Formal aspects of Floer theory

Morse Theory

$f : M \to \mathbb{R}$

critical points: $df = 0$.

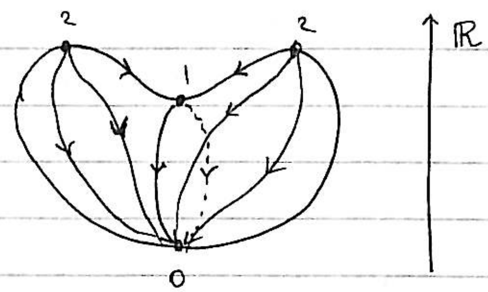
In local coordinates near a critical point p

$$f = -x_1^2 - x_2^2 - \dots - x_i^2 + x_{i+1}^2 + \dots + x_n^2$$

$\Rightarrow p$ is of index i.

Choose a metric g on M.

E.g.



$M = S^2$
with ~~kes~~ f = height function

Define the gradient flow of f, $\dot{u}(t) = -\nabla f$. Time-t flow is φ_t.
For a critical point p, define

$W^s(p) = \{ z \in M \mid \lim_{s\to\infty} \varphi_s(z) = x \}$ "stable manifold" - dim $= n-i$

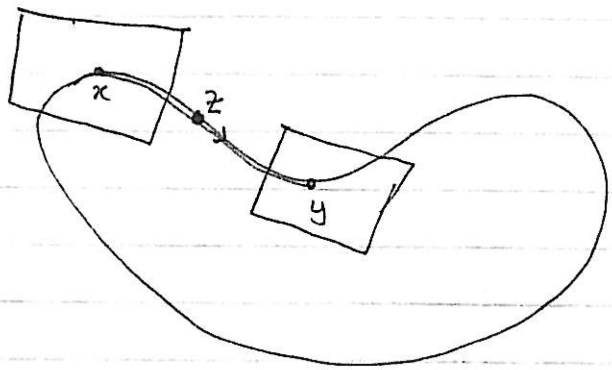
$W^u(q) = \{$ " $\quad {}_{s\to-\infty}$ " " " "unstable manifold". - dim = index

$\mathcal{M}(x,y) = W^s(y) \cap W^u(x)$ = flow lines from x to y.
If (f,g) is Morse-Smale, $\mathcal{M}(x,y)$ is a smooth mfd of dim $\mathrm{ind}(x) - \mathrm{ind}(y)$.

$\hat{\mathcal{M}}(x,y) = \mathcal{M}(x,y)/\mathbb{R}$ - moduli space of flow lines from x to y.

Defn: $CM_k = \bigoplus_{\mathrm{ind}(p)=k} \mathbb{Z}\langle p \rangle$

Stable framing: we can flow the tangent space at x along to y:



$z \in \mathcal{M}(x,y)$. $\quad T_z W^u(x) \big/ \text{direction of flow lines} \quad \cong \quad T_y W^u(x)$.

$$T_z W^u(x) \cong T_x W^u(x) \qquad \text{(as } W^u(x) \text{ is a disc)}$$
$$\nu_z W^u(x) \cong T_x W^s(x)$$
$$\begin{aligned} T_z \mathcal{M}(x,y) &= T_z W^u(x) \cap T_z W^s(y) \\ &= \ker\left(T_z W^u(x) \longrightarrow \nu_z W^s(y) \right) \\ &= \ker\left(T_x W^u(x) \underset{\substack{\text{surjective map} \\ \text{depends on } z}}{\longrightarrow} T_y W^u(y) \right) \end{aligned}$$

$$T\mathcal{M} \oplus T_y W^u(y) \cong T_x W^u(x)$$

Now define the Morse differential:

$$\partial\langle x \rangle = \sum_{\text{ind}(y) = \text{ind}(x) - 1} \varepsilon \langle y \rangle \qquad \text{where } \varepsilon = \text{some sign.}$$

To determine ε, choose an orientation on $W^u(y)$ and choose to orient $T\mathcal{M}$ by flowing 'down'. Then let ε be $+1$ if the decomposition $T\mathcal{M} \oplus T_y W^u(y) \cong T_x W^u(x)$ matches orientations, -1 otherwise.

$\underline{\partial^2 = 0}$: $\langle \partial^2 x, z \rangle$ is counted by broken trajectories $x \to y \to z$. These appear as boundary points of the 1-mfd $\widehat{\mathcal{M}}(x,z)$, which is a union of circles and intervals. The boundary points of intervals have opposite sign.

A Morse function leads to a handle construction of M, from which one can show Morse homology is $\cong$ singular homology.

Note: $HM^*(M)$ is

① $\mathbb{Z}$-graded

② has $\mathbb{Z}$-coefficients

③ captures topology

④ nice moduli spaces of flows.

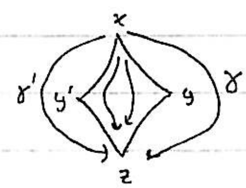
Suppose instead of df use a closed 1-form α.

Problems: - may have orbits

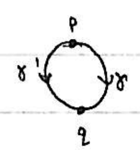
Solution: Go to some cover of M, so that α is exact.

Use Novikov ring, $\Lambda = \left\{ \sum_{\substack{i\in\mathbb{Z} \\ \nu_i \to \infty}} a_i t^{\nu_i} \right\}$

Now $\partial\langle x\rangle = \sum_{x\to y} \varepsilon\, t^{\int\alpha} \langle y\rangle$

$\partial^2 = 0$ still holds because  $\int_\gamma \alpha = \int_{\gamma'} \alpha$ by Stokes.

E.g. $M = S^1$, $\alpha = d\theta \Rightarrow HM^* = 0$

or S^1 with α,  $\int_\gamma \alpha \neq \int_{\gamma'} \alpha$

$\Rightarrow \partial\langle p\rangle = \left(t^{c_1} - t^{c_2}\right)\langle q\rangle \quad \Rightarrow HM^* = 0$

(You recover the homology of the cover modulo a nbhd of $-\infty$).

(truly it's a direct limit of $\tilde{M}/_{f^{-1}(-\infty,-c)}$ as $c\to\infty$,

$\tilde{M}$ = cover of M, $\pi^*\alpha = df$).

<u>Defn</u>: E is a polarised Hilbert space if it has a $\mathbb{P}$ such that

$\mathbb{P}^2 = I + \text{cpct}$ ($\mathbb{P}$ is chosen up to compact operator)

$E = E_+ \oplus E_-$ (+1 and −1 eigenspaces, defined up to finite-dim spaces)

E.g. $\gamma \in \mathcal{L}M$

vector fields along γ lie in $T_\gamma \mathcal{L}M$.

$\mathbb{P}: \ X \longmapsto J \cdot \nabla_{\frac{\partial}{\partial\theta}} X$ (J = a.c. structure on M).

Defn: $$GL_{res}(E) = \left\{ \begin{array}{c|cc} & E_+ & E_- \\ E_+ & A & B \\ E_- & C & D \end{array} \;\middle|\; \begin{array}{l} A, D \text{ Fredholm} \\ B, C \text{ compact} \end{array} \right\}$$

$\|$

$\mathbb{Z} \times BO$

Defn: M is a Hilbert manifold if $T_x M$ have polarisation (i.e. structure group of TM is reduced to GL_{res}).

So there is a map

$$M \longrightarrow BGL_{res} = \mathcal{U}(\infty)/O(\infty)$$

$$\pi_1(\mathcal{U}(\infty)/O(\infty)) = \mathbb{Z}.$$

Moving along a loop in M may mix up E_+ and E_- ... this is called "spectral flow" ... we can only define a grading ~~mod~~ in

$$\mathbb{Z}/\operatorname{im}(\pi_1(M) \to \pi_1(\mathcal{U}(\infty)/O(\infty))).$$

in Floer theory.

Furthermore we need to orient our moduli spaces in order to give signs in the Floer complex $\Rightarrow$ a lot of the time we must use $\mathbb{Z}_2$ coefficients.

Defn (Floer theory): (M, ω) - symp. mfd. L_0, L_1 Lagrangians

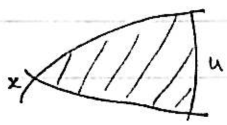
$$\Omega(L_0, L_1) = \{ u \in C^\infty(([0,1], 0, 1), (M, L_0, L_1)) \}$$

$X \in T_u \Omega(L_0, L_1)$ is a vector field in $u^* TM$.

Define $\alpha(X) = \int_0^1 \omega(\dot{u}(t), X)\, dt$

$\alpha = d\mathcal{A}$ $\mathcal{A}$ = multivalued in general

$x \in L_0 \cap L_1$



$\Phi : \mathbb{R} \times [0,1] \to M$

$\mathcal{A}(u) = \int \Phi^* \omega.$

Floer theory = doing Morse theory with this $\mathcal{A}$.