

$$\begin{aligned} ch(L)Td^*(TX) &= (1 + D)(1 - \frac{1}{2}K) \\ &= 1 + D - \frac{1}{2}K \end{aligned}$$

and plugging this into the formula from the theorem, we get

$$\begin{aligned} \chi(\mathscr{L}(D)) &= \int_M (1 + D - \frac{1}{2}K) \\ &= deg(D - \frac{1}{2}K) \end{aligned}$$

Now, taking $D = 0$, this says $1 - g = \chi(\mathscr{O}_X) = -\frac{1}{2}degK$, and plugging this in we get the classical formula:

$$\chi(\mathscr{L}(D) = degD + 1 - g$$

6 Conclusion

There is a lot more to say about complex K-theory, but I will leave off here for now. I hope these notes prove an accessible and easy to use introduction to the main features of K-theory, and at the very least, that they make it easier to navigate the various texts and articles which lay out the theory.

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