talk #4
Jacob Lurie
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Lie*-algebras

Recall: (right) (cplxs of) D-modules on $X/\mathbb{C}$ $\qquad$ sheaves on X

$$\begin{array}{ccc} \cup\!\!\text{l} & & \cup\!\!\text{l} \\ \text{regular holonomic D-modules} & \xleftrightarrow{\text{Riemann-Hilbert}} & \text{constructible sheaves} \\ M & \longrightarrow & M \otimes_{D_X} \mathcal{O}_X^{an} \\ \boxtimes & & \boxtimes \\ \pi_+ & \longleftrightarrow & \pi_* \\ \pi^+ & & \pi^! \text{ (right adj. to } \pi_!) \end{array}$$

analytic functions

note: this version of R-H is covariant

But constructible sheaves have more operations:

$$\left.\begin{array}{l} \pi_*,\ \pi_! \\ \pi^*,\ \pi^! \\ \otimes,\ R\mathrm{Hom} \end{array}\right\} \text{Grothendieck's 6 operations}$$

(Note: Don't have these in full context of all D-modules.)

In sheaves, $F \otimes G \cong \Delta^*(F \boxtimes G)$

& can define for D-modules $\Delta^+(M \boxtimes N)$ $\sim$ for left D-modules $M \otimes N$

for right D-modules

$$F \otimes^! G := \Delta^!(F \boxtimes G)$$

(verdier dual to usual $\otimes$)
but don't have this in full context of D-mod

An algebra A
is a (sheaf/whatever)
w/ a map $A \otimes A \longrightarrow A$

$$\begin{array}{c} A \otimes A \\ \text{is} \\ \Delta^*(A \boxtimes A) \\ \Updownarrow \\ A \boxtimes A \longrightarrow \Delta_* A \end{array}$$

& this is in our dictionary.

Def: a pseudo-tensor category is ① a collection of objects
② "multilinear maps"
"$\mathrm{Mult}(P_1 \otimes \dots \otimes P_n, Q)$" not actually constructed in terms of a $\otimes$, which may not exist.
③ composition laws, associativity...

Ex: $*$-pseudotensor str. on D_X-mods

$$\mathrm{Mult}(M_1, \dots, M_k; N) \simeq \mathrm{Hom}_{D_{X^k}}(M_1 \boxtimes \dots \boxtimes M_k, \Delta_+ N)$$

Def: A Lie*-alg on X is a Lie-algebra in D_X-mods,
i.e., a D-module L w/ a map $L \boxtimes L \longrightarrow \Delta_+ L$ satisfying skew sym. & Jacobi.

Via Riemann-Hilbert corr.,

$$\begin{array}{c}\text{Lie}^*\text{-algebras } L \\ \text{w/ } L \text{ regular holomic}\end{array} \Longleftrightarrow \begin{array}{c}\text{constructible sheaves of} \\ \text{Lie-algebras on } X.\end{array}$$

but the Lie*-algebras we like aren't reg. holonomic.

Algebraic analogue of R-H:

$$\begin{array}{ccc}\text{right } D_X\text{-mods} & \xrightarrow[h]{} & \text{sheaves on } X \\ \rotatebox{90}{$\in$} & & \\ M & \longrightarrow & M \otimes_{D_X} \mathcal{O}_X\end{array}$$

Remark: h commutes w/ pushford along immersions, &
h commutes w/ $\boxtimes$.

So, if L is a Lie*-algebra: $L \boxtimes L \longrightarrow \Delta_+ L$
$\Longrightarrow$ $h(L) \boxtimes h(L) \longrightarrow \Delta_* h(L)$
$\Longrightarrow$ $h(L)$ is a sheaf of Lie algebras on X.

More generally, if A is a q-coh. sheaf of $\mathcal{D}_X^{\text{comm.}}$-algebras (a com. alg. in cat. of left $\mathcal{D}_X$-mods)
define $h_A(M) = h(M \otimes A)$ "coeff. twisted by A"
& same remarks apply to h_A,

$$h_A : \text{Lie}^*\text{-alg} \longrightarrow \begin{matrix}\text{sheaves of} \\ \text{Lie alg on } X\end{matrix}$$

& if we use "all A at once," then this functor is fully faithful.

<u>Construction</u>:

Induction $\mathcal{O}_X\text{-mods} \xrightarrow{\text{Ind}} \mathcal{D}_X\text{-mods}$

$$M \longmapsto M \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

Observe that Ind commutes w/ $\boxtimes$ & π_+:

$X \xrightarrow{\pi} Y$, $\quad \pi_+ M = \pi_*\left(M \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\right)$

$$= \pi_*\left(M \otimes_{\mathcal{D}_X} \pi^*\mathcal{D}_Y\right)$$

& If $M = F \otimes_{\mathcal{O}_X} \mathcal{D}_X$ then

$$= \pi_*\left(F \otimes_{\mathcal{O}_X} \pi^*\mathcal{D}_Y\right)$$

$$= \pi_*\left(F \otimes_{\mathcal{O}_Y} \mathcal{D}_Y\right)$$

by push-pull formula ("projection")

Upshot is that if F a q.coh sh. of Lie algs on X then $\text{Ind}(F)$ is a Lie*-algebra

$$F \otimes F \xrightarrow{[\,,\,]} F$$

$$\Delta^*(F \boxtimes F) \longrightarrow F$$

$$F \boxtimes F \longrightarrow \Delta_* F$$

$$\Downarrow$$

$$\text{Ind}(F) \boxtimes \text{Ind}(F) \longrightarrow \Delta_+ \text{Ind}(F)$$

<u>Example</u>: $\mathfrak{g}$ fin. dimensional semi-simple Lie algebra

Then $\mathfrak{g}_{\mathcal{O}} := \mathfrak{g} \otimes \mathcal{O}_X$ a q. coh. sh. of Lie algs on X.

$\mathfrak{g}_{\mathcal{D}} := \mathrm{Ind}(\mathfrak{g}_{\mathcal{O}}) = \mathfrak{g} \otimes D_X$

is a Lie*-alg

What is $h(\mathfrak{g}_{\mathcal{D}})$?

Answer: $X \xrightarrow{\pi} *$

$$\Gamma(X; h(\mathfrak{g}_{\mathcal{D}})) = \pi_+ \mathfrak{g}_{\mathcal{D}}$$
$$= \pi_* \mathfrak{g}_{\mathcal{O}}$$
$$= \mathfrak{g} \otimes \text{functions on } X$$

If X = formal disk, $h(\mathfrak{g}_{\mathcal{D}}) = \mathfrak{g}[[t]]$

& X = punctured disk, $h(\mathfrak{g}_{\mathcal{D}}) = \mathfrak{g}((t))$

In setting of Lie*-alg, <u>can consider central extensions</u>:

$$0 \to \mathbb{C} \to \widehat{\mathfrak{g}((t))} \to \mathfrak{g}((t)) \to 0$$

<u>Q</u>: how to get corresponding Lie*-algebra for $\widehat{\mathfrak{g}((t))}$?

<u>Idea</u>: Make central extension of Lie*-alg $\mathfrak{g}_{\mathcal{D}}$ by something whose cohomology is $\mathbb{C}$. By ω_X (in degree zero).

X a curve:

Fix invariant sym. bilinear form $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, eg., Killing form

<u>Observation</u>: $\exists$ map of sheaves $\mathfrak{g}_{\mathcal{O}} \boxtimes \mathfrak{g}_{\mathcal{O}} \xrightarrow{b} \Delta_* \omega_X$

$$f, f' \longmapsto \langle df, f' \rangle$$

<u>problems</u>: b <u>not</u> a map of $\mathcal{O}_X$-modules, (1)
since d involved

& <u>not</u> skew symmetric. (2)

By def., D_X = alg of diff. ops $\mathcal{O}_X \to \mathcal{O}_X$

more generally for M, N $\mathcal{O}_X$-mods

then $\mathrm{Hom}_{\mathcal{O}_X}(M, N \otimes D_X)$ = diff. ops from M to N

is

$\mathrm{Hom}_{D_X}(M \otimes_{\mathcal{O}_X} D_X, N \otimes_{\mathcal{O}_X} D_X)$

The map b comes from a map of D-mods

$$\mathfrak{g}_D \boxtimes \mathfrak{g}_D \longrightarrow \Delta_+(\omega_X \otimes_{\mathcal{O}_X} D_X) \leftarrow \mathrm{Ind}(\omega_X)$$

This is one of our "multilinear operations".

fixes problem(1).

ω_X is already a right D-module, so $\exists$ a counit of adjunct.

$$0 \longrightarrow \mathcal{O}_X \otimes_{\mathcal{O}_X} D_X \xrightarrow{d} \omega_X \otimes_{\mathcal{O}_X} D_X \xrightarrow{\text{counit}} \omega_X$$

(like Poincaré Lemma)

get an exact sequence

& compose this

& claim is that $\Pi_+ \omega_X$

the composition is skew symmetric.

(uses $\langle df, f' \rangle + \langle f, df' \rangle = d\langle f, f' \rangle$)

& It defines a 2-cocycle of $\mathfrak{g}_D$ w/coef. in ω_X

$\Longrightarrow$ gives a central extension

$$0 \longrightarrow \omega_X \longrightarrow \widetilde{\mathfrak{g}_D} \longrightarrow \mathfrak{g}_D \longrightarrow 0$$

(this is an example of something hard to do w/ q.coh. sheaves of Lie alg)

Ex: X = formal disk

$h(\omega_X)$ is cokernel of "differentiation wrt to x"

$\omega_X = \{dx, x\,dx, \ldots\}$ basis $\Rightarrow h(\omega_X) = 0$

so $h(\widetilde{\mathfrak{g}_D}) = h(\mathfrak{g}_D) = \mathfrak{g}[[t]]$

Ex: X = punc. disk

$\omega_X = \{\ldots, x^{-1}dx, dx, x\,dx, \ldots\}$

then $h(\omega_X) = \mathbb{C}$

& $0 \longrightarrow \mathbb{C} \longrightarrow h(\widetilde{\mathfrak{g}_D}) \longrightarrow \mathfrak{g}((t))$

Kac-Moody central extension

manifestly splits over usual power series

Lie*-modules

Def: an L-Lie*-module is a D_X-module $\mathcal{M}$ w/ a map $L \boxtimes \mathcal{M} \xrightarrow{\psi} \Delta_+ \mathcal{M}$ satisfying "identities"...

For simplicity, $L = \mathfrak{g}_{\mathcal{D}}$, $X = \mathbb{A}^1$, & $\mathcal{M}$ is supported at $\{0\}$

Note Kashiwara's thm: $Y \overset{\pi}{\hookrightarrow} X$ then π_+ induces an equiv.

$$D_Y\text{-mods} \xrightarrow{\simeq} D_X\text{-mods}$$

set theoret. supp. on Y

so $i : \{0\} \hookrightarrow \mathbb{A}^1$

then $\mathcal{M} = i_+ V$, V some vector space

ψ is determined by a D_X-module map

$$\mathfrak{g}_{\mathcal{D}} \otimes V \longrightarrow i_+ V$$

in terms of global sections, looks like

$$\mathfrak{g} \otimes V \otimes D_{\mathbb{A}^1} \xrightarrow[D_X\text{-mod map}]{} V \cdot \partial \oplus V \cdot \partial^2 \oplus \dots$$

$$\text{is } V[t, t^{-1}]/V[t]$$

by adjunction $\Updownarrow$

$$(\mathfrak{g} \otimes V)[t] \xrightarrow[\mathcal{O}_X\text{-mod map}]{} V[t, t^{-1}]/V[t]$$

~~for $x \in \mathfrak{g}$, $v \in V$, apply this map to $x \otimes V$~~

$\downarrow$ Residue

$$\mathfrak{g}[t] \otimes V \xrightarrow{m} V$$

Observe: for fixed $v \in V$, v is annhilated by t^n for some n.

$\Longrightarrow$ m extends to a continuous mult.

$$\mathfrak{g}[[t]] \otimes V \longrightarrow V$$

encodes the Lie*-action of L on $\mathcal{M}$

& "identities" translates to usual Lie-module ...

In this case, $M \longleftrightarrow$ a continuous $\mathfrak{g}[[t]]$ - module

$\exists$ a notion of a chiral module, using other pseudo-tensor str. defined as a map

$$j_+ j^+(L \boxtimes M) \longrightarrow \Delta_+ M$$

& same calculation shows in previous case

chiral mod. $M \longleftrightarrow$ a continuous $\mathfrak{g}((t))$ - module.

Example: vacuum rep. of $\mathfrak{g}((t))$:

$$V := U(\mathfrak{g}((t))) \otimes_{U(\mathfrak{g}[[t]])} \mathbb{C}$$

give geometric description of V as a chiral mod.

(assume X affine $x \in X$)

$$\begin{tikzcd} \Gamma_X h(\mathfrak{g}_{\mathcal{D}}) \arrow[r] \arrow[d] & \Gamma_{X-\{x\}} h(\mathfrak{g}_{\mathcal{D}}) \arrow[r] \arrow[d] & \mathrm{coker} \arrow[d, "?"] \\ \mathfrak{g}[[t]] \arrow[r] & \mathfrak{g}((t)) \arrow[r] & \mathrm{coker} \end{tikzcd}$$

can describe V as $\mathrm{Ind}_{\Gamma_X h(\mathfrak{g}_{\mathcal{D}})}^{\Gamma_{X-x} h(\mathfrak{g}_{\mathcal{D}})} (\mathbb{C})$

same result by PBW

(X aff.) & can now suppose $I = \{x_1 \ldots x_n\}$

define $\underset{\text{vacuum}}{V_I} = \mathrm{Ind}_{\Gamma_X h(L)}^{\Gamma_{X-\{x_1 \ldots x_n\}} h(L)} \mathbb{C}$

what does V_I look like?

V_I depends only on $X - \{x_1, \ldots x_n\}$

& for $I = I' \sqcup I''$ $\quad V_I \simeq V_{I'} \otimes V_{I''}$

$\Longrightarrow$ get a factorization algebra $V^{(n)}$ sheaf of $\mathcal{O}_{X^n}$-mods
w/ fiber at $I = \{x_1 \dots x_n\}$
$= V_I$

This construction takes

$$\left\{\begin{matrix}\text{Lie}^*\text{-algs}\\ \text{on } X\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{fac. alg}\\ \text{on } X\end{matrix}\right\}$$

$$L \xrightarrow{\text{"chiral envelope"}} \text{"vacuum rep" w/ fac. str.}$$

Claim: This construction is left adjoint to the forgetful functor

$$\text{Fac algs} \longleftrightarrow \text{chiral algs} \longrightarrow \text{Lie}^*\text{-algs}$$

what does chiral env. look like?

$$\{x\} \xhookrightarrow{i} X \overset{j}{\hookleftarrow} X - \{x\}$$

$\Rightarrow$ ex. seq. $\quad i_+ i^+ L \longrightarrow L \longrightarrow j_+ j^+ L$
& passing to cohomology get

$$\Gamma_X h(L) \longrightarrow \Gamma_{X-\{x\}} h(L) \longrightarrow \text{fiber of } L \text{ at } x,\ L_x$$

& Upshot = vacuum rep of L at x has a filtration
w/ ass. gr. $\simeq \text{Sym}^*(L_x)$