Discussion.

Curved A_∞-structures

$\mathrm{Ob}\,\mathcal{A}$

$\hom_{\mathcal{A}}(X, Y)$.

$\mu^d_{\mathcal{A}} : \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \longrightarrow \hom(X_0, X_d)[2-d]$

for $d \geq 0$

$\mu^1_{\mathcal{A}}(\mu^0_{\mathcal{A}}) = 0 \in \hom^3(X, X)$

$\mu^1_{\mathcal{A}}(\mu^1_{\mathcal{A}}(x)) = \mu^2_{\mathcal{A}}(\mu^0_{\mathcal{A}} x) \pm \mu^2_{\mathcal{A}}(x, \mu^0_{\mathcal{A}})$

e.g. $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ v.b. w/ curvature. $\mathcal{A} = \Omega^*(M, \mathrm{End}\, E)$

$\mu^1 = d_\nabla \quad \mu^2 = \wedge \quad \mu^0 = F_\nabla$

($d_\nabla F_\nabla = 0$ is Bianchi)

e.g. $\begin{array}{c} E \\ \downarrow \\ M \end{array}$ $\mathbb{C}$ v.b. / $\mathbb{C}$-mfd. $\mathcal{A} = \Omega^{0,*}(M, \mathrm{End}\, E)$

$\mu^1 = \bar\partial_\nabla \quad \mu^0 = F^{0,2}_\nabla$ — whole point not nec. holomorphic.

Technical (important though) $\mathcal{A}$ defined over $\mathbb{Z}[[t]]$, $\mu^0_{\mathcal{A}}$ order t "small"
($\sim$ Like to deform connection to get flat connection).

curved
$\neq$

not-curved
$\|$

$\mathcal{A}$ obstructed $\Rightarrow$ $\tilde{\mathcal{A}}$ unobstructed.

$\mathrm{Ob}\,\tilde{\mathcal{A}} = \{ (X, \alpha) \mid X \in \mathrm{Ob}\,\mathcal{A},\ \alpha \in \hom^1_{\mathcal{A}},\ \mu^0 + \mu^1(\alpha) + \mu^2(\alpha, \alpha) + \cdots = 0 \}$

of order ε

inhomogeneous MC

$\| $

$\mu^0_{\tilde{\mathcal{A}}}$

or

$\tilde{\mathcal{A}}_\rho$ deforms

order ε ?

enough to set

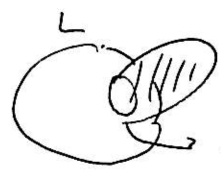
$\rho \in CC^2(\mathcal{A}, \mathcal{A})$

"the curvature"

$d\rho = 0$.

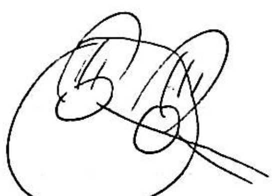
(M, ω).

Lets assume $c_1(M) = 0$



μ^0 = boundaries of holo$^=$ discs.

Suppose we have 2-disks.



boundaries Betype
w/ opposite signs. , could engineer to cancel. ($\sim$ complex str correct).

~~Algebraically~~,
example. $L^3 \subset M^6$, $c_1(M) = 0$

$H^1(L) = H^2(L) = 0 \Rightarrow \exists$ unique obj $\in \mathcal{F}(-)$

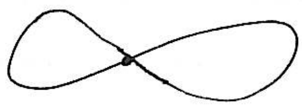
solve the obstruction problem.

(in general, could be impossible to solve / ∞ solutions)
Moduli space empty / not 0-dim'l

(in physics, $(L, A \in \Omega^1(L;\mathbb{R}))$ $\omega|_L = 0$, $\frac{i}{2\pi} F_A = -PD(\mu^0)$)

Immersed Lagrangians

transverse double pts



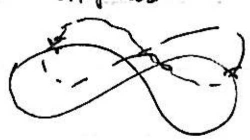
$L \xhookrightarrow{i} M$ Lagrangian immersions

formal name

$$CF^*(L,L) = C^*(L) \oplus \bigoplus_{\substack{(x,y)\in L \\ i(x)=i(y) \\ x\neq y}} \mathbb{Z}\langle x,y\rangle$$

Morse complex

displaced



2 pts generated

2 new.

μ^0 = holo discs + 

~~$u : \mathbb{H} \to M$~~

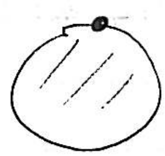
$u : \mathbb{H} \to M$ J-hol$^{=}$

$\tilde{u} : \mathbb{R} \to L$

$i(\tilde{u}) = u|\mathbb{R}$

$\lim_{s\to\infty} \tilde{u}(s) = x$

$\lim_{s\to-\infty} \tilde{u}(s) = y$.



[redacted]

Signs & Gradings

$L_0, L_1 \subset M$

$P = \{x : [0,1] \to M \mid u(0) \in L_0,\ u(1) \in L_1\}$
path space

$P \to U_\infty / O_\infty$ infinite Lag. Gr.

① $H^1(U_\infty/O_\infty) = \mathbb{Z} \longrightarrow H^1(P)$ obstruction to grading

② $H^2(U_\infty/O_\infty) = \mathbb{Z}/2 \longrightarrow H^2(P; \mathbb{Z}/2)$ obstruction to signs.

(need index theory) others to obstruct of eg. Floer-K theory

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① $2c_1^{rel} \in H^2(M \times [0,1],\ L_0 \times \{0\} \cup L_1 \times \{1\}) \xrightarrow[\text{transgression}]{} H^1(P)$

② $w_2(L_0) \oplus w_2(L_1) \in H^2(L_0; \mathbb{Z}/2) \oplus H^2(L_1; \mathbb{Z}/2) \xrightarrow[\text{evaluation}]{} H^2(P; \mathbb{Z}/2)$

Stieffel Whitney classes •

eg ①: Killing $2c_1$

(M, ω, J). $(\Lambda^n_{\mathbb{C}} TM)^{\otimes 2}$ (K^{-2} for alg. geometers)

$\cong \mathbb{C}$ - trivial by ($\cancel{2}c_1 = 0$)

fixed iso$^{\underline{m}}$.

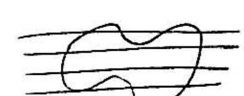
complex n-form η (eg [redacted] C-Y, have uniqueness)
in general, different choices give somewhat different results

Given oriented $L \subset M \Rightarrow \mu_L \in H^1(L)$.
Maslov class

$\mu_L = [\alpha_L : L \to S^1]$

$\alpha_L(x) = \dfrac{\eta(TL_x)}{\cancel{\|TL_x\|}}$ take onb of TL.

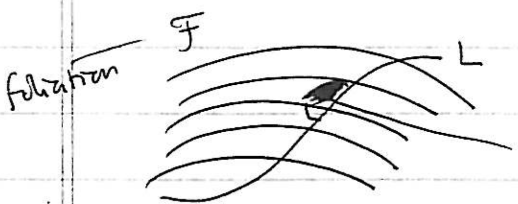
known as phase



rotation vs. foliation

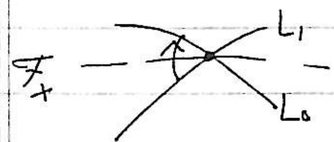
Choose real valued phase $\tilde{\alpha} : L \longrightarrow \mathbb{R}$ of α

If M a surface, A trivialization of TM = oriented foliation.



$\alpha_L(x)$ measures this angle

$\tilde{\alpha}_L(x)$ gives a way of rotating TL into the foliation



$$\mathcal{F}_x \xrightarrow{\text{real valued phase of } L_0} (TL_0)_x \longrightarrow (TL_1)_x$$ in that way.

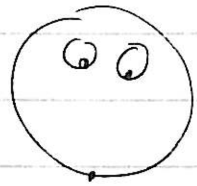
$$\mathcal{F}_x \xleftarrow{\text{real valued phase of } L_1} (TL_1)_x$$

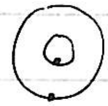
Count winding # & that's degree

defn : M CY ; α_L – constant
then L is special Lagrangian.

Formal theory of open/closed string theory.

Consider framed little disc operad $\mathcal{O}$



$\mathcal{O}(1) = \{$  $\} \simeq S^1$ homotopy eq.

(relative angle)

$\mathcal{O}(2) = \{$  $\} \simeq T^3$

all different: $x \cdot y$
$x \cdot \Delta y$
$\Delta x \cdot y$
$\Delta(x \cdot y)$

$\mathcal{O}(3) = \ldots$

Algebra over ~~this operad~~ $H_*(\mathcal{O})$ BV-algebra. — graded vector space

$\Delta : V \longrightarrow V[-1]$.

& product $V \times V \longrightarrow V$ commutative, associative

$\Delta^2 = 0$.

The failure of Δ to be a derivation is a Lie bracket $\{x, y\} = \Delta(xy) - x \cdot \Delta y - \Delta x \cdot y$

Jacobi identity comes from $\mathcal{O}(3)$.

— approximately a closed string genus zero ~~topological~~ field theory.

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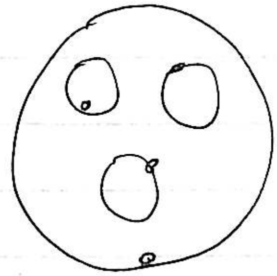
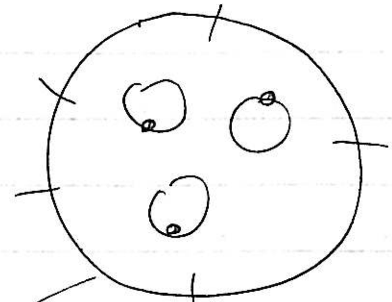
$d=3$:



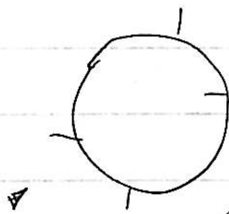
$V^{\otimes d} \longrightarrow V$

still have nonunital BV-algebra

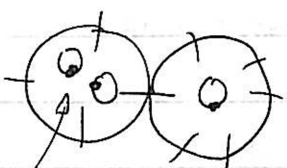


$V^{\otimes d} \longrightarrow W^{\otimes e}$

W – carries a graded symmetric pairing.

In particular  gives a map $\mathbb{C} \longrightarrow W^{\otimes e}$.

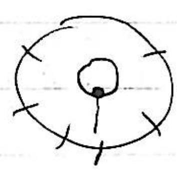
degenerate / compactify



disks still not allowed to collide

"open-closed" topological string theory.

Gives W the structure of an A_∞-algebra which is cyclic (Calabi-Yau)

Have a map $V \longrightarrow CC^*(W,W)$ 

cyclic chain?

$HH^*(W,W) \ni$ Connes' Boundary operator, Δ.

Sheel's product

not in general a q-iso (eg $W=0$)

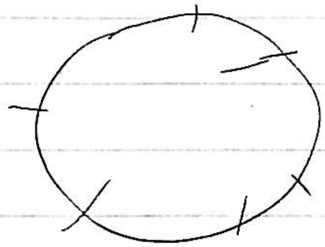
Physicists "reconstruction" $V = CC^*(W,W)$ & map is identity

4 closed string theory.

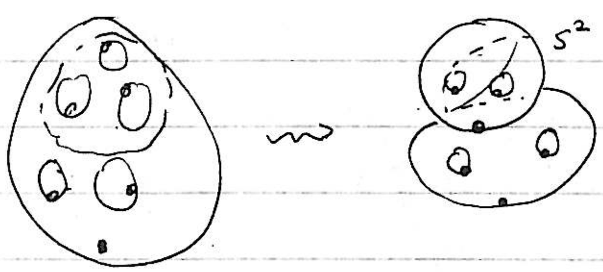
$\longleftarrow$ Givental's twisted loop group

What actually happens?

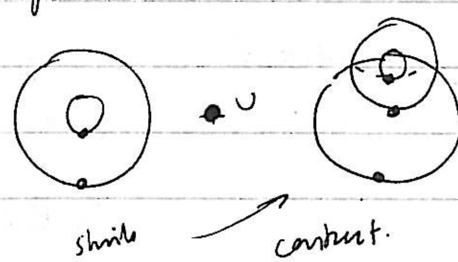
Moduli spaces are further compactified, to Deligne-Mumford spaces.



nothing happens



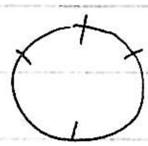
Closed string operad $\bar{\mathcal{O}}$

$\bar{\mathcal{O}}(1)$ = 

$\simeq pt$

$\bar{\mathcal{O}}(2)$ = bubble  $\simeq pt.$

should be D–M spaces.

$\bar{\mathcal{O}}(3) \simeq \bar{\mathcal{M}}_{0,4} \simeq S^2$ 

genuinely new 3-fold product.

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In the open string sector, we still have cyclic A_∞-str. $\qquad W$

On the closed string sector, we no longer have BV str. $\qquad V$

Now: Cohomological field theory (∞ # of generators)

open-closed string sector. ?

rmk
can't do reconstruction any more

We still have a map $V \longrightarrow CC^*(W,W)$
Cohomological field theory str. $\qquad$ homotopy BV

analyzed by
- Konbonth + Manin
- Dubrovin
- Givental.

If this is a quasi-iso$^{\underline{m}}$ then the Connes Bouchey operator vanishes.
In fact, the spectral sequence from $HH^*(W,W)$ to $HC^*_-(W,W)$ degenerates
negative

In particular $\Delta = 0 \Rightarrow \{-,-\} = 0$ on V
$V \longrightarrow CC^*(W,W)$ map of dgla's.
bracket is zero

—given A_∞
$\Rightarrow$ have family of A_∞-algebras over V

the Big Fukaya Category

fibre at 0 is usual Fukaya category

rmk: A_∞ categorys carry natural deformation.
(has to do w/ c_1)

$\mu^d \longrightarrow \hbar^{d-2}\mu^d.$
const $\qquad$ degree of map.