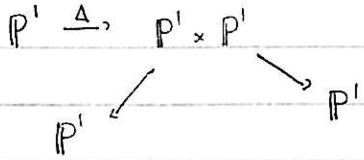
Exceptional Collections

$D^b(\mathbb{P}^1)$ – derived category associated with Abelian category of coherent sheaves on $\mathbb{P}^1$.

$\mathrm{Id} : D^b(\mathbb{P}^1) \to D^b(\mathbb{P}^1)$



$\mathrm{Id} \cong R\pi_{2*}(\mathcal{O}_\Delta \overset{L}{\otimes} \pi_1^* \_\_)$

↖ functions along Δ

$G: 0 \to \mathcal{O}(-1) \boxtimes \mathcal{O}(-1) \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to 0$

↗ deg -1 — pull back and tensor; deg 0

$\Phi_{G^{-1}}(F) := R\pi_{2*}(\pi_2^* \mathcal{O}(-1) \otimes \pi_1^* \mathcal{O}(-1) \otimes \pi_1^* F)$

$= R\pi_{2*}(\pi_2^* \mathcal{O}(-1) \otimes \pi_1^* F(-1))$

$= R\Gamma(F(-1)) \otimes \mathcal{O}(-1) \cong R\mathrm{Hom}(\mathcal{O}(+1), F) \otimes \mathcal{O}_{\mathbb{P}^1}(-1)$

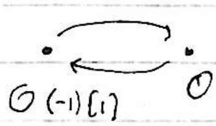
$\Phi_{G^0}(F) = R\Gamma(F) \otimes \mathcal{O} \cong R\mathrm{Hom}(\mathcal{O}, F) \otimes \mathcal{O}_{\mathbb{P}^1}$

$\mathrm{Id} \cong \Phi_G(\Sigma) = \{ R\mathrm{Hom}(\mathcal{O}(1), F) \underset{-1}{\otimes} \mathcal{O}_{\mathbb{P}}(-1) \to R\mathrm{Hom}(\mathcal{O}, F) \otimes \mathcal{O} \}$

$\mathcal{A}$ = Ext cat of $\mathcal{O}$ $\mathcal{O}(-1)[1]$

$D^b(\mathbb{P}^1) \cong D^b(\mathcal{A})$.

Quiver picture:



Kronecker quiver.

Defn: An <u>exceptional collection</u> of a triangulated category T consists of the following:

I ordered (finite) set

$\{Y_i\}_{i \in I}$ $Y_i \in T$

s.t. $RHom^k(Y_i, Y_j) = \begin{cases} 0 & \text{if } i > j \\ \mathbb{K} & \text{if } i = j,\ k = 0 \\ \text{~~0 if i=j, k~~ bounded in } k, \text{ finite dim'l} & \\ 0 & \text{if } i = j,\ k \neq 0 \end{cases}$

$End(\oplus Y_i, \oplus Y_j)$ is upper triangular.

Defn: A <u>full</u> exceptional collection is an exceptional collection s.t. $T(Y_i) \cong T$ where $T(Y_i)$ is the triangulated <u>hull</u>.

So the example we did for $\mathbb{P}^1$ gives a full exceptional collection.

E.g. $\{\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(1)\}$ $\{\mathcal{O}(-1)[1], \mathcal{O}\}$

Defn: Left mutation of an object Y by X

$$L_X Y = \left(\{ \bigoplus_i Hom(X[-i], Y) \otimes X[-i] \xrightarrow{ev} Y \} \right)$$

Defn: A left mutation by the ith object on an exceptional collection is the following collection:

$$\begin{array}{ccccccccc} & Y_0 & Y_1 & \cdots & Y_{i-1} & Y_i & \cdots & & Y_n \\ \sigma_{i-1} & \downarrow & \downarrow & & \swarrow & \searrow & & & \downarrow \\ \{Z_i\} & Y_0 & Y_1 & & L_{Y_{i-1}} Y_i & Y_{i-1} & Y_{i+1} & \cdots & Y_n \end{array}$$

(doesn't necessarily extend to automorphism)

Results: 1) $\exists$ The concept of a right mutation (If (A, B) are an exceptional pair)

2) $R\mathrm{Hom}(A, L_A B) = 0$

3) $R_A L_A B \cong B$

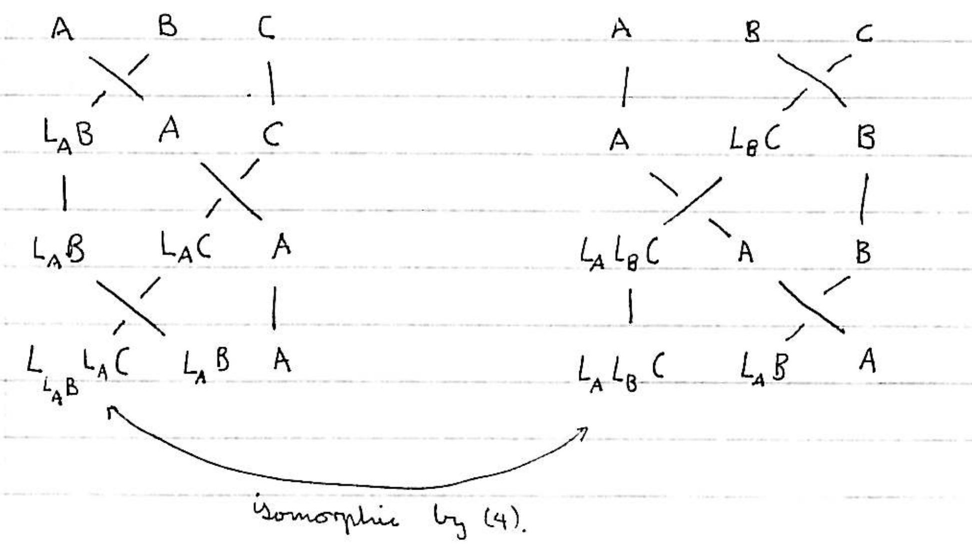
4) $L_{L_A B} L_A C \cong L_A L_B C$

$(A, B) \longrightarrow (L_A B, A)$

So a mutation takes an exceptional collection to an exceptional collection.

So the left mutation σ_{i-1} has an inverse.

(A, B, C)



E.g. from beginning: $(\mathcal{O}(-1)[1], \mathcal{O})$ $\qquad (\mathcal{O}, \mathcal{O}(1))$

$$\begin{array}{c} (\mathcal{O}(-1)[1], \mathcal{O}) \\ \parallel \\ L_{\mathcal{O}}(\mathcal{O}(1), \mathcal{O}) \end{array}$$

$\mathcal{O}(-1) \to \mathcal{O} \oplus \mathcal{O} \xrightarrow[y]{x} \mathcal{O}(1)$ take cone of morphism

$(\{\mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1)\} \cong \mathcal{O}(-1)[1]$

$\left(\mathcal{O}(-1)[1],\ \mathcal{O}(-n)\right)$ $n>0$ full exceptional collection

$\overset{\bullet}{\mathbb{C}}$ $\overset{\bullet}{\mathbb{C}}$ no ~~...~~ morphisms $\Rightarrow$ can't get to full thing...?

Defn: A directed A_∞ category, $A^{\to}(Y_i)$ is a strictly unital A_∞ category with a finite ordered set of objects $\{Y_i\}$

$$\mathrm{Hom}_{A^{\to}(Y_i)}(Y_i, Y_j) = \begin{cases} 0 & i>j \\ \mathbb{K}e_{Y_i} & i=j \\ \text{fin dim over } \mathbb{K} & i<j \end{cases}$$

In $H(A^{\to}(Y_i))$ this is an exceptional collection.

F: A_∞ functor

$F: A^{\to}(Y_i) \longrightarrow T$

$\Rightarrow \exists$ A_∞ functor from twisted complexes to T, (isomorphic onto hull of image?).

$Tw\, A^{\to}(Y_i)$

$\{Y_i\} \subset Tw\, A^{\to}(Y_i)$

mutate: $\downarrow$ $\cup$

Full subcat $\{Z_i\}$

There is a braid group action on the collection of directed A_∞ cat.

$A^{\to}(Y_0, \ldots Y_n) \longrightarrow A^{\to}(Y_0, \ldots, L_{Y_{i-1}} Y_i, Y_{i-1}, \ldots)$