Lecture 8 Combinatorial Fukaya Categorys

$(\underbrace{M}, J, \omega, \theta)$ $\omega = d\theta$ exact

Riemann surface w/ boundary

We compute directed Fukaya Category

$\theta|_{L_i} = df_i$

Def[n] Let $(L_1, \dots, L_n)$ be a collection of Lagrangians, exact.

Here: S^1's embedded in M.

(allows us to ignore bubbling & set $t=1$ in Novikov ring, i.e. forget).

Note exactness $\Rightarrow$ L_i not nullhomologous.

$\mathcal{F}(L_1, \dots, L_n)^{\rightarrow}$ over field $\mathbb{K}$, char $\mathbb{K} = 2$ transverse double intersections

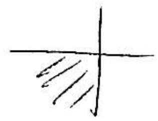
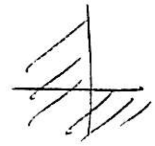
$ob(\mathcal{F}) = \{L_i\}$ (in general position) & no triple intersections

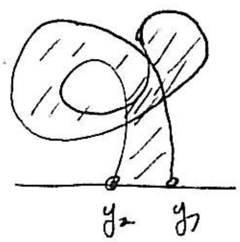
$$hom(L_i, L_j) = \begin{cases} \mathbb{K}^{L_i \cap L_j} & i<j \\ \mathbb{K}e & i=j \\ 0 & i>j \end{cases}$$

Let's define A_∞-structure maps μ^d's combinatorially.

Given $y_0 \in L_{i_0} \cap L_{i_d}$

$y_k \in L_{i_{k-1}} \cap L_{i_k}$ $k = 1, \dots, d$,

Let $n_d(y_0, \dots, y_d)$ = # immersed d-gons in M like this : (PTO) corners  not 



(immersed!)

Structure maps $\mu^d: \hom(i_{d-1}, i_d) \otimes \cdots \otimes \hom(L_{i_0}, L_{i_1}) \longrightarrow \hom(L_{i_0}, L_{i_d})$

$$\mu^d(y_d \otimes \cdots \otimes y_1) = \sum_{y_0} n_d(y_0, \ldots, y_d)\, y_0$$

$$(= 0 \text{ unless } i_0 < i_1 < \cdots < i_d)$$

This is an A_∞-category.

Consider the non-unital, non-commutative k-algebra

$$T(L_1, \ldots, L_n) = \bigoplus_{d \geq 1} T_d$$

$$T_d = \bigoplus_{i_0 < \cdots < i_d} \hom(L_{i_0}, L_{i_1})^* \otimes \cdots \otimes \hom(L_{i_{d-1}}, L_{i_d})^*$$

Define $\alpha \otimes \beta = 0$ if wrong ordering.

really dual.

T_d has a basis $\{ a_1 \otimes \cdots \otimes a_d \mid a_j \in L_{i_j} \cap L_{i_{j+1}} \quad i_1 < \cdots < i_d \}$

Define $\delta: T \longrightarrow T$ by $\delta(a_0) = \sum_d n_d(a_0, \ldots, a_d)\, a_1 \cdots a_d$

$(= \mu^{1*} + \mu^{2*} + \cdots)$
gives different # terms

$\delta: T_1 \to T$

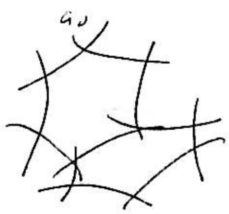
extend δ to T by Leibniz's rule

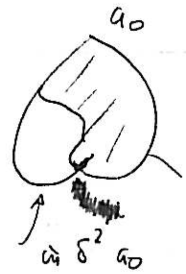
Prop^{n} $\delta^2 = 0$

$$\delta^2 a_0 = \delta\Big(\sum_{\text{polygons}} a_1 \cdots a_d\Big)$$

$$= \sum_i \sum_{\text{polygons}} a_1 \cdots \delta a_i \cdots a_d$$

$$= \sum_{\text{polygons}} \sum_{\text{polygons}} a_1 \cdots a_{i-1}\, b_1 \cdots b_\ell\, a_{i+1} \cdots a_d$$





(cancels ... ?)

$\Rightarrow$ terms in $\delta^2 a_0$ cancel in pairs

$\Rightarrow$ $\delta^2 a_0 = 0$

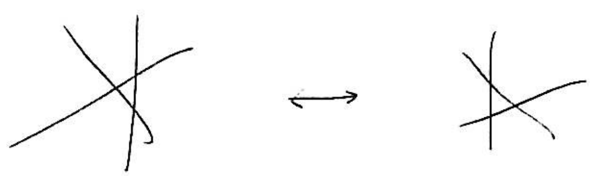
$\Rightarrow$ $\delta^2 = 0$

Corollary μ^d define A_∞-structure

Prop^{n} Hamiltonian Isotopy of L_i changes $\overrightarrow{\mathcal{F}(L_1, \ldots, L_n)}$ by a quasi-isomorphism.

proof

I



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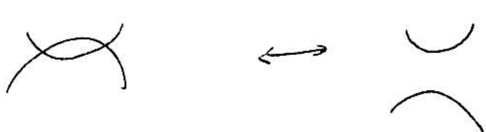
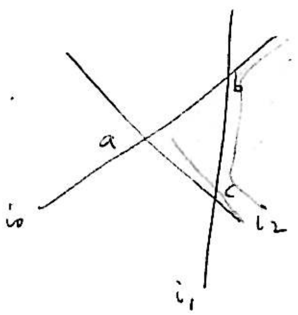
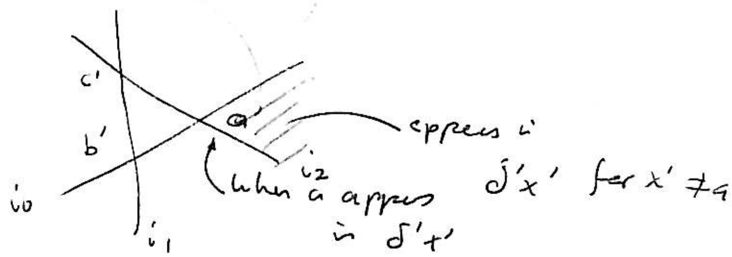


I.





WLOG i_0 is smallest

if not $(i_0 < i_1 < i_2)$ $\quad a \longmapsto a'$, $b \longmapsto b'$, $c \longmapsto c'$

If $i_0 < i_1 < i_2$,

$f: a \longmapsto a' + b'c'$

$x \longmapsto x'$ all other x

$a + bc \longleftarrow a'$: g

$x \longleftarrow x'$

this means $f(a) = a + bc$

$f_1(a^*) = a^* + \cdots$

$f_2(b^*, c^*) = a^* + \cdots$

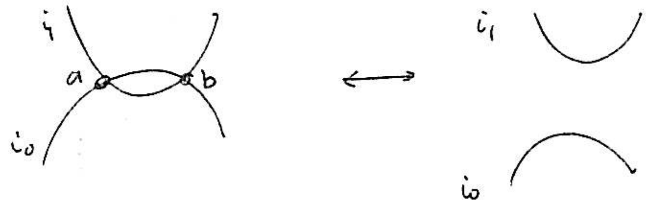
claim: f, g are chain maps

for $x \neq a_0$, $\quad \delta'(fx) = \delta' x' = f \delta x$

$\Rightarrow$ chain map

$\Rightarrow$ A_∞ morphism

$\Rightarrow$ f, g strict inverses & are A_∞ - isomorphisms

II 

since, $\delta a = b + v$,

Define $f: a \longmapsto 0$

$b \longmapsto v'$

$x \longmapsto x'$ all other x

Let $h_1 : T_1 \longrightarrow T_1[1]$

$h_1(b) = a$

$h_1(x) = 0$, else

$g_1 : T_1' \longrightarrow T_1$

$g_1(x') = x + h_1 \delta_1 x$

f is an A_∞-morphism.

$f_1 g_1 = \mathrm{Id}$

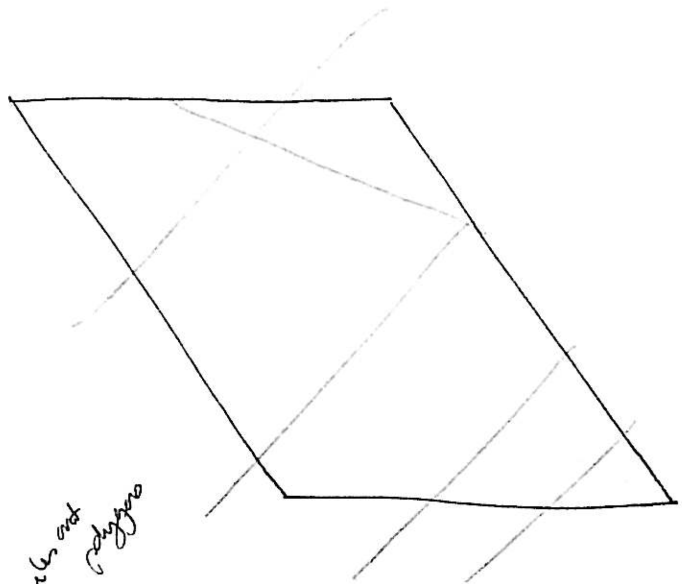
$g_1 f_1 = \mathrm{Id} + h_1 \delta_1 + \delta_1 h_1$

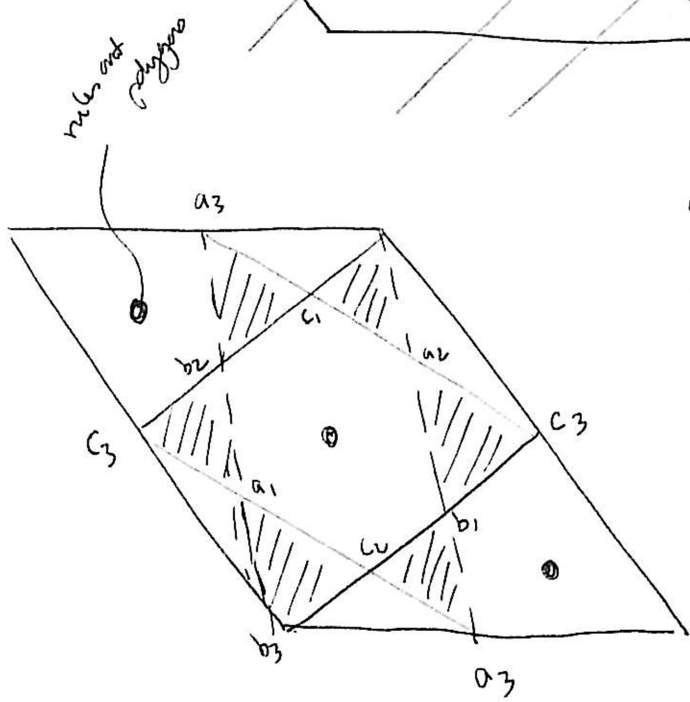
$\Big\}$ f_1, g_1 inverses on level of homology

Apply perturbation Lemma

— can extend g_1 to a $g: T' \longrightarrow T$ s.t.

$f \circ g$, $g \circ f$ are homotopic to id example





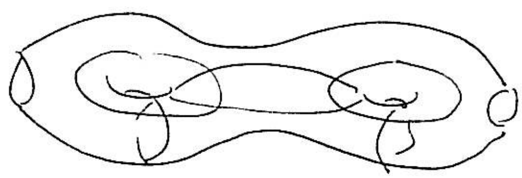
on torus

$$a_i b_j = \varepsilon_{ijk} c_k$$

$$\begin{tikzcd} \underset{L_1}{\circ} \arrow[r, "V"'] \arrow[rr, bend right, "\Lambda^2 V"'] & \underset{L_2}{\circ} \arrow[r, "V"'] & \underset{L_3}{\circ} \end{tikzcd}$$

$V \cong \mathbb{C}^3$

Warning!



full subcategory ~~with obj~~ $L_1, \ldots, L_5 \subset \mathcal{F}(M)$

<u>Thm</u> This is not formal