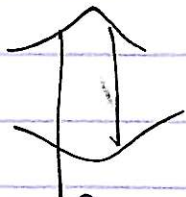


# D-modules: JL lect #2

$X$  a smooth variety  
(affine  $X = \text{Spec } A$ )

{ an algebraic vector bundle on  $X$  + flat connection }



{ a projective fin. gen.  $A$ -module  $M$  + for every (algebraic) tangent vector field }  
 a map  $\nabla_{\mathbb{Z}}: M \rightarrow M$   
 s.t.  $\nabla_{f\mathbb{Z}} = f\nabla_{\mathbb{Z}} + \langle \mathbb{Z}, df \rangle m$  &  $\nabla_{\mathbb{Z}} + \nabla_{\mathbb{Z}'}$   
 + Leibniz  $\nabla_{\mathbb{Z}}(fm) = f\nabla_{\mathbb{Z}}m + \langle \mathbb{Z}, df \rangle m$

Can be summarized as the action of something on  $M$

$A[\nabla_{\mathbb{Z}}] / (\text{relations})$  encodes flatness of  $\nabla$

$X$  proper, no non-constant global functions

In general, can define  $\mathcal{D}_X$  sheaf of rings on  $X$

s.t.  $\mathcal{D}_X(U) = A[\nabla_{\mathbb{Z}}] / (\text{rel.})$   
 $U = \text{Spec } A$

Def.: a  $\mathcal{D}$ -module on  $X$  is a sheaf of  $\mathcal{D}_X$ -modules  
 (usually assume  $q$ -coh. as sheaf of  $\mathcal{O}_X$ -modules)

Ex:  $X = \mathbb{A}^n = \text{Spec } \mathbb{C}[x_1, \dots, x_n]$

&  $\mathcal{D}_X = \mathbb{C}\langle x_1, x_2, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle$

s.t.  $[x_i, x_j] = 0$ , commutative.

&  $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = \begin{cases} [\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i}] & \text{for } i \neq j \\ = 0 & i = j \end{cases}$

encodes flatness

Examples of left  $\mathcal{D}$ -mod:  $\mathcal{O}_X$   
 (things like functions)

Ex:  $X = \mathbb{A}^1$ ,  $M = \mathcal{O}_X \{e^x\}$  free module on 1-gen.  
 set  $\frac{\partial}{\partial x}(fe^x) = (\frac{\partial f}{\partial x})e^x + fe^x$  "section" not algebraic so  $\mathcal{O}_X[e^x] \neq \mathcal{O}_X$

right D-modules: like dual functions, e.g., distributions  
 $\langle a, f \rangle$

f a function, a a distribution, has action of  $D_x$  s.t.

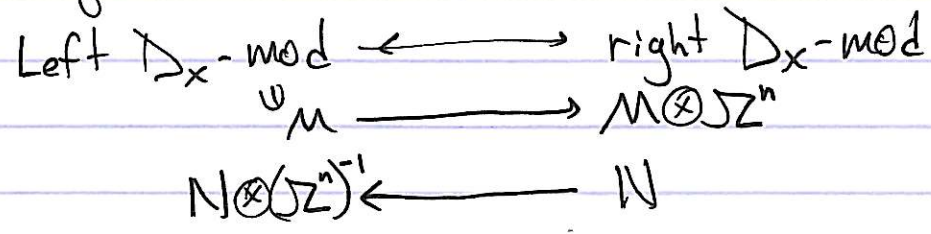
$$\langle a D, f \rangle = \langle a, Df \rangle$$

one way to get distributions = differential forms of top degree  $\mathcal{J}Z^n$   
 $n = \dim X$

more precisely: f a function,  $\nabla_{\xi} f = \langle \xi, df \rangle = \xi_j df^j$  can do w/  $\omega$ -forms

& if  $\omega$  an n-form, define  $\omega(\nabla_{\xi}) = d(\xi_j \omega^j)$   
right action

There is an equivalence



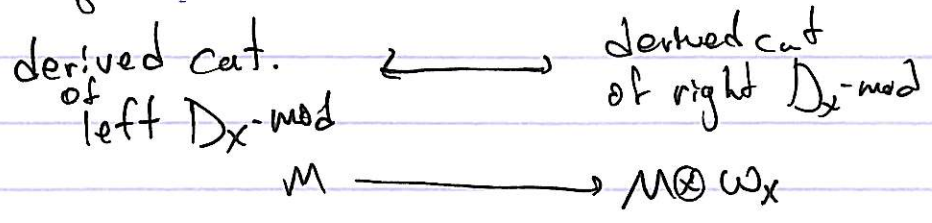
[but really should be working w/ complexes of  $D_x$ -mod]

think of  $\mathcal{J}Z^n_x \approx \text{"det } \mathcal{J}Z'_x \text{"}$  but comes w/ grading.

so define  $\omega_x = \text{"graded det of } \mathcal{J}Z'_x \text{"}$

$$= \mathcal{J}Z^n_x[n] \quad \begin{array}{l} \uparrow \\ \text{n-forms} \\ \text{in deg n} \end{array}, \text{ homological degree n}$$

& get equiv.



Operations on D-modules: (everything derived) -

$$X \xrightarrow{f} Y \quad X, Y \text{ smooth (not nec. proper)}$$

no assump. on f

If  $M$  is a  $D_Y$ -module get  $f^+M \in D_X\text{-mod}$ .

- If  $M$  a left  $D$ -mod., then  $f^+M \cong f^*M$  as  $\mathcal{O}_X$ -modules ~~is~~

- If  $M$  a right  $D$ -mod., then  $f^+M = \omega_X \otimes f^*(M \otimes \omega_Y^{-1})$   
 $= f^*M \otimes \omega_{X/Y}$   
 $\cong f^!M$

in the sense of g-coh sheaves  
 when  $f$  proper,  $f^!$  is right adj to  $f_*$

tensor product:

If  $X, Y$  smooth:  $M \in D_X\text{-mod}$   $N \in D_Y\text{-mod}$   
 make  $M \boxtimes N \in D_{X \times Y}\text{-mod}$

for  ~~$\mathbb{A}^1$~~   $X=Y$

define  $M \otimes N = \Delta^+(M \boxtimes N)$

$$\Sigma \xrightarrow[\text{diag.}]{\Delta} \Sigma \times \Sigma$$

- for left modules, this is the usual  $\otimes$ -prod. of  $\mathcal{O}_X$ -mods.

Remark:  $D_X^{\text{left-mod}} \subset D_X^{\text{right-mod}}$

by formula  $(M \otimes N) \nabla_3 = M \otimes N \nabla_3 - \nabla_3 M \otimes N$   
 equivalent to  $\Delta^+((M \otimes \omega_X) \boxtimes N)$

Def:  $D_{X \rightarrow Y} := \text{pullback} f^+ D_Y$   
 in left  $D$ -mod

have bimod str.  $D_X \subset D_{X \rightarrow Y} \supset f^{-1} D_Y$

just sheaf pullback

upshot: If  $M$  is a right  $D_X$ -module

$f_+ M := f_* (M \otimes_{D_X} D_{X \rightarrow Y})$  is a right  $D_Y$ -module  
 note, derived

Example 1:  $Y = *$ . Say  $M$  a left  $D_X$ -module.

One way to phrase having a connection

is 
$$0 \rightarrow M \xrightarrow{\exists d} M \otimes \Omega'_X \rightarrow M \otimes \Omega^2_X \rightarrow \dots$$
  
 de Rham cplx  
 w/ coeff. in  $M$

$$= (0 \rightarrow D_X \rightarrow \Omega'_X \otimes D_X \rightarrow \dots \rightarrow 0) \otimes_{D_X} M$$
  
 universal de Rham cplx

translate for right  $D_X$ -modules:  
 $N \in \text{right } D_X\text{-mod}$

$$N \otimes_{D_X} (0 \rightarrow \omega_X^{-1} \otimes D_X \rightarrow \dots \rightarrow D_X \rightarrow 0)$$

version of Poincaré lemma:

$\exists$  exact seq.

$$0 \rightarrow \dots \rightarrow D_X \rightarrow \mathcal{O}_X \rightarrow 0$$

which is exactly seq. above.

$\implies$  is a proj. res. of  $\mathcal{O}_X$

but  $\mathcal{O}_X = f^* \mathcal{O}_Y = f^* D_Y \cong f^+ D_Y = D_{X \rightarrow Y}$

$\implies$  this complex computes  $N \otimes_{D_X} D_{X \rightarrow Y}$

so for  $f: X \rightarrow Y$

$f_+$  is "de Rham cohomology"

Ex 2: If  $X = *$ ,  $Y = \mathbb{A}^1$

&  $f: \{0\} \hookrightarrow \mathbb{A}^1$  closed immersion, so no derived functors to worry about here

What is  $f_+(\mathbb{C}) = f^* (\mathbb{C} \otimes_{D_X} D_{X \rightarrow Y})$   
 free mod. on  $*$  of rk=1

recall  $D_{X \rightarrow Y} := f^+ D_Y = f^* D_Y$

free  $\mathbb{C}[y, \frac{\partial}{\partial y}]$  module gen. by sections s.t.  $y \delta = 0 \implies \mathbb{C}[y, \frac{\partial}{\partial y}] / \mathbb{C}[y, \frac{\partial}{\partial y}]$  weyl algebra

think of a Dirac delta function

this has a basis  $(\delta, \frac{\partial}{\partial y} \delta, \frac{\partial^2}{\partial y^2} \delta, \dots)$

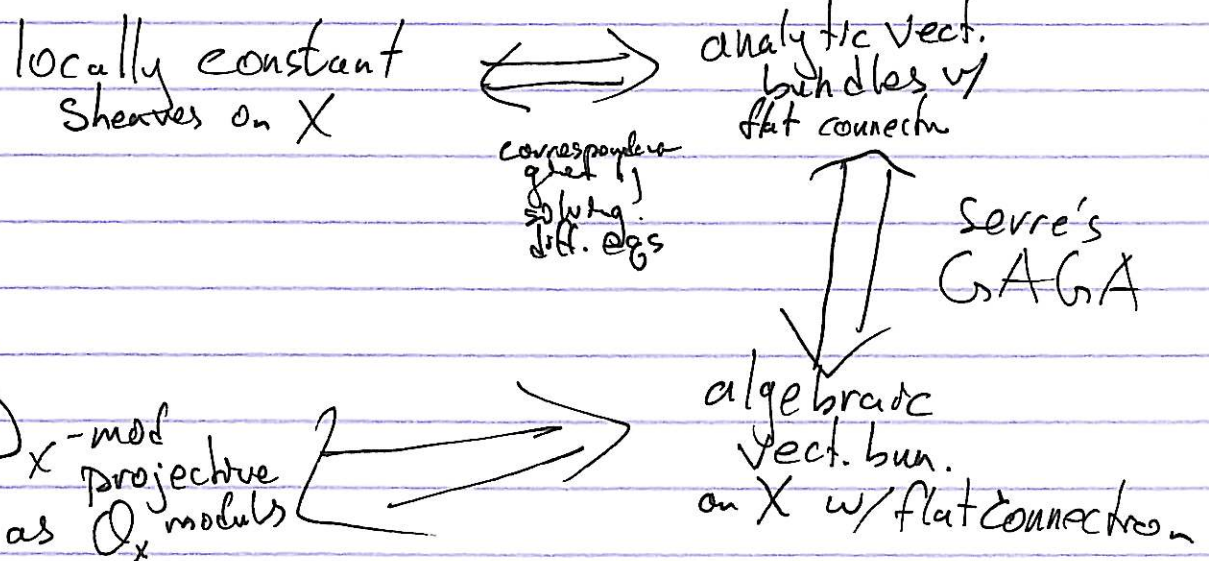
$$y \frac{\partial}{\partial y} (\delta) = - (1 - \frac{\partial}{\partial y} y) \delta = -\delta$$

upshot:  $f_+ (\mathbb{C}) = "$ D-mod. generated by  $\delta$ "  
 set-theoretically supp. at origin  $\{0\}$   
 but not scheme " supp. at origin!

looks like  $\mathbb{C}[y, y^{-1}] / \mathbb{C}[y] \xrightarrow{\partial} \frac{1}{y}$  ← not quite type as  $\mathcal{O}_X$ -mod but finite in some sense as a  $D_X$ -mod.

more generally:  $X \xrightarrow{f} Y$   $\mathbb{C}^0$   
 closed immersion  $f_+$  behaves like above

Riemann-Hilbert correspondence:  
 $X$  smooth variety



(all sheaves on  $X$ )

$D^b_{\text{constructible}}(X)$

Riem. Hilbert

$D^b(\text{all } D_X\text{-mod})$

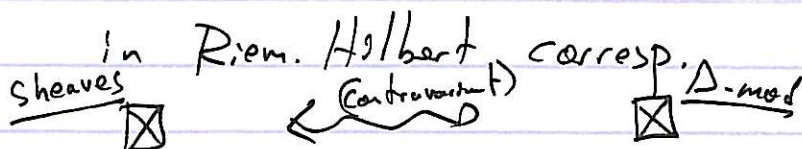
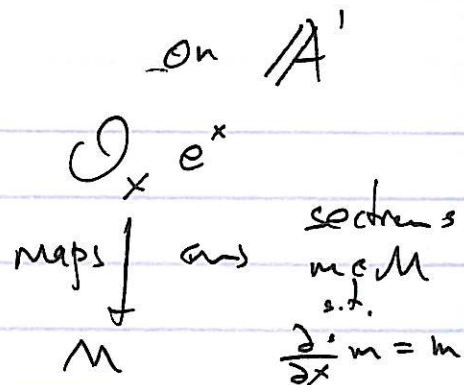
regular holonomic  $D_X$ -mod

$D^b(\text{reg hol. } D_X\text{-mod}) \xrightarrow{\sim} D^b_{\text{constructible}}(\text{sheaves on } X)$   
 a contravariant functor in this setup

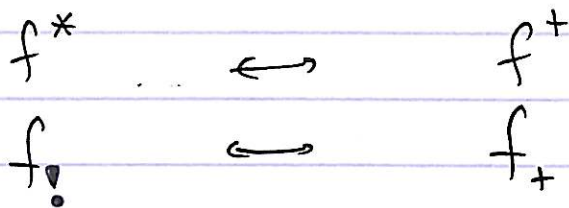
If  $M$  is a left  $D_X$ -module

$$\underline{\text{Hom}}_{D_X}(M, \mathcal{O}_X^{\text{analytic}})$$

inner Hom in derived cat.



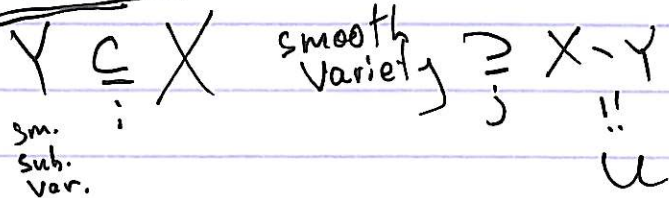
$j_*$  should be left adj to  $j^*$



now,  $\mathcal{D}^b$  (reg. holonomic  $D_X$ -mod)

purely algebraic conditions, so makes sense as a general substitute for a category of sheaves like  $\text{Sh}_{\text{const}}(X)$  not an algebraic condition

final remark: but big cat. of all  $D_X$ -modules behaves differently.



Remark 1: if  $j$  an open immersion (or étale) then  $j_+$  is right adjoint to  $j^+$

since for  $j: U \hookrightarrow X$

$$\text{then } D_{U \rightarrow X} = D_X|_U = D_U$$

$$\begin{aligned} \otimes j_+(M) &= j_*(M \otimes_{D_U} D_{U \rightarrow X}) = j_* M \\ \otimes j^+ N &= N|_U \end{aligned}$$

Remark 2: If  $i: Y \rightarrow X$  is proper then  $i_+$  is left adjoint to  $i^*$  (much less obvious).

" $i_+$  left adj to  $i^*$ "  
" $i_+$ "

$M \in \mathcal{D}_X\text{-mod}$   
but counit + unit of adj.  
 $\Rightarrow i_+ i^* M \rightarrow M \rightarrow j_+ j^* M$

Thm: This is a fiber sequence of  $\mathcal{D}_X\text{-mods}$

proof: can check after restriction to  $Y$  & complement to  $Y = U$   
□

consequence:  $M$  is the fiber of a connecting map  
 $j_+ j^* M \rightarrow i_+ i^* M [1]$

$\Rightarrow$  can view a  $\mathcal{D}$ -mod on  $X$  as data on  $Y, U$  + gluing data?

example: in chiral alg.  
↓