## $A_\infty$ structures

~~$A^\cdot$ = graded algebra~~

Defn: A differential graded algebra is a graded algebra $A^\cdot$ with a differential $d$ s.t.

$$d(a.b) = da.b + (-1)^{|a|} a.db$$

A morphism of DGAs is an algebra + chain morphism

Defn: A quasi-isomorphism is a morphism inducing iso on homology

Problem: quasi-isomorphisms aren't necessarily invertible.

E.g. $k[\alpha]/\alpha^2$, $d=0$ $\Rightarrow k \oplus 0 \oplus \alpha$

E.g. $k[\beta_1, \beta_2, \beta_3]/\beta_1^2$ $\deg \beta_1 = \deg \beta_2 = 2$ $\deg \beta_3 = 1$

$d\beta_1 = 0$

$d\beta_2 = 0$

$d\beta_3 = \beta_1$

$k \oplus 0 \oplus k \oplus H^3 \oplus H^4 \oplus \ldots$

kill $H^3$ by adding vars of deg 2

kill $H^4$ by adding vars of deg 3.

$\rightsquigarrow$

$k[\beta_1, \beta_2, \ldots]/\beta_1^2$

$\downarrow$ homology

$k \oplus 0 \oplus k$

E.g. Rational homotopy theory

Study rat. homotopy type ~~iff~~ of $X$ by studying $C^\cdot(X, \mathbb{Q})$ w/ cup product.

Thm (roughly): $X, Y$ have same rat. htpy type iff $\exists$ chain of q-isoms

$$C^\cdot(X,\mathbb{Q}) \to D_1 \leftarrow D_2 \to \ldots \leftarrow D_n \to C^\cdot(Y,\mathbb{Q})$$

To fix: $A_\infty$-algebras $\supset$ dgas (not full subcategory)

Another example: de Rham cohomology $\cong$ singular cohomology but not as rings.

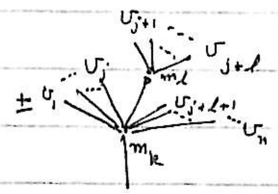
$A_\infty$-algebra

Def 1: Given by following data:

(1) $V^\bullet$ $\mathbb{Z}$-graded vec. space

(2) operations $m_n : V^{\otimes n} \to V[2-n]$ $\quad n \geq 1$

(3) satisfying

$$\sum_{k+l=n+1} \sum_{j=0}^{k-1} \pm$$  $= 0$



$n=1$: $m_1 = d$ $\qquad m_1(m_1(v_1)) = 0 \iff d^2 = 0$

$n=2$: Leibniz rule, $m_2$ = multiplication

$n=3$: $v_1(v_2 \cdot v_3) - (v_1 \cdot v_2)\cdot v_3 = d\, m_3(v_1, v_2, v_3) \pm m_3(dv_1, v_2, v_3)$

$\pm\, m_3(v_1, dv_2, v_3)$

$\pm\, m_3(v_1, v_2, dv_3)$

E.g. A dga is an $A_\infty$-alg. with $m_n = 0$ $\forall\ n \geq 3$.

Defn 2: Given by following data

(1) $V^\bullet$ $\mathbb{Z}$-graded vector space $\qquad$ "formal manifold"

Let $C(V) = \bigoplus_{n \geq 1} V[1]^{\otimes n}$ $\qquad$ "dual or predual to algebra of functions"

cofree coalgebra cogenerated by $V[1]$.

coproduct: $\Delta(v_1 \otimes \dots \otimes v_n) = \sum (v_1 \otimes \dots \otimes v_k) \otimes (v_{k+1} \otimes \dots \otimes v_n)$

(2) coderivation

$C(V) \xrightarrow{Q} C(V)[1]$ $\qquad$ "vector field with $[Q, Q] = 0$"

Prop: Def 1 + Def 2 are equivalent

Pf: $V[1]^{\otimes n} \longrightarrow V[1][1]$

$\Updownarrow$

$V^{\otimes n} \longrightarrow V[2-n]$

"Koszul duality of operads".

Defn: $A_\infty$-morphism is a map of co-free co-algebras that commutes with the differential $Q$

$$\begin{tikzcd} C(V_1) \arrow[r, "Q_1"] \arrow[d, "F"'] & C(V_1)[1] \arrow[d, "F[1]"] \\ C(V_2) \arrow[r, "Q_2"] & C(V_2)[1] \end{tikzcd}$$

equivalently:

$$F : C(V) \to C(W)$$

$$F_n : V^{\otimes n} \to W[1-n] \qquad \text{"Taylor coefficients of } F\text{"}$$

satisfying:

$$d \circ F_1 = F_1 \circ d$$

$$F_1(v_1) \cdot F(v_2) = F_1(v_1 \cdot v_2) \pm F_2(d(v_1), v_2) \pm F_2(v_1, d(v_2)).$$

Perturbation Lemma / Transfer Lemma:

Let $(A, m_n)$ be an $A_\infty$-algebra, $\pi : A \to A$ morphism of complexes satisfying $\pi^2 = \pi$, $d\pi = \pi d$.

Assume there is a chain homotopy $H : A \to A[-1]$.

$$1 - \pi = dH + Hd.$$

Let $B$ be the image of $\pi$. Then the $A_\infty$ structure on $A$ induces one on $B$.

E.g. $A^\bullet = (\ker d^\bullet) \oplus \mathrm{comp}_1^\bullet$

$\qquad = \cancel{\ker d} \ \mathrm{im}\, d^{\bullet-1} \oplus \underbrace{\mathrm{comp}_2^\bullet}_{\cong H^\bullet(A,d)} \oplus \mathrm{comp}_1^\bullet$

$$\pi : A^\bullet \xrightarrow{p} H^\bullet(A, d) \xrightarrow{i} A$$

$$H : (\mathrm{im}\, d^{\bullet-1}) \oplus \mathrm{comp}_2^\bullet \oplus \mathrm{comp}_1^\bullet \longrightarrow \mathrm{im}\, d^{\bullet-2} \oplus \mathrm{comp}_2^{\bullet-1} \oplus \mathrm{comp}_1^{\bullet-1}$$

$$= d^{-1} : (\mathrm{im} \to \mathrm{comp}_1^{\bullet-1}) \oplus 0 \oplus 0$$

$$B = H^\bullet(A, d).$$

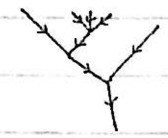
Let $i : B \hookrightarrow A$ be inclusion, $p : A \to B$ projection.

$$m_1^B = p \circ m_1 \circ i$$

$$m_2^B = p \circ m_2 \circ (i \otimes i)$$

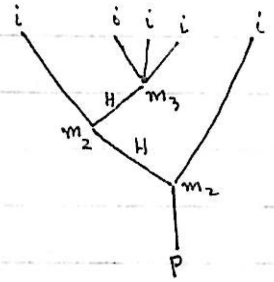
$$m_n^B = \sum_T m_T$$ where $T$ is an oriented planar rooted tree with $n$ tail vertices an s.t. all int. vertices have # incoming edges $\geq 2$.

E.g.



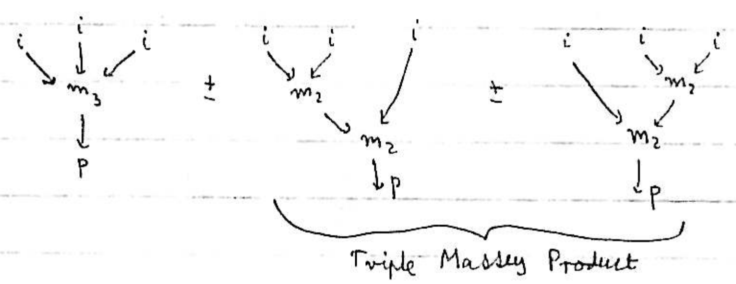
$m_T$ = for each tail plug in an $i$
for each internal vertex plug in an $m_k$
for each internal edge put an $H$
put a $p$ on the root.

E.g.




E.g. $A$ = dga
$B = H^\cdot(A)$

$m_3^B =$




<u>Prop</u>: There are $A_\infty$-quasi-isomorphisms $F: A \to B$, $G: B \to A$ s.t.
$F_1: A \to B$ is $p$, $G_1: B \to A$ is $i$.

Cor: $\mathcal{A}_\infty$ quasi-isomorphisms are invertible.

Pf: $A \xrightarrow[\mathcal{A}_\infty\text{-quasi-iso}]{} B$

Any morphism of $\mathcal{A}_\infty$-algebras which is an iso on vector spaces (generators) is invertible ~~as~~ (i.e. an iso) in $\mathcal{A}_\infty$ category.

Thus we have

$$\begin{tikzcd} A \arrow[r, "\mathcal{A}_\infty \text{ quasi-iso}"'] \arrow[d, leftrightarrow, "\cong"'] & B \arrow[d, "\cong"] \\ H^\cdot(A) \arrow[r, leftrightarrow, "\cong"] & H^\cdot(B) \end{tikzcd}$$

E.g. Thm: $X$ compact Kähler mfd, $\underset{\text{dga}}{(\Omega^\cdot(X), d)} \xrightarrow[\text{quasi-iso}]{\text{dga}} Q \xleftarrow[\text{quasi-iso}]{\text{dga}} \underset{\text{dga}}{\left(H^\cdot(X), d=0\right)}$

$\Rightarrow (\Omega^\cdot(X), d)$ quasi-isomorphic to $(H^\cdot(X), d=0)$ as $\mathcal{A}_\infty$ algebras, hence as dgas.

Cor: The $\mathcal{A}_\infty$ structure ~~of~~ on $H^\cdot(X)$ is trivial.

$\Rightarrow$ Massey products are all $0$.

Cor: The rational homotopy type of a compact Kähler is a formal consequence of its cohomology ring.