

INTRO TO CHIRAL ALGEBRAS

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1. INTRODUCTION

“This is our seminar on chiral algebras. Me learning this material prior to the seminar would have defeated the point. I’m about to talk. Other people (Scott, Dennis, John) will then talk later, and so will I, again, and then other people, again. And then there will be the second half of the seminar.”

2. MOTIVATION FROM THEORY OF LOOP GROUPS

Some motivation for the study of chiral algebras comes from the theory of loop groups. Here let G be compact, simple Lie group. Define the loop group, LG as the space of unbased loops $\text{Map}(S^1, G)$.

The loop group LG itself doesn’t have many interesting representations. However, one can define a central extension of the loop group $S^1 \rightarrow \widetilde{LG} \rightarrow LG$ for each $n > 0$, and then study representations of \widetilde{LG} at level n . This category $\text{Rep}(\widetilde{LG})$ is semi-simple, and has finitely many simple objects. Additionally, there exists a “fusion” tensor structure, \otimes , on $\text{Rep}(\widetilde{LG})$ which is associative and also *braided* monoidal.

We can also study the Lie algebra $\text{Lie}(LG) = \text{Map}(S^1, \mathfrak{g})$. This can be studied by Fourier series, though this depends on what types of loops we were considering, which I didn’t mention. The most algebraic version of these loops are Laurent series, and in this version our loop Lie algebra is $\mathfrak{g}((t))$. We have a corresponding Lie algebra central extension $\mathbb{C} \rightarrow \widetilde{\mathfrak{g}((t))} \rightarrow \mathfrak{g}((t))$. However, there are *more* representations of $\widetilde{\mathfrak{g}((t))}$ than there are of \widetilde{LG} . To obtain a representation of the group from one of the algebra, we need an integrability condition. (In fact, this is visible just at the level of characters.)

In this setting of Lie algebras (which works over any field of characteristic 0) we can ask what the analogue of *fusion* is? To answer, we’ll use chiral algebras.

Denote the unit of the fusion tensor structure on $\text{Rep}(\widetilde{LG})$ by 1 , this is the “vacuum” representation. As usual, there exists a morphism $1 \otimes 1 \rightarrow 1$ in $\text{Rep}(\widetilde{LG})$ making 1 an algebra in this category. However, this does not make 1 an algebra in vector spaces (since the forgetful functor isn’t monoidal). What structure does the underlying vector space of the vacuum 1 inherit from being an algebra in $\text{Rep}(\widetilde{LG})$?

It is a chiral (or vertex) algebra.

3. REPRESENTATIONS

How can we construct representation of loop groups? In finite dimensions, for G , we can use Borel-Weil. Take a parabolic subgroup P in $G_{\mathbb{C}}$. The space of flags $X = G_{\mathbb{C}}/P$ is a compact algebraic variety (the flag variety) since P was a parabolic. (This is essentially the definition of what a parabolic subgroup is.)

Choose a line bundle $\mathcal{L} \rightarrow X$. If G is simply-connected then there exists an action of G on \mathcal{L} . From this we get a representation of G on the sheaf cohomology $H^0(X; \mathcal{L})$.

There is an analogue of this construction for loop groups. The analogue of a parabolic is a *parahoric*. The example to keep in mind is $G[[t]]$ in $G((t))$. Think of $\text{Spec } \mathbb{C}[[t]]$ as the disk, and $\text{Spec } \mathbb{C}((t))$ as the punctured disk. So the inclusion of the parahoric above you can think of as

the inclusion $\text{Map}(\text{disk}, G) \rightarrow \text{Map}(\text{punc. disk}, G)$. (For $G = GL_n$, then $GL_n[[t]]$ is equivalent to $n \times n$ -matrices with entries in power series in t .)

The analogue of the flag variety used above is now $G((t))/G[[t]] = Gr_G$, the affine Grassmannian. How to implement the construction of representations? Given a level $n \in H^4(BG; \mathbb{Z})$? We can construct a line bundle $\mathcal{L} \rightarrow Gr_G$ with an action of \widetilde{LG} . (For simplicity, we'll ignore the central extension in what follows.)

The idea is to get representation occurring on $H^0(Gr_G; \mathcal{L})$, but this doesn't quite work given how everything here is infinite dimensional. To get the right answer, we should write this as a limit over finite dimensional things: $\lim H^0(\text{fin. dimnsl.}; \mathcal{L})$. But this still doesn't work; we get something of uncountable dimension.

The right answer is to consider $H^0(Gr_G; \mathcal{L})_{\text{cont.}}^* \cong \text{colim } H^0(\text{fin. dimnsl.}; \mathcal{L})^*$, which works good. This construction produces the vacuum representation, which we'll now denote by A .

4. GEOMETRY (FACTORIZATION)

Let X be an algebraic curve, $x \in X$ a point. Define Gr_x as the moduli space of G -bundles on X equipped with a choice of trivialization of the bundle on $X - \{x\}$. To describe a point of Gr_x , you can fix a trivialization over the formal disk, which has a parameter space of $G((t))$, and then quotient by ambiguity of the choice, given by $G[[t]]$. Thus $Gr_x \cong G((t))/G[[t]] \cong Gr_G$. This describes a family of spaces Gr_G parametrized by points of X . Let us denote this space $Gr^{(1)}$.

That is, there exists a map $Gr^{(1)} \rightarrow X$ whose fiber over a point x is exactly Gr_x . The construction of the line bundle \mathcal{L} above can also be carried out in families, which produces a line bundle (also denoted \mathcal{L}) $\mathcal{L} \rightarrow Gr^{(1)}$. Let's denote the associated quasi-coherent sheaf of sections on X as $A^{(1)} \rightarrow X$, with fibers A .

Definition 4.1. Define $Gr_{x,y}$ as the parameter space for G -bundles on X together with a choice of trivialization on $X - \{x, y\}$. Again, these spaces assemble into a family of spaces $Gr^{(2)}$ parametrized by X^2 , sitting in a pullback square:

$$\begin{array}{ccc} Gr^{(2)} & \longleftarrow & Gr_{x,y} \\ \downarrow & & \downarrow \\ X^2 & \longleftarrow & \{x, y\} \end{array}$$

It's obvious that for $x = y$, you get that $Gr_{x,y} = Gr_x = Gr_y$. Also for $x \neq y$, you get that $Gr_{x,y} = Gr_x \times Gr_y$. (Editor's note: the second statement isn't obvious, if you didn't already know it.)

This structure is called *factorization*. Note that this structure looks weird from the point of view of finite dimensional algebraic geometry, in which you expect the dimensions of the fibers to be upper semi-continuous. Here, it's the opposite.

Back to the line bundle: we get another line bundle $\mathcal{L} \rightarrow Gr^{(2)}$, with associated sheaf we'll denote $A^{(2)} \rightarrow X^2$. We see a similar factorization structure here: in the diagram $X \xrightarrow{\Delta} X^2 \xleftarrow{j} U := X^2 - \Delta_X$, we have that $\Delta^* A^{(2)} \cong A^{(1)}$ and that $j^* A^{(2)} \cong j^*(A \boxtimes A)$. This is a "factorization algebra." This is *like* having a multiplication on A .

$A^{(1)}$ has a flat connection. In differential geometry, connections give rise to parallel transport, i.e., some way of identifying the bundle fibers at different points given a choice of path connecting them. Grothendieck had the idea that in algebraic geometry, a flat connection is something that gives an isomorphism between fibers of infinitesimally close points. (x and y are infinitesimally close means that $X - x \simeq X - y$.)

Thus $Gr_x \simeq Gr_y$ if x and y are infinitesimally close, and this gives a flat connection on $A^{(1)}$ and $A^{(2)}$.

Remark 4.2. For $X = \mathbb{A}^n = \text{Spec } \mathbb{C}[x_1, \dots, x_n]$, a flat connection on $M \in \text{QCoh}_X$ is module structure on M for the noncommutative ring $\mathbb{C}\{x_1, \dots, x_n, \delta_{x_1}, \dots, \delta_{x_n}\} = D_X$. (In this ring, the derivations commute with each other (i.e., “mixed partials commute”) and act on the polynomial variables x_i by the rules of calculus (i.e., Leibnitz rules).) This makes M an algebraic D -module.

So $A^{(2)}$ is an algebraic D -module on X^2 , and we have a short exact sequence that I won’t explain going $\Delta_! \Delta^! \rightarrow A^{(2)} \rightarrow j_* j^* A^{(2)}$. (This describes the sheaf $A^{(2)}$ in terms of its values on the diagonal, complement of the diagonal, plus gluing data in the SES.) From this, we get $A^{(2)} \rightarrow j_* j^* A^{(1)} \boxtimes A^{(1)} \rightarrow \Delta_* A^{(1)}$. This last map is a *chiral bracket*.

Incomplete Definition 4.3. A chiral algebra on a curve X is a D -module A with a *chiral bracket*.

This is closely related to a vertex algebra, which jfd will talk about.

5. BACK TO LIE ALGEBRAS

Back to $\widetilde{\mathfrak{g}((t))}$, the Kac-Moody algebra. What structure did it inherit from all this?

For a A a chiral algebra, we had this composition $A \boxtimes A \rightarrow j_* j^* A \boxtimes A \rightarrow \Delta_* A$, where the composition anti-commutes.

This looks similar to the picture with the sheaf of differential operators, $D_X \boxtimes D_X \cong D_{X \times X} \rightarrow \Delta_* D_X$, but this is commutative not anti-commutative. So D_X doesn’t quite give a chiral algebra.

However, if we start with a Lie algebra \mathfrak{g} , then $D_X \otimes \mathfrak{g}$ has a *Lie*-bracket*, very similar to the anti-commuting chiral bracket above, making $D_X \otimes \mathfrak{g}$ a Lie*-algebra.

There is a forgetful functor from chiral algebras to Lie*-algebras, $\text{ch-alg}_X \rightarrow \text{Lie}^*\text{-alg}_X$ that admits a left adjoint, the *chiral envelope*.

$D_X \otimes \mathfrak{g}$ “is like” the Kac-Moody algebra $\mathfrak{g}((t))$. Modestly more precisely, consider $D_X \otimes \mathfrak{g}$ over the formal disk $\text{Spec } \mathbb{C}[[t]]$. Take the “horizontal/flat” sections (which are like the kernel of covariant differentiation, in differential geometry), and you’ll get $\mathfrak{g}[[t]]$. Do the same thing for the punctured disk $\text{Spec } \mathbb{C}((t))$, and the flat sections produce $\mathfrak{g}((t))$.

5.1. Modules. Given $L \in \text{Lie}^*\text{-alg}_X$, you can talk about Lie* L -modules and chiral L -modules on X . This is a D_X -module M together with an action of L . If M is supported at a point $x \in X$ and $L = D_x \otimes \mathfrak{g}$, then chiral L -modules are equivalent to $\mathfrak{g}((t))$ -modules, and Lie* L -modules are equivalent to $\mathfrak{g}[[t]]$ -modules. (These modules are discrete, continuous, as usual.) This produces families of categories on curves, behaving in a certain way when points collide.

6. CHIRAL HOMOLOGY

The second main topic of the seminar is chiral homology. Let \mathcal{C} denote the category of loop group representations with fusion monoidal structure \otimes . This is a modular tensor category (i.e., rigid, balanced, braided plus condition). This means that for every surface (with incoming δ^{in} , outgoing boundaries δ^{out}) Σ , we get a functor $F_\Sigma : \mathcal{C}^{\otimes \delta^{in}} \rightarrow \mathcal{C}^{\otimes \delta^{out}}$. For instance, for the pair-of-pants, we get a functor corresponding to fusion.

“Rigid” means each object in \mathcal{C} has a dual with respect to the fusion product \otimes . This is encoded by the functor F_Σ for Σ the surface with no outgoing boundary component.

This implies that $\text{Hom}(F_\Sigma(C \otimes C'), D \otimes D') \cong \text{Hom}(F_\Sigma(C \otimes C' \otimes D^* \otimes D'^*), k)$.

Given Σ with marked points, together with objects of \mathcal{C} , M_i , at each marked point, we get a vector space denoted $\langle M_1, M_2, M_3 \rangle_\Sigma$. We also have this structure for the category of modules over $\mathfrak{g}((t))$ or $\mathfrak{g}[[t]]$. The construction that assigns to the collection M_i the vector space $\langle M_1, M_2, M_3 \rangle_\Sigma$ is *chiral homology*.

This is a very useful idea in geometric Langlands: for G a group, and a curve X , the conjecture is that there is an equivalence $D\text{-mod}(Bun_G X) \cong \text{QCoh}(Bun_G^{flat} X)$. If you understood this for the disk $X = \text{Spec } \mathbb{C}[[t]]$ and you wanted to build the equivalence for more general curves X , you might want a procedure of “integrating” over X , very similar to chiral homology. If you could prove

geometric Langlands, then you'd be very popular. There exists a definition of a functor that is supposed to implement this equivalence.