Fukaya Categories of Torus Fibrations

$p : X \to Y$

X symplectic, Y compact

p locally trivial Lag. torus fibration

Kont. - Soib. 2001

Relate $\mathcal{F}(X)$ to $\mathcal{O}_Y$ - modules, $\mathcal{O}_Y$ = sheaf of Λ-algebras on Y (Λ = Novikov ring).

$$D\mathcal{F}(X) \simeq D^b(\text{Coh} \; \mathcal{O}_Y \text{ - modules})$$

Affine structure on Y

<u>Defn</u>: An integral affine structure on Y consists of an atlas $\{\varphi_i : \mathcal{U}_i \subset Y \to \mathbb{R}^n\}$

$$\varphi_i \circ \varphi_j^{-1} \in GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$$

Y integral affine $\rightsquigarrow$ $p : X \to Y$.

$$\begin{array}{l} X = T^*Y \big/ \underbrace{T^*_{\mathbb{Z}}Y}_{\text{local system of integral cotangent vectors.}} \\ \;\; p \big\downarrow \\ \;\;\;\; Y \end{array}$$

Conversely, given $p : X \to Y$, $H_1(\text{fibre}, \mathbb{Z})$ = local system on Y. Take a local basis of sections. ~~Represent them to (this can't always be done).~~ $\gamma_1 \ldots \gamma_n$

Represent them by $(n+1)$-dim submflds $\Sigma_1, \ldots, \Sigma_n \subset X$.

Define $y_i(x) = \displaystyle\int_{p^{-1}(\gamma) \cap \Sigma_i} \omega$ coords on affine Y.

$$\Lambda = \left\{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} \;\middle|\; c_i \in \mathbb{C} \quad \lambda_i \in \mathbb{R}, \quad \lim_{i \to \infty} \lambda_i = \infty \right\}$$

$\text{val}(\sum c_i T^{\lambda_i}) = \min\{\lambda_i\}$

To construct $\mathcal{O}_Y$, construct $\mathcal{O}_{\mathbb{R}^n}$ along with an action $GL(n, \mathbb{Z}) \ltimes \mathbb{R}^n$ on $(\mathbb{R}^n, \mathcal{O}_{\mathbb{R}^n})$.

$$\mathcal{O}_{\mathbb{R}^n}(U) = \left\{ \sum_{\underline{k}} a_{k_1 \ldots k_n} z_1^{k_1} \cdots z_n^{k_n} \;\middle|\; a_{\underline{k}} \in \Lambda, \; \forall y \in U, \lim_{N \to \infty} \inf_{\|k\| \leq N} [\text{val}(a_k) + \langle k, y \rangle] \right\} = \infty$$

$k \in \mathbb{Z}^n$

$y \in \mathbb{R}^n$.

$f \in \mathcal{O}_{\mathbb{R}^n}(U)$ formally a Laurent series on $(\Lambda^*)^n$. The last condition says that
$f(z_1, \dots, z_n)$ converges in Λ (i.e. lies in Λ) whenever $(\mathrm{val}(z_1), \dots, \mathrm{val}(z_n)) \in U$.

E.g. $\mathcal{O}_{\mathbb{R}}(\mathbb{R}) = \{ \sum_{k \in \mathbb{Z}} a_k z^k \mid a_k \in \Lambda$, as $k \to \pm\infty$, the order (valuation) of a_k should grow more than linearly.

$A \in GL(n, \mathbb{Z})$, $y \longmapsto Ay$

$$\sum a_k z^k \longmapsto \sum a_{A^T k} z^k$$

LHS converges in U $\Leftrightarrow$ RHS converges in AU.

$b \in \mathbb{R}^n$, $y \longmapsto y + b$

$$\sum a_k z^k \longmapsto \sum a_k T^{-\langle k, b \rangle} z^k.$$

Coherent $\mathcal{O}_Y$-modules
want $\mathcal{F}(X) \longrightarrow \mathcal{O}_Y$-mod
Consider Lagrangian submfds ~~of~~ L of X s.t. $P|_L : L \to Y$ is an unramified covering (think of a section or multisection)

$L \in \mathcal{F}_{unram}(X)$
$F(L)$ - locally free $\mathcal{O}_Y$-module. Suppose $P|_L : L \to Y$ is one to one.
Over a sufficiently fine open cover ~~of~~ $\{U_i\}$ of Y
$F(L)(U_i) \cong \mathcal{O}_Y(U_i)$
choice of isomorphism corresponds to choice of $f \in C^\infty(U_i)$ s.t. $L \cap p^{-1}(U_i) =$ graph of df mod $T^*_{\mathbb{Z}} Y$
Change $f \longmapsto f + l$ $\Rightarrow$ dl is section of $T^*_{\mathbb{Z}} Y$ $\Rightarrow$ loc. const.
$\Rightarrow l = c + \langle m, y \rangle$ = affine linear
$T^*_{\mathbb{Z}} Y$.

$\Rightarrow$ multiply by $\exp(\ell) = T^c \prod z_i^{m_i} \in \mathcal{O}_Y^*$

If $p|_L : L \to Y$ is not 1-1 do the same with direct sum locally of sheaves associated to each sheet.

E.g. $Y = \mathbb{R}^n/\mathbb{Z}^n$ $\quad X = \mathbb{C}^n/\mathbb{Z}^{2n}$ $\quad c_1 = 0$

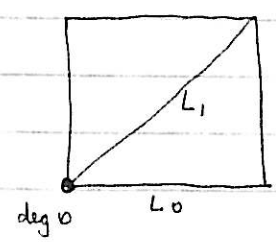
L_1 = section given by df, f = nondeg. quadratic form on Y s.t. df takes integral values on $\mathbb{Z}^n$.

L_0 = 0-section

$CF^*(L_0, L_1)$ concentrated in degree = index of f.

We would like to see that rank $HF^*(L_0, L_1)$ = rank $Ext^*(F(L_0), F(L_1))$.

E.g.



$f(y) = \frac{1}{2} y^2$

$F(L_0) = \mathcal{O}_Y$ $\quad (f = 0)$

$HF^0(L_0, L_1) = \Lambda$

$HF^1(L_0, L_1) = 0$

$Ext^0_{\mathcal{O}_Y}(F(L_0), F(L_1)) = \Lambda$?

$= H^0(F(L_1))$

Suppose $\sum a_k z^k$ is some section. In passing from nbhd of 0 to nbhd of 1, shift $y \mapsto y+1$

$\frac{1}{2} y^2 \longrightarrow \frac{1}{2}(y-1)^2 = \frac{1}{2} y^2 - y + \frac{1}{2}$

$\sum a_k z^k \longrightarrow \sum a_k T^{-k} z^k \longmapsto \sum a_k T^{-k} z^{k-1} T^{\frac{1}{2}}$

$\exp(-y + \frac{1}{2}) = z^{-1} T^{\frac{1}{2}}$

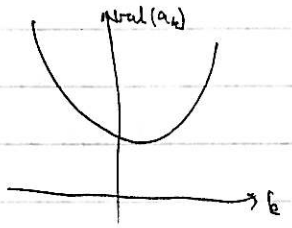
In order to define global section need

$$a_{k-1} = a_k T^{-k} T^{\frac{1}{2}}$$

$\Rightarrow$ gives recurrence determining all coeffs from a single one, say a_0.

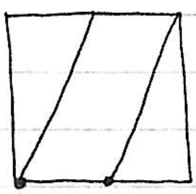
$h^0(\mathcal{F}(L)) \leq 1$.

$\mathrm{val}(a_{k-1}) = \mathrm{val}(a_k) - k + \frac{1}{2}$



$\Rightarrow$ convergence.

$\Rightarrow$ get global section $\Rightarrow h^0(\mathcal{F}(L)) = 1 \Rightarrow \mathrm{rk}\, HF^0(L_0, L_1)$



- $\mathrm{rk}\, HF(L_0, L_1) = 2$
- recurrence relation is $a_k \leftrightarrow a_{k+2} \Rightarrow 2$ global sections $\Rightarrow h^0(\mathcal{F}(L)) = 2$.

$\mathcal{F}_{\mathrm{unram}}(X) \longrightarrow \mathcal{O}_Y\text{-mod}.$