D-modules: JL lect #2

X a smooth variety
(affine $X = \operatorname{Spec} A$)

{an algebraic vector bundle on X + flat connection}

$\Updownarrow$

{a projective fin. gen. A-module M + for every (algebraic tangent) vector field ξ
a map $\nabla_\xi : M \longrightarrow M$
s.t. $\nabla_{f\xi} = f\nabla_\xi$ & $\nabla_\xi + \nabla_{\xi'}$
+ Leibniz $\nabla_\xi(fm) = f\nabla_\xi m + \langle \xi, df\rangle m$ }

Can be summarized as the action of some ring on M
$A[\nabla_\xi]/(\text{relations})$ encodes flatness of ∇

X proper, no non-constant global functions

In general, can define D_X sheaf of rings on X
s.t. $D_X(U) = A[\nabla_\xi]/(\text{rel.})$
$U = \operatorname{Spec} A$

Def.: a D-module on X is a sheaf of D_X-modules
(usually assume q-coh. as sheaf of $\mathcal{O}_X$-modules)

Ex: $X = \mathbb{A}^n = \operatorname{Spec}\mathbb{C}[x_1 \dots x_n]$
& $D_X = \mathbb{C}\{x_1, x_2 \dots x_n, \frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}\}$
s.t. $[x_i, x_j] = 0$, commutative.
& $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] \begin{cases} = [\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}] & \text{for } i \neq j \\ = 0 & i = j \end{cases}$ encodes flatness

Examples of left D-mod: $\mathcal{O}_X$
(things like functions)

Ex: $X = \mathbb{A}^1$, $M = \mathcal{O}_X\{e^x\}$ free module on 1-gen.
set $\frac{\partial}{\partial x}(fe^x) = \left(\frac{\partial f}{\partial x}\right)e^x + fe^x$ "section" not algebraic so $\mathcal{O}_X\{e^x\} \neq \mathcal{O}_X$

<u>right D-modules</u> : like dual of functions, e.g., distributions

$\langle a, f \rangle$

f a function, a a distribution, has action of D_X s.t.

$$\langle aD, f \rangle = \langle a, Df \rangle$$

one way to get distributions = differential forms of top degree Ω^n

$n = \dim X$

more precisely : f a function, $\nabla_\xi f = \langle \xi, df \rangle = i_\xi df$ can do w/ 0-forms

& if ω an n-form, define $\omega(\nabla_\xi) = d(i_\xi \omega)$

right action

There is an equivalence

$$\text{Left } D_X\text{-mod} \longleftrightarrow \text{right } D_X\text{-mod}$$

$$M \longrightarrow M \otimes \Omega^n$$

$$N \otimes (\Omega^n)^{-1} \longleftarrow N$$

[but really should be working w/ complexes of D_X-mod]

think of $\Omega^n_X \simeq$ "$\det \Omega^1_X$" but comes w/ grading.

so define $\omega_X =$ "graded det of Ω^1_X"

$= \Omega^n_X[n]$ n-forms in deg n, homological degree n

& get equiv.

$$\text{derived cat. of left } D_X\text{-mod} \longleftrightarrow \text{derived cat of right } D_X\text{-mod}$$

$$M \longrightarrow M \otimes \omega_X$$

Operations on D-modules: (everything derived)

$$X \xrightarrow{f} Y \qquad X, Y \text{ smooth} \quad (\text{not nec. proper})$$
no assump. on f

If M is a D_Y-module get $f^+ M \in D_X$-mod.

- If M a left D-mod., then $f^+M \simeq f^* M$ as $\mathcal{O}_X$-modules
- If M a right D-mod, then $f^+ M = \omega_X \otimes f^*(M \otimes \omega_Y^{-1})$

$$= f^* M \otimes \omega_{X/Y}$$
$$\cong f^! M$$

[in the sense of q-coh sheaves]

[when f proper, is right adj to f_*]

tensor product:

If X, Y smooth: $M \in D_X$-mod $N \in D_Y$-mod

make $M \boxtimes N \in D_{X \times Y}$-mod

for $X = Y$ $\qquad X \xrightarrow[\text{diag.}]{\Delta} X \times X$

define $M \otimes N = \Delta^+ (M \boxtimes N)$

- for left modules, this is the usual $\otimes$-prod. of $\mathcal{O}_X$-mods.

Remark: D_X^{left}-mod $\circlearrowright$ D_X^{right}-mod

$\qquad\qquad m \qquad\qquad N$

by formula $(m \otimes n) \nabla_\xi = m \otimes n \nabla_\xi - \nabla_\xi m \otimes n$

equivalent to $\Delta^+ ((M \otimes \omega_X) \boxtimes N)$

Def: $D_{X \xrightarrow{f} Y} := f^+ D_Y$
in left D-mod

have bimod str. $D_X \circlearrowright D_{X \to Y} \circlearrowleft f^{-1} D_Y$

just sheaf pullback

upshot: If M is a right D_X-module

$f_+ M := f_* (M \otimes_{D_X} D_{X \to Y})$ is a right D_Y-module

note, derived

Example 1: $Y = *$. Say M a left D_X-module.

one way to phrase having a connection

is $0 \to M \xrightarrow[d]{\exists c} M \otimes \Omega^1_X \to M \otimes \Omega^2_X \to \cdots$

de Rham cplx w/ coeff. in M $\quad \cdots \to M \otimes \Omega^n_X \to 0$

$$= \big(0 \to D_X \to \Omega^1 \otimes D_X \to \cdots \to 0\big) \underset{D_X}{\otimes} M$$

universal de Rham cplx

translate for right D_X-modules:

$N \in$ right D_X-mod

$$N \underset{D_X}{\otimes} \big(0 \to \omega_X^{-1} \otimes D_X \to \cdots \to D_X \to 0\big)$$

version of Poincaré lemma:

$\exists$ exact seq.

$$0 \to \quad \cdots \to D_X \twoheadrightarrow \mathcal{O}_X \to 0$$

which is exactly seq. above.

$\Longrightarrow$ is a proj. res. of $\mathcal{O}_X$

but $\quad \mathcal{O}_X = f^*\mathcal{O}_Y = f^* D_Y \simeq f^+ D_Y = D_{X \to Y}$

$\Longrightarrow$ this complex computes $N \underset{D_X}{\otimes} D_{X \to Y}$

so for $f: X \to Y$

f_+ is "de Rham cohomology"

Ex 2: If $X = *$, $Y = \mathbb{A}^1$

& $\quad f: \{0\} \hookrightarrow \mathbb{A}^1$ $\quad$ closed immersion, so no derived functors to worry about here,

what is $f_+(\mathbb{C}) = f_*(\mathbb{C} \underset{D_X}{\otimes} D_{X \to Y})$ $\quad$ recall $D_{X \to Y} := f^+ D_Y = f^* D_Y$

free mod. on $*$ of $rk = 1$

free $\mathbb{C}[y, \frac{\partial}{\partial y}]$ module gen. by section δ s.t. $y\delta = 0$ $\longrightarrow$ $\mathbb{C}[y, \frac{\partial}{\partial y}]\,y \backslash \mathbb{C}[y, \frac{\partial}{\partial y}]$ $\quad$ weyl algebra think of a Dirac delta function

this has a basis $\left(\delta, \frac{\partial}{\partial y}\delta, \frac{\partial^2}{\partial y^2}\delta, \dots\right)$

$$y\frac{\partial}{\partial y}(\delta) = -\left(1 - \frac{\partial}{\partial y}y\right)\delta$$
$$= -\delta$$

upshot: $f_+(\mathbb{C})$ = "D-mod. generated by δ"

$X \xrightarrow{f} \mathbb{A}^1$ set-theoretically supp. at origin $\{0\}$

but not scheme " supp. at origin!

looks like $\mathbb{C}[y, y^{-1}]/\mathbb{C}[y]$ ← not finite type as $\mathcal{O}_Y$-mod but finite in some sense as a D_Y-mod.

$\delta \longrightarrow \frac{1}{y}$

more generally: $X \overset{f}{\hookrightarrow} Y$ closed immersion, f_+ behaves like above

Riemann-Hilbert correspondence:

X smooth variety

locally constant sheaves on X $\Longleftrightarrow$ analytic vect. bundles w/ flat connectn

correspondence given by solving diff. eqs

Serre's GAGA

D_X-mod projective as $\mathcal{O}_X$ moduls $\Longleftrightarrow$ algebraic vect. bun. on X w/ flat connection

(all sheaves on X) ⊃ $D^b_{constructible}(X)$

$\updownarrow$ Riem. Hilbert

$D(\text{all } D_X\text{-mod})$ ⊇ regular holonomic D_X-mod

$$D^b(\text{reg hol. } D_X\text{-mod}) \xrightarrow{\sim} D^b_{construble}\binom{\text{sheaves}}{\text{on } X}$$

a contravariant functor in this setup

If $\mathcal{M}$ is a left D_X-module

$$\underline{\mathrm{Hom}}_{D_X}(\mathcal{M}, \mathcal{O}_X^{\text{analytic}})$$

inner Hom in derived cat.

on $\mathbb{A}^1$

$\mathcal{O}_X \; e^x$

maps $\downarrow$ $\mathcal{M}$ ↭ sections $m \in \mathcal{M}$ s.t. $\frac{\partial}{\partial x} m = m$

in Riem. Hilbert corresp.

$$\underset{\text{sheaves}}{\boxtimes} \xleftrightarrow{\text{(contravariant)}} \underset{\text{D-mod}}{\boxtimes}$$

$$f^* \longleftrightarrow f^+$$

$$f_! \longleftrightarrow f_+$$

$j_!$ should be left adj to j^*

now, D^b(reg. holonomic D_X-mod)

purely algebraic conditions, so makes sense as a general substitute for a category of sheaves like $Sh_{\text{constructible}}(X)$

not an algebraic condition

final remark: but big cat. of all D_X-modules behaves differently.

$Y \overset{i}{\subseteq} X$ smooth variety $\overset{j}{\supseteq} X - Y = \mathcal{U}$

sm. sub. var.

Remark 1: if j an open immersion (or étale) then j_+ is right adjoint to j^+

since for $j : \mathcal{U} \hookrightarrow X$

then $D_{\mathcal{U} \to X} = D_X|_{\mathcal{U}} = D_{\mathcal{U}}$

& $j_+(\mathcal{M}) = j_*(\mathcal{M} \otimes_{D_{\mathcal{U}}} D_{\mathcal{U} \to X}) = j_* \mathcal{M}$

& $j^+ N = N|_{\mathcal{U}}$

Remark 2: If $i : Y \longrightarrow X$ is proper

then i_+ is left adjoint to i^+

(much less obvious).

" $i_!$ left adj to i^* "

" i_*

$\mathcal{M} \in D_X$-mod

but counit + unit of adj.

$$\Rightarrow \quad \begin{tikzcd} i_+ i^+ \mathcal{M} \arrow[r] \arrow[rr, bend right, "0"'] & \mathcal{M} \arrow[r] & j_+ j^+ \mathcal{M} \end{tikzcd}$$

Thm: This is a fiber sequence of D_X-mods

proof: Can check after restriction to Y &

complement to $Y = U$

$\square$

Consequence: $\mathcal{M}$ is the fiber of a connecting map

$$j_+ j^+ \mathcal{M} \longrightarrow i_+ i^+ \mathcal{M}[1]$$

$\Rightarrow$ Can view a D-mod on X as

data on Y, U + gluing data.

example: in chiral alg.