TALBOT 2006

Automorphisms of Manifolds

with host Michael Weiss

and organizers Chris Douglas
John Francis
Andre Henriques
Mike Hill

Schedule of Talks:

Monday
1. Overview of the method. [Chris Douglas]
2. Surgery theory and G/BlockAut, Part I. [Andre Henriques]
3. Surgery theory and G/BlockAut, Part II. [Stacy Hoehn]
4. Orthogonal calculus and BlockTop/Top. [Tibor Macko]
5. BlockTop/Top and the Whitehead space. [Jacob Lurie]

Tuesday
6. On a train from Beilefeld to Oberwolfach. [Michael Weiss]
7. Waldhausen K-theory. [Teena Gerhardt]
8. Waldhausen K-thy, TOP-Whitehead space, concordances, Part I. [Allegra Berliner]
9. Waldhausen K-theory, DIFF-Whitehead space, concordances, Part II. [Vigleik Angeltveit]

Wednesday
10. Diffeomorphisms of disks and spheres. [Mike Hill]
11. Algebraic Surgery, LA-theory. [Marcus Meyer]
12. Controlled topology, Part I. [Qayum Khan]
13. Controlled K- and L-Theory, Part II [Michael Weiss]

Thursday
14. Dwyer-Weiss-Williams, Part I. [Jeffrey Giansiracusa]
15. Dwyer-Weiss-Williams towards Weiss-Williams III, Part II. [Julia Weber]
16. Weiss-Williams III. [Michael Weiss]
17. Discussion session.

Friday
18. Higher torsion invariants, Part I. [Scott Carnahan]
19. Higher torsion invariants, Part II. [John Francis]
20. Igusa's singularity approach. [Michael Weiss]
21. Discussion session.

AUTOMORPHISMS OF MFLDS

27.02.06

TALBOT 2006

Overview

Chris

$$\begin{tikzcd}
 & & & & & \text{Alg. K-thy of Rings} \\
\text{Diff} \arrow[r, squiggly, "\text{Surgery th.}"] & \dfrac{\widetilde{\text{Diff}}}{\text{Diff}} \arrow[r, squiggly, "\text{Conc. Spec. Seq.}"] & \text{Conc} \arrow[r, squiggly, "\text{Stab. Thm}"] & \text{Stab Conc} \arrow[r, squiggly, "\text{Sp. of h-Cob.}"] & \text{Whitehead} \arrow[r, squiggly, "\text{Waldh. spl.}"] & \text{Alg K-thy} \arrow[u, "\text{linearization}"'] \arrow[d, "\text{cycl. trace}"'] \\
 & & & & & \text{Top cyclic hom}
\end{tikzcd}$$

We are interested in:

Htpy type of Diff groups

Motivation

Back in the day:

Classify mfld $\longrightarrow$ find exotics

$$M \underset{top}{\cong} N \,, \quad M \underset{diff}{\ncong} N$$

$$\text{e.g.: } M \underset{top}{\neq} S^7 \,, \quad M \underset{diff}{\ncong} S^7$$

Nowadays:

Classify bundles of mflds $\rightsquigarrow$ find exotics

$$\begin{tikzcd}
M \arrow[r] & E \arrow[d] \\
 & X
\end{tikzcd} , \quad
\begin{tikzcd}
N \arrow[r] & F \arrow[d] \\
 & X
\end{tikzcd} \qquad E \underset{top}{\cong} F \qquad E \underset{diff}{\neq} F$$

$$\text{e.g. } \exists \quad \begin{tikzcd}
D^{15} \arrow[r] & E \arrow[d] \\
 & S^4
\end{tikzcd} \quad \text{nontriv.}$$

$$\begin{tikzcd} \text{Class. bundles with fiber } M \arrow[r, leftrightarrow] & \pi_* \mathrm{Diff}(M) \arrow[d, leftrightarrow] \\ {[X, B\mathrm{Diff}(M)]} \arrow[r, leftrightarrow] & \pi_* B\mathrm{Diff}(M) \end{tikzcd}$$

From Diff to $\widetilde{\mathrm{Diff}}/\mathrm{Diff}$

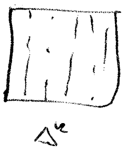
$$\boxed{\begin{tikzcd} \mathrm{Diff}(M) \arrow[r] \arrow[d, dash] & H(M) \arrow[r] \arrow[d, dash] & H(M)/\mathrm{Diff}(M) \arrow[d, dash, squiggly] \\ \text{want} & \text{htpy thy ✓} & \end{tikzcd}}$$

to understand $H(M)/\mathrm{Diff}(M)$: compare to block aut. groups

Block groups:

$$H(M) = |H_\cdot(M)|$$

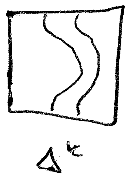
$$H_k(M) = \left\{ \begin{tikzcd} M \times \Delta^k \arrow[rr] \arrow[dr] & & M \times \Delta^k \arrow[dl] \\ & \Delta^k \arrow[u, phantom, "="] & \end{tikzcd} \right\}$$



"enlarge things to get simpler things we understand better":

$$\widetilde{H}(M) = |\underbrace{\widetilde{H}_k(M)}_{\overset{\shortparallel}{H(M \times \Delta^k, \text{faces})}}|$$

(faces mapped to faces, not nec. identity)



$$\widetilde{\mathrm{Diff}}(M) = |\widetilde{\mathrm{Diff}}_k(M) = \mathrm{Diff}(M \times \Delta^k, \text{faces})|$$

$$\begin{tikzcd} H(M) \arrow[r, "\sim"] & \widetilde{H}(M) & \text{"because diff-htpy eqn. are already so flabby"} \arrow[l, bend right, shift right=2] \\ \mathrm{Diff}(M) \arrow[u, hook] \arrow[r, hook] & \widetilde{\mathrm{Diff}}(M) \arrow[u, hook] & \end{tikzcd}$$

$$\boxed{\begin{tikzcd} \dfrac{\widetilde{\mathrm{Diff}}(M)}{\mathrm{Diff}(M)} \arrow[r] & \dfrac{H(M)}{\mathrm{Diff}(M)} \arrow[r] & \dfrac{\widetilde{H}(M)}{\widetilde{\mathrm{Diff}}(M)} \end{tikzcd}}$$

$$\begin{tikzcd} & \text{want} \arrow[u] & \text{surgery theory} \arrow[u] \end{tikzcd}$$

27.02.06
②

Surgery Theory:

"Space of smooth manifold structures on M" $\mathcal{S}^s(M)$

h-cobordism theorem: $\mathcal{S}(M) \simeq \widetilde{H}(M) / \widetilde{\mathrm{Diff}}(M) + \pi_0 \ldots$

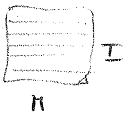
Surgery Exact Sequence:

$$\begin{tikzcd}
\mathcal{S}^s(M) \arrow[r] & (G/O)^M \arrow[r] \arrow[d, dash] & \mathcal{L}(M) \arrow[d, dash] \\
 & \text{easy } \checkmark & \text{surgery obs sp. } \checkmark
\end{tikzcd}$$

[Attention: might be difficulties with the decorations here.]

From $\frac{\widetilde{\mathrm{Diff}}}{\mathrm{Diff}}$ to Conc:

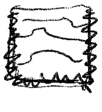
Consider



$$\begin{tikzcd}
\Omega \mathrm{Diff}(M) \arrow[r] \arrow[d] & P\mathrm{Diff}(M) \arrow[r] \arrow[d] & \mathrm{Diff}(M) \arrow[d] \\
\Omega \widetilde{\mathrm{Diff}}(M) \arrow[r] & P\widetilde{\mathrm{Diff}}(M) \arrow[r] & \widetilde{\mathrm{Diff}}(M)
\end{tikzcd}$$

now: change middle & left terms to get something interesting.

Instead:



$C(M)$ – concordance space (= pseudo-isotopy space)

implicitly: id on bdry

$$\begin{tikzcd}
 & \mathrm{Diff}(M\times I) \arrow[r] \arrow[d] & \mathrm{Diff}(M\times I, \underset{M\times 1}{\cancel{\partial M\times I \cup M\times 0}}) \arrow[r] \arrow[d] & \mathrm{Diff}(M) \arrow[d] \\
\Omega\widetilde{\mathrm{Diff}}(M) \arrow[r, phantom, "="] & \widetilde{\mathrm{Diff}}(M\times I) \arrow[r] \arrow[d] & P\widetilde{\mathrm{Diff}}(M) \arrow[r] \arrow[d] & \widetilde{\mathrm{Diff}}(M) \arrow[d] \\
 & \frac{\widetilde{\mathrm{Diff}}(M\times I)}{\mathrm{Diff}(M\times I)} \arrow[r] & BC(M) \arrow[r] & \frac{\widetilde{\mathrm{Diff}}(M)}{\mathrm{Diff}(M)}
\end{tikzcd}$$

with $C(M) = \mathrm{Diff}(M\times I, M\times 1)$

wrap this up in a spectral sequence!

Have

$$E^1_{pq} = \pi_q\left(BC(M\times I^p)\right) \Longrightarrow \pi_{p+q}\left(\frac{\mathrm{Diff}(M)}{\mathrm{Diff}(M)}\right)$$

$$d : C(M\times I^p) \longrightarrow C(M\times I^{p-1})$$

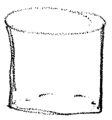
restr. to top.

(Hatcher spectral sequence)

From Conc to Stab Conc:

Note:

$$(M\times I, \partial M\times I \cup M\times 0) \cong (M\times I, M\times 0)$$

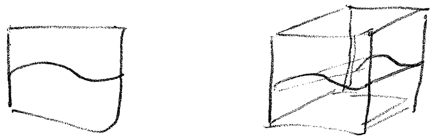


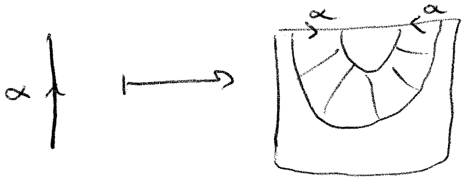
(push down using a collar neighborhood)

$$\underset{C(M)}{\underset{\shortparallel}{\mathrm{Diff}(\;\text{———}\;''\;\text{———}\;)}} \cong \underset{C'(M)}{\underset{\shortparallel}{\mathrm{Diff}(\;\text{—}\;''\;\text{—}\;)}}$$

Stabilization Maps:

$$C'(M) \xrightarrow{\sigma} C'(M\times I) \qquad\qquad C(M) \xrightarrow{\sigma} C(M\times I)$$





Thm (Igusa):

σ is k-connected for $k \sim< \frac{\dim(M)}{3}$

so define: stable concordance space

$$\mathcal{C}(M) := \operatorname{colim} C(M\times I^p).$$

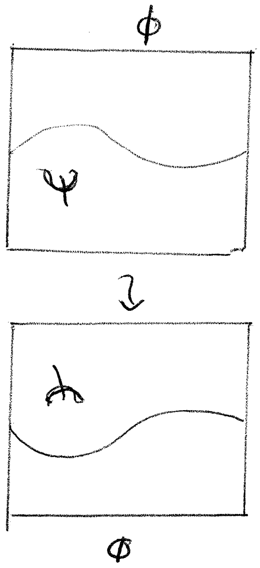
$$\boxed{\text{This is a homotopy functor on compact manifolds !}}$$ Weird, amazing!

⇝ try to define on all spaces

use htpy thy

27.02.06

Involution χ:



$$\rightsquigarrow \quad (\phi^{-1} \times id_I)(h) =: \chi(\psi)$$

$$d^1_{pq}(\sigma(\psi)) = \psi + \chi(\psi)$$

$$\Rightarrow d^1_{pq}(\underset{\mathcal{C}(M)}{\underset{\cap}{\psi}}) = \psi + (-1)^p \chi(\psi)$$

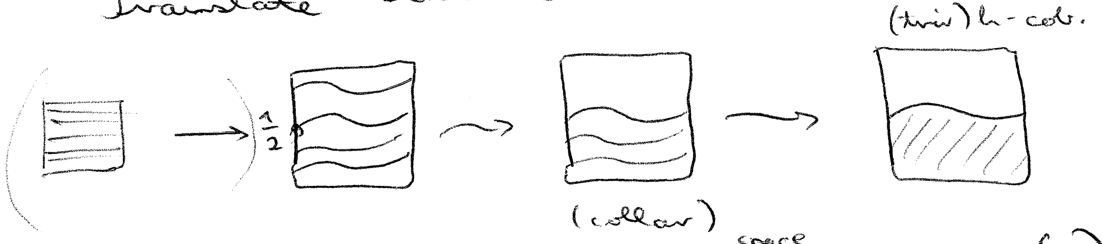
(Involution only commutes with stab. up to a sign; when you define it on the stable space, you have to introduce a sign.)

Conclusion:

$$\boxed{E^2_{pq} = H_p(\mathbb{Z}/2\,;\ \pi_{q-1}\,\mathcal{C}(M)) \qquad \text{for } q \lesssim p/3}$$

From Stab Conc to Whitehead:

Translate Stab Conc to h Cobordisms:



$$\begin{tikzcd} C_{1/2}(M) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & E(M) \arrow[r, "\text{space of collars}"] \arrow[d, phantom, "="] & \mathcal{H}(M) \\ C(M) & \{\alpha \in Emb(M\times I, M\times I) \mid \alpha|_{M\times 0} = id,\ \alpha|_{\partial M\times I} = id_{\partial M} \times \tfrac{1}{2} id_I\} & \end{tikzcd}$$

$\mathcal{H}(M)$ — space of triv h-cob.

$E(M) \simeq *$

$$\Rightarrow \boxed{BC(M) \simeq \mathcal{H}(M)}$$

Concordance
≙ bdles of h-cob on I
trivialized at the ends
≙ loop in $\mathcal{H}(M)$

Translate $\mathcal{H}(M)$ to Whitehead:

S-Cob. Theorem:

$$\left\{\begin{matrix} h\text{-Cob.} \\ \text{on } M \end{matrix}\right\} \Big/ {\scriptstyle \substack{\text{diff} \\ \text{rel. base}}} = Wh_1(\pi_1(M))$$

stable version:

$$\mathcal{H}(M) \simeq \Omega\, Wh(M)$$

Whitehead space will be mentioned later

$$\boxed{B^2\mathcal{C}(M) \simeq Wh(M)}$$

From Whitehead to Alg. K-theory

Classical: $Wh_1(\pi_1 M) = K_1(\mathbb{Z}[\pi_1 M]) / \pm \pi_1 M$

(it is also a summand of K_1, ~~more~~ difficult to prove)

Modern: "$Wh(M) = K(M) / Q(M_+)$"

$$A(M) = K(M) = \mathbb{Z} \times BGL(\underbrace{Q(\Omega M_+)}_{})^+$$

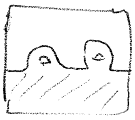
htpy version of $\mathbb{Z}\pi_1 X$

$$= \Omega\, |wS.R_f(M)|$$

Thm (Waldh.):

group completion of some geometric maps.

space of rigid tube partitions



$\exists$ htpy fiber sequence

$$\underline{\Omega B}(B\mathcal{C}(M)) \longrightarrow D(M) \longrightarrow A(M)$$

group completion

rewrite this as

$$\begin{tikzcd} D(M) \arrow[r] & A(M) \arrow[l, bend left, dashed] \arrow[r] & Wh(M) \end{tikzcd}$$

Thm (Waldh.): $\exists$ splitting ↑

Thm (Waldh.): $D(M) \simeq Q(M_+)$

$$\boxed{\text{Cor.:} \quad A(M) \simeq Wh(M) \times Q(M_+)}$$

Aside: $Wh(M) = Wh^{Diff}(M)$

27.02.06

Now: compute algebraic K-theory, using

- Linearization
- cyclotomic trace to TC
  (we won't talk about it, but that's what people actually do these days)

<u>Results</u>: $\pi_* \mathrm{Diff}(D^n)$

$\pi_* \mathrm{Diff}(S^n)$

$$\begin{tikzcd}
 & & & & & & \text{Alg K-th of Rings} \\
\mathrm{Diff} \arrow[r, "\substack{2+3 \\ \text{Smng.}}"] & \widetilde{\mathrm{Diff}}/\mathrm{Diff} \arrow[r, "\substack{4+5 \\ \text{Conc SS}}"] \arrow[rrr, dashed, bend right, "\mathrm{WW}"'] & \mathrm{Conc} \arrow[r, "\text{Stab}"] & \text{Stab Conc} \arrow[r, "\substack{8+9 \\ \text{Eq. of h-cob}}"] & \mathrm{Wh} \arrow[r, "\substack{8+9 \\ \text{Waldh. Eqv.}}"] & \overset{7}{\text{Alg K-th}} \arrow[ur, "\substack{10 \\ \text{linearization}}"] \arrow[dr, "\text{cyclo trace}"'] & \\
 & & & & & & \text{Top. cyc.}
\end{tikzcd}$$

1 st half of the week

Alg. Surgery 11
Controlled Top. 12+13
Index Thy 14+15
→ WWW $\underline{\text{III}}$ 16

Higher Torsion Inv. 17+18

Singularity App. 19

27.02.06

⑤

Introduction to Surgery Theory I

André Henriques

$$\cdots \hookrightarrow S^{\alpha,\beta}_{n+1}(X\times I, X\times\partial I) \longrightarrow \mathcal{N}^{\alpha}_{n+1}(X\times I, X\times\partial I) \longrightarrow L^{\beta}_{n+1}(\mathbb{Z}[\pi_1(X)]) \hookrightarrow$$

$$S^{\alpha,\beta}_{n}(X) \longrightarrow \mathcal{N}^{\alpha}_{n}(X) \longrightarrow L^{\beta}_{n}(\mathbb{Z}[\pi_1(X)])$$

structure set — set of normal invariants — quadratic L-groups

$\alpha \in \{TOP, PL, DIFF\}$, $\beta \in \{s, h\}$.

X : Poincaré complex, (n-dim)

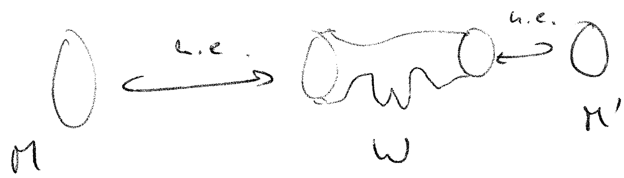
(space that cohomologically looks like a manifold)

<u>structure set</u>:

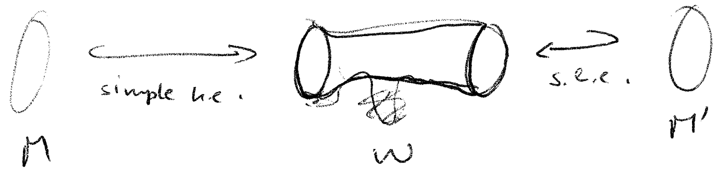
$$S^{h}_{n}(X) = \left\{ f: M \longrightarrow X \;\middle|\; \begin{array}{l} M \text{ is a manifold } \left(\begin{Bmatrix} TOP \\ PL \\ DIFF \end{Bmatrix}\right) \\ f \text{ is a homotopy equ.} \end{array} \right\} \Big/ \sim_{h\text{-cob.}}$$

$$S^{s}_{n}(X) = \text{———————}\text{''} - f \text{ is a } \underline{\text{simple}} \text{ homotopy equ. } \text{—''—} \Big/ \sim_{s\text{-cob.}}$$

<u>h-cob.</u>:



<u>s-cob.</u>:



$(\Rightarrow W \cong M\times I)$

What is a simple homotopy equivalence?

$$f: X \rightleftarrows Y : g \qquad \begin{array}{l} h: X\times I \longrightarrow X \\ k: Y \times I \longrightarrow Y \end{array}$$

$$A = h+k+f+g+d_X+d_Y : C_{even}(\tilde{X}) \oplus C_{odd}(\tilde{Y}) \longrightarrow C_{odd}(\tilde{X}) \oplus C_{even}(\tilde{Y})$$

$$A \in K_1(\mathbb{Z}[\pi_1 X]) \Big/ \{\pm g \mid g\in\pi_1\} =: Wh_1(\pi_1(X)).$$

simple htpy equi: $[A] = 0$.

This is the Whitehead torsion of the map.

(56)

Now: $\mathcal{N}_n^{\sim}(X)$.

we build the "normal bundle" of X, even if X is not a mfld.

$X \hookrightarrow \mathbb{R}^N$ $\qquad N \supseteq X$

$\partial N \longrightarrow X$ $\qquad$ corr. to the sphere bundle of the normal bdlg on a mfld.

We can do this, this is not a fibration in general, but

$p : E_X := \partial N \times_X X^I \longrightarrow X$ is a fibration, $\boxed{\text{SPIVAK NORMAL FIBRATION}}$

This shares properties with the sphere bundle of the normal bdle on a mfld.
~~(e.g. the homotopy fibre is~~ $\qquad$ this is a homotopy inv. constr.

Lemma:

X Poincaré complex $\Longleftrightarrow$ fib $(p) \simeq S^{N-n-1}$

$\ulcorner$ Spanier-Whitehead dual of X: $\mathbb{R}^N / \mathbb{R}^N - N$ $\lrcorner$

Def.: X Poincaré complex :$\Leftrightarrow$ complex equipped with a choice of

$[X] \in H_n(X, \{\mathbb{Z}\})$ s.th. $\qquad$ (essentially unique)

$\cap [\widetilde{X}] : H_c^{n-i}(\widetilde{X}) \xrightarrow{\simeq} H_i(\widetilde{X})$

$\uparrow$ lives in Borel-Moore homology of $\widetilde{X}$

Def.:

$$\mathcal{N}(X) = \left\{ \begin{tikzcd} \text{Sphere}(V) \arrow[r, "\eta"] \arrow[dr] & E_X \arrow[d, "p"] \\ & X \end{tikzcd} \;\middle|\; \begin{array}{l} V \text{ is a vector bundle} \\ \eta \text{ is a fib. h.e.} \end{array} \right\} \Big/ \sim$$

$\cong$ ~~$G/\!\!X$~~ ~~G/O~~ (if $\uparrow$ non-empty)

$[X, G/O]$

"vector bundle": DIFF : usual one

TOP : homeom of $\mathbb{R}^n$ that fix origin

PL : PL — — "

mod out by: — homotopies

— stabilization $(V \longmapsto V \oplus \mathbb{R})$

$$\begin{tikzcd} & G/O \arrow[d] \\ & BO \arrow[d] \\ X \arrow[ur, "?"] \arrow[r] & BG \end{tikzcd}$$

poss. choices of lifts $\rightarrow G/O$

27.02.06

⑥

$$\begin{tikzcd}
 & V \arrow[dl] \arrow[dr, dashed] \arrow[ddr, bend left, "i"'] & \\
E_M \arrow[rr] \arrow[d] & & E_X \arrow[d] \\
M \arrow[rr, dash, "\simeq"'] & & X
\end{tikzcd}$$

can. element in the set of normal invariants

Comment on the relative sets:

We already need X to be a mfld to define the relative sets

idea of explanation: $S_n^{\alpha,\beta}(X)$ is π_0 of a space, the space of mfld structures,

$S_{n+1}^{\alpha,\beta}(X \times I, X \times \partial I)$ is π_1, ~~for that~~ you need a basepoint to define ~~that~~, i.e. a certain mfld structure on X

Rmk.: in TOP, everything in the ~~sequence~~ is π_0 of an infinite loop space, so a group

in DIFF, this is not the case

Now: L-groups:

$L_n(R)$ = Cobordism group of n-dimensional quadratic Poincaré complexes

(complexes of R-modules equipped with additional data)

- algebraically, every complex is cobordant to a rather simple complex:

n even: concentrated in dim $\frac{n}{2}$

n odd: — " — $\frac{n-1}{2}, \frac{n+1}{2}$

[This statement "is false" if I were to replace quadratic by symmetric.]

$$L_n(R) = \left\{ \begin{matrix} \text{non-degenerate} \\ \varepsilon\text{-quadratic free} \\ R\text{-module} \end{matrix} \right\} \Big/ \sim \quad \leftarrow \text{cob + hyperbolic ones} \sim 0.$$

n even:

$$\varepsilon = \begin{cases} +1 & n \equiv 0 \ (4) \\ -1 & n \equiv 2 \ (4) \end{cases}$$

~~Def.: ε quadratic module: $\exists$ q:~~

Def.:

ε-symmetric module M;

$\exists \; q : M \times M \longrightarrow R$

$q(a,b) = \varepsilon \overline{q(b,a)}$

$\tilde{q} : M \xrightarrow{\simeq} M^*$

ε-quadratic module M;

ε-symmetric module equipped with a quadratic refinement, i.e.

$\mu : M \longrightarrow R/_{r = \varepsilon \bar{r}}$ s.th.

$\mu(ra) = r\mu(a)\bar{r}$

$\widetilde{\mu(a)} + \varepsilon \overline{\widetilde{\mu(a)}} = q(a,a)$ ($\sim$: left to R.)

$\mu(a+b) - \mu(a) - \mu(b) = [q(a,b)]$.

---

hyperbolic module: corr. to a matrix $\begin{pmatrix} 0 & 1 \\ \varepsilon & 0 \end{pmatrix}$.

Now: surgery obstruction class

map from $\mathcal{N}_n^{\alpha}(X)$ to $L_n^{\beta}(\mathbb{Z}\pi_1(X))$.

~~start~~ with $\left(\begin{tikzcd} \text{Sphere}(V) \arrow[r, "\eta"] \arrow[dr] & E_X \arrow[d, "p"] \\ & X \end{tikzcd} \right) \in \mathcal{N}_n(X)$.

$$S^N \longrightarrow N/_{\partial N} \simeq \text{Thom}(E_X) \overset{\eta}{\simeq} \text{Thom}(V)$$

$M := f^{-1}(\text{0-section of }) \subseteq S^N$

(comes with a map to X)

$$\begin{tikzcd} \nu(M) \arrow[r] \arrow[d, dash] & V \arrow[d, dash] \\ M \arrow[r] & X \\ {[M]} & {[X]} \arrow[l, mapsto] \end{tikzcd}$$

this is called a degree 1 normal map.

now: make the map $M \to X$ more & more connected, see when we are stuck.

Surgery below the middle dim $\Rightarrow$ $H_*(\tilde{M}) \xrightarrow{\simeq} H_*(\tilde{X})$ $\forall *$ but $n/2$.

27.02.06

for $* = n/2$: $H_{\cancel{*}\, n/2}(\tilde{M}) \xrightarrow{\not\cong} H_{n/2}(\tilde{X})$

Define the surgery kernel

$$K := \ker\left(H_{n/2}(\tilde{M}) \longrightarrow H_{n/2}(\tilde{X}) \right) .$$

This is an R-module,

you obtain a symmetric str. by filling in spheres, looking at their self-intersection (~~[illegible]~~ and intersections with its translates) ~~[illegible]~~ ,

actually gives quadr. str,

Now: The connecting map:

action of $\pi_1(X)$ on ..., called plumbing construction .

---

for topological mflds, you need:

- TOP - transversality
- ~~handlebody dec. for top. mflds~~ Whitney trick

(need to know when you can embed sphere × disks)

27.02.06

⑧

Surgery II

Stacy

Connections between Surgery & $\widetilde{G}^s(M)/\widetilde{TOP}(M)$ or $\widetilde{G}^s(M)/\widetilde{DIFF}(M)$

G: space of homotopy equivalences.

M^n = connected oriented closed CAT-manifold

CAT = TOP or DIFF

① Construct a homotopy fibration sequence of Δ-sets

$$\widetilde{\mathcal{S}}^s_{CAT}(M) \longrightarrow \underset{\substack{\uparrow \\ \text{space of} \\ \text{normal maps}}}{\mathcal{N}_{CAT}(M)} \longrightarrow \underset{\substack{\uparrow \\ \text{surgery space}}}{\mathbb{L}^s_n(\pi_1 M)}$$

② $$\widetilde{\mathcal{S}}^s_{CAT}(M) \simeq \coprod_{\substack{\text{CAT-iso} \\ \text{classes of} \\ \text{mflds } N \text{ s.h.e. to } M}} \widetilde{G}^s(N)/\widetilde{CAT}(N)$$

③ CAT = TOP

$\widetilde{\mathcal{S}}^s_{TOP}(M) \simeq$ ho-fiber of a certain assembly map.

CAT = TOP

$\widetilde{\mathcal{S}}^s_{TOP}(M)$ = block structure space of M

0-simplex (N^n, f) N^n s.h.e. to M $\left\{ \begin{array}{l} N^n \text{ closed top. mfld} \\ M \xrightarrow[\text{s.h.e.}]{f} M \end{array} \right.$

1-simplex (W^{n+1}, f) W = cob. bdw. $\partial_0 W$ & $\partial_1 W$

$W \xrightarrow[\text{s.h.e.}]{f} M \times I$

$(W; \partial_0 W, \partial_1 W) \xrightarrow{f} (M\times I, M\times 0, M\times 1)$

k - simplex (Q^{n+k}, f) $\qquad$ Q^{n+k} = "manifolds modelled on Δ^k"

$$f : Q \xrightarrow[s.h.e.]{} M \times \Delta^k$$

(correspondence between faces of Δ^k and submanifolds of Q, corr. is intersection preserving)

Alternatively:

$\mathcal{S}^s_{TOP}(M)$ is the diagonal nerve of the simplicial category whose objects in degree k are pairs ~~$(N \times \Delta^k$~~

$$\left(\underset{\substack{\uparrow \\ s.h.e. \\ to\ M}}{N} \times \Delta^k,\ \alpha : N \times \Delta^k \underset{\substack{\uparrow \\ block\text{-}pres. \\ s.h.e.}}{\longrightarrow} M \times \Delta^k \right)$$

maps in degree k are block-pres. homeos

$$\begin{tikzcd}
N \times \Delta^k \arrow[rr] \arrow[dr, "\alpha"'] & \equiv & N' \times \Delta^k \arrow[dl, "\alpha'"] \\
& M \times \Delta^k &
\end{tikzcd}$$

s.th. $\qquad$ commutes.

$\mathcal{N}_{TOP}(M)$ = space of normal maps

0- simplex $(f, \hat{f})$

$$\begin{tikzcd}
\nu_N \arrow[r, "\hat{f}"] \arrow[d] & \eta \\
N^n \arrow[r, "f", "deg 1"'] & M \arrow[from=1-2, to=2-2]
\end{tikzcd}$$

η = Euclidean bundle over M

n - simplex $(f, \hat{f})$

$$\begin{tikzcd}
\nu_Q \arrow[r, "\hat{f}"] \arrow[d] & \eta \arrow[d] \\
Q^{n+k} \arrow[r, "f", "deg 1"'] & M \times \Delta^k
\end{tikzcd}$$

η = Euclidean bundle over $M \times \Delta^k$

$$\mathcal{N}_{TOP}(M) \simeq \left(G/TOP \right)^M .$$

27.02.06

⑨

Surgery Space $\mathbb{L}^s_n(K)$ (Quinn)

(If $K = K(\pi, 1)$, write $\mathbb{L}^s_n(\pi)$ or $L^s_n(\mathbb{Z}\pi)$)

0-simplex

$$\begin{tikzcd} \nu_N \arrow[r] \arrow[d] & \eta \arrow[d] \\ (N^n, \partial N) \arrow[r, "f", "\deg 1"'] & (X, A) \end{tikzcd}$$

Euclidean bundle over X

n-dim. Poincaré pair

reference map $w: X \longrightarrow K$

$$f: \partial N \xrightarrow[\text{s.h.e.}]{} A$$

1-simplex: bordism of such things in domain & codomain

(normally, we only allow product cobordisms in the codomain, here we allow all)

For more info: Quinn "Geometric Formulation of Surgery"

Nicas "Induction Thms ..."

Nice Properties of $L^s_n(K)$

1) $\pi_0(\mathbb{L}^s_n(K)) = L^s_n(\mathbb{Z}\pi_1 K)$

$\pi_j(\mathbb{L}^s_n(K)) = L^s_{n+j}(\mathbb{Z}\pi_1 K)$ $(n+j \geq 5)$

2) $\mathbb{L}^s_n(K) \simeq \Omega \mathbb{L}^s_{n-1}(K)$

3) $\mathbb{L}^s_{n+4}(K) \simeq \mathbb{L}^s_n(K)$ (Periodicity)

given geometrically by $\times \underline{id_{\mathbb{CP}^2}}$!

homotopy fibration sequence

$$\mathcal{S}^s_{TOP}(M) \longrightarrow \mathcal{N}_{TOP}(M) \longrightarrow L^s_n(\pi_1 M)$$

$$\mathcal{N}_{TOP}(M) = (G/TOP)^M$$

(9b')

② $\mathcal{S}^s_{TOP}(M) \simeq \coprod\limits_{\substack{\text{homeom.}\\ \text{classes of mflds } N\\ \text{s.h.e. to } M}} \widetilde{G}^s(N) \big/ \widetilde{TOP}(N)$

<u>The "over-category" for simplicial categories:</u>

$\mathcal{C}, \mathcal{D} : \Delta^{op} \longrightarrow \underline{\mathcal{C}at}$ simplicial category

$\mathcal{C}_K, \mathcal{D}_K$ = K-th level categories

$F : \mathcal{C} \longrightarrow \mathcal{D}$

D obj in $\mathcal{D}_0$

$\exists!$ map $p : [k] \longrightarrow [0]$ in Δ

which induces a map

$p_* : \mathcal{D}_0 \longrightarrow \mathcal{D}_k$

$\quad D \longmapsto p_* D$

<u>Def.:</u> <u>Over-Category</u> (F/D)

$(F/D)_k = \Big\{$ objects $\big(C_k,\ \alpha : F(C_k) \longrightarrow p_* D\big)$

C_k obj. in $\mathcal{C}_k$ — map in $\mathcal{D}_k$

maps $f : C_k \longrightarrow C_k'$

map in $\mathcal{C}_k$

s.th.

$$\begin{tikzcd} F(C_k) \arrow[rr, "F(f)"] \arrow[dr, "\alpha"'] & \arrow[d, phantom, "="] & F(C_k') \arrow[dl, "\alpha'"] \\ & p_* D & \end{tikzcd}$$

given D, D' in $\mathcal{D}_0$ and map $D \to D'$,

get induced map $B(F/D) \xrightarrow[\uparrow]{} B(F/D')$

transition map.

<u>Quillen's Thm B:</u>

If all transition maps are h.e's,

$$\begin{tikzcd} B(F/D) \arrow[r] \arrow[d] & B\mathcal{C} \arrow[d, "BF"] \\ * \simeq B(Id_{\mathcal{D}}/D) \arrow[r] & B\mathcal{D} \end{tikzcd}$$

homotopy pullback.

27.02.06

⑩

Application:

$\widetilde{\mathcal{G}}^s_M$, $\widetilde{Top}_M$

$(\widetilde{\mathcal{G}}^s_M)_k$ = category whose obj. $N^n \times \Delta^k$ N s.h.e to M

maps $N \times \Delta^k \underset{\substack{\text{block pres.}\\ \text{s.h.e.}}}{\longrightarrow} N' \times \Delta^k$

$(\widetilde{Top}_M)_k$ = same objects

maps : block-pres. homeos

$F : \widetilde{Top}_M \longrightarrow \widetilde{\mathcal{G}}^s_M$

$M = M \times \Delta^0 =$ obj in $(\widetilde{\mathcal{G}}^s_M)_0$

(F/M) $\quad \left(N \times \Delta^k, N \times \Delta^k \xrightarrow[\text{s.h.e.}]{\text{homeo-pres,}} M \times \Delta^k\right)$

Quillen's Thm B $\Rightarrow$

$$\begin{tikzcd}
 & & B(F/M) \arrow[r] \arrow[dl, equal] & B(\widetilde{Top}_M) \arrow[r, "BF"] \arrow[dd, "\cong"] & B(\widetilde{\mathcal{G}}^s_M) \arrow[dd, "\cong"] \\
 & \widetilde{\mathcal{G}}^s_{TOP}(M) \arrow[dl, "\simeq"] & & & \\
\coprod\limits_{\substack{\text{homeo classes}\\ \text{of mflds s.h.e to}M}} \widetilde{G}^s(N)/\widetilde{TOP}(N) \arrow[rrr] & & & \coprod\limits_{\substack{\text{homeo classes}\\ \text{of mflds }N\\ \text{s.h.e. to}M}} B\widetilde{TOP}(N) \arrow[r] & B\widetilde{G}^s(M)
\end{tikzcd}$$

So $\quad \widetilde{\mathcal{G}}^s_{TOP}(M) \simeq \coprod\limits_{\substack{\text{homeo classes}\\ \text{of mflds}\\ \text{s.h.e to}M}} \widetilde{G}^s(N)\Big/\widetilde{TOP}(N)$

Rmk.: All of this could have been done for DIFF instead of TOP.

Now: Only possible in TOP.

③ For CAT = TOP only:

simply connected L-theory spectrum $\mathbb{L}_*(pt) = \mathbb{L}_*(\mathbb{Z})$.

$$\mathbb{H}_n(M; \mathbb{L}_n(\mathbb{Z})) \xrightarrow[\text{assembly map}]{A} \mathbb{L}_n(\mathbb{Z}\pi_1 M)$$

0th space of $\mathbb{L}_*(\mathbb{Z}) = \mathbb{Z} \times G/TOP$. (We would like G/TOP only.)

1- connected cover of $\mathbb{L}_0(\mathbb{Z}) =: \underline{\mathbb{L}_\cdot}(\mathbb{Z})$

$$\mathbb{H}_n(M; \underline{\mathbb{L}}_\cdot(\mathbb{Z})) \xrightarrow[\text{assembly map}]{\tilde{A}} \mathbb{L}_n(\mathbb{Z}\pi_1 M)$$

0th space of $\underline{\mathbb{L}}_\cdot(\mathbb{Z}) = G/TOP$.

$$\begin{tikzcd}
\tilde{\mathcal{S}}^s_{TOP}(M) \arrow[r] \arrow[ddd, phantom, "\cong" rotate=90] & \mathcal{N}_{TOP}(M) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \mathbb{L}_n(\mathbb{Z}\pi_1 M) \arrow[ddd, equal] \\
 & (G/TOP)^M \arrow[d, phantom, "=" rotate=90] & \\
 & \mathbb{H}^0(M; \underline{\mathbb{L}}_\cdot(\mathbb{Z})) \arrow[d, phantom, "\cong\ \text{Poincaré Duality}" rotate=90] & \\
\text{hom. fiber of } \tilde{A} \arrow[r] & \mathbb{H}_n(M; \underline{\mathbb{L}}_\cdot(\mathbb{Z})) \arrow[r, "\tilde{A}"] & \mathbb{L}_n(\mathbb{Z}\pi_1 M)
\end{tikzcd}$$

commutativity of this is not completely clear (if you just use any assembly map) you need some explicit descr. of the ass map to see this is hom. com—.

(does also not quite work for PL)

27.02.06

Orthogonal Calculus

Tibor

① Motivation

$\mathrm{Aut}(X)$ modulo $G(X)$

(TOP, DIFF)

$$\mathrm{Aut}(X) \rightsquigarrow \widetilde{\mathrm{Aut}}(X)/\mathrm{Aut}(X) \xrightarrow[\text{sp. sequ.}]{\text{concordance}} \mathcal{C}(X)$$

$$\widetilde{\mathrm{Aut}}(X)/\mathrm{Aut}(X) \Leftarrow \pi_{*-1}(\mathcal{C}(X)) \circlearrowleft \mathbb{Z}/2$$

$$\mathcal{C}(X) \simeq \Omega\mathcal{H}(X)$$

$$\mathcal{H}(X) \simeq \Omega^\infty \mathbb{H}(X).$$

[WW1]

$$\widetilde{\mathrm{Aut}}(X)/\mathrm{Aut}(X) \xrightarrow{(*)} \Omega^\infty\left(\mathbb{H}_s^{\mathrm{Aut}}(X)_{h\mathbb{Z}/2}\right)$$

$(*)$ is roughly $\frac{\dim(X)}{3}$ -conn.

Rmk.: There is always a s.s. ass. to a htpy orbit space, the above s.s. is what is ass to this htpy orbit space (in the stable range)

- This is a special case of a map from orthogonal calculus.
- Suggests a description of $\widetilde{\mathrm{Aut}}(X)/\mathrm{Aut}(X)$ outside the stable range.

② Setup

Informally:

- tool for studying spaces with a coordinate free ~~spectrum~~ filtration indexed by finite-dim. vector spaces

$$BO \longrightarrow \dim_{n\in\mathbb{N}} BO(n)$$

$$BO(V) \longrightarrow BO$$

- Important to have the filtration indexed by vector spaces rather than natural numbers.

Formally:

$\mathcal{J}$ $\mathrm{Ob}(\mathcal{J})$ – real fin. dim. vector spaces with inner products

$\mathrm{Mor}_{\mathcal{J}}(V,W)$ – lin. isometries (need not be isomorphisms, can be embeddings)

$E: \mathcal{J} \longrightarrow \mathcal{T}_*$ continuous

ev: $\mathrm{Mor}(V,W)_+ \wedge E(V) \longrightarrow E(W)$ is cont. $\forall$ V, W.

Rem.: · normally interested in $E(\mathbb{R}^\infty) = \operatorname{hocolim}_{n\in\mathbb{N}} E(\mathbb{R}^n)$

or $E_0(0) = \operatorname{hofiber}_*(E(0) \longrightarrow E(\mathbb{R}^\infty))$

Ex.: 1. $E: V \longmapsto BO(V)$ $E(\mathbb{R}^\infty) = BO$ $E_0(0) = O$ $(=\Omega BO)$

2. ~~E:~~ $\mathrm{Aut}^b(X\times V)$: bounded autom. of $X\times V$.

bounded: $f: X\times V \longrightarrow X\times V$

$f(x,t) = (f_1(x,t), f_2(x,t))$

$\exists K \quad \|f_2(x,t) - t\| < K$.

$E: V \longmapsto B\mathrm{Aut}^b(X\times V)$

$E(\mathbb{R}^\infty) = B\mathrm{Aut}^b(X\times\mathbb{R}^\infty)$

$E_0(0) = \mathrm{Aut}^b(X\times\mathbb{R}^\infty)/\mathrm{Aut}(X)$

3. $E: V \longmapsto BTOP(V)$ $E(\mathbb{R}^\infty) = BTOP$ $E_0(0) = TOP$

$TOP(V)$

28.02.06

③ Taylor Tower

$$\begin{tikzcd}
 & \vdots \arrow[d, dash, dotted] \\
E \arrow[r] \arrow[rd] \arrow[rddd] & T_k^d E \arrow[d] \\
 & T_{k-1}E \arrow[d] \\
 & \vdots \arrow[d] \\
 & T_0 E
\end{tikzcd}$$

polynomial of deg. $\leq k$
with a certain univ. prop.
(best approx. by such a functor)

- $T_0 E(V) = \operatorname{hocolim}_{n \in N} E(V \otimes R^n) \simeq E(R^\infty)$
- $D_k E(V) = \operatorname{hofiber}_*(T_k E(V) \to T_{k-1} E(V))$

classified by certain spectra with action of $O(k)$

- $E_0(V) = \operatorname{hofiber}(E(V) \to T_0 E(V))$ $\quad T_0E(V) \simeq E(R^\infty)$

$$\begin{tikzcd}
E_0(V) \arrow[r, dashed] \arrow[d] & D_1 E(V) \arrow[d] \\
E(V) \arrow[r] \arrow[d] & T_1 E(V) \arrow[d] \\
T_0 E(V) \arrow[r, "="] & T_0 E(V)
\end{tikzcd}$$

$\rightsquigarrow$ $E_0 \longrightarrow D_1 E$ (nat. transf.)

turn out to be interesting

$D_k E$ are homs of certain characteristic classes

④ The Derivative Spectra

$\ominus$ $E^{(1)}$

$E^{(1)}(V) = \operatorname{hofiber}_*(E(V) \longrightarrow E(V \oplus R))$

$\sigma_1 : U^c \wedge E^{(1)}(V) \longrightarrow E^{(1)}(V \oplus U)$

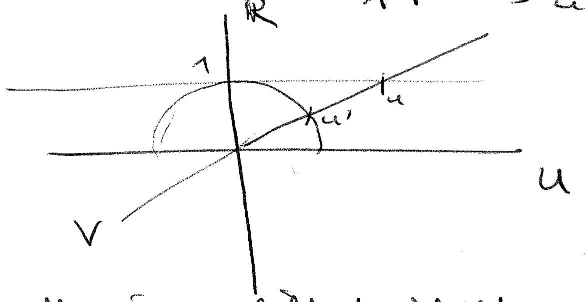
has certain assoc. prop.

$V \longmapsto BO(V)$
V^c
$U^c \wedge V^c \to (U \oplus V)^c$

$$\begin{tikzcd}
V \arrow[r, "id\oplus 0"] \arrow[d, "id\oplus 0"'] & V\oplus\mathbb{R} \arrow[d, dashed, "\zeta_u"] \\
V\oplus U \arrow[r] & V\oplus U\oplus\mathbb{R}
\end{tikzcd}$$

$\forall u\in U\ \exists\, \zeta_u : V\oplus\mathbb{R} \longrightarrow V\oplus U\oplus\mathbb{R}$

$1 \longmapsto u'$.



If $u\to\infty$, then $u'\in U$, so there's a lift to $V\oplus U$.

Rem.: does not satisfy assoc. prop.

Induction:

$O(1) \circlearrowleft \mathbb{R} \Longrightarrow O(1)\circlearrowleft E^{(1)}(V)$

(by flip)

$\theta E^{(k)}$

$\sigma : (kU)^c \wedge E^{(k)}(V) \longrightarrow E^{(k)}(V\oplus U)$

$k\cdot U = U\oplus\mathbb{R}^k$

- $k=2$

$F(V) := \Omega^V E^{(1)}(V)$ ⊛

$F^{(1)}(V) \to \Omega^V F^{(1)}(V\oplus U)$

$E^{(1)}(V) = \text{hofiber}_*\left(E^{(1)}(V)\longrightarrow \Omega E^{(1)}(V\oplus\mathbb{R})\right)$

$F^{(1)}(V) = \Omega^V E^{(2)}(V)$

$\Omega^V E^{(2)}(V) \longrightarrow \Omega^U\Omega^{V\oplus U} E^{(2)}(V\oplus U)$

$E^{(2)}(V) \longrightarrow \Omega^{2\cdot U} E^{(2)}(V\oplus U)$

$O(2)\circlearrowleft E^{(2)}(V)$

(on the loop coord. & the other of $\mathbb{R}$ inside)

⊛ Defining F like that, we have to check that F is a functor: we have to use the $\sigma_1 : U^c\wedge E^{(1)}(V)\to E^{(1)}(V\oplus U)$

$F(V)\to F(V\oplus U)$

$\Omega E^{(1)}(V)\to \Omega^{V\oplus U} E^{(1)}(V\oplus U)$

$\Omega^U E^{(2)}(V)\to \Omega^V\Omega^U E^2(V\oplus U)$

27.02.06

Ex.:

① $E: V \longmapsto BO$

$\Theta E^{(1)} \simeq S^0$ action of $O(1)$ is trivial

$\Theta E^{(2)} \simeq S^{-1}$

[diverse]

② $E: V \longmapsto B\,\mathrm{Diff}^b(X \times V)$

$\Theta E^{(1)} \simeq H^{\mathrm{diff}}(X)$

③ $E: V \longmapsto BTOP(V)$

$\Theta E^{(1)} \simeq A(*)$

---

⑤ Taylor Tower revisited

$$\begin{tikzcd}
 & \vdots \arrow[d] & \\
E \arrow[r] \arrow[dr] & T_kE \arrow[d] & D_kE \arrow[l] \\
 & T_{k-1}E \arrow[d] & \\
 & \vdots &
\end{tikzcd}$$

Class. Thm.:

~~$D_kE(V) \simeq$~~

$$D_kE(V) \simeq \Omega^\infty\left[(k\cdot V)^c \wedge \Theta E^{(k)}{}_{hO(k)}\right] \qquad k\cdot V = \mathbb{R}^k \otimes V$$

$$E_0(0) = \mathrm{hofiber}_*\big(E(0) \to \underset{\overset{\shortparallel}{E(\mathbb{R}^\infty)}}{T_0E(0)}\big)$$

Ex.:

① $E: V \longmapsto BO$

$E_0(0) = O$

$\Theta E^{(1)} = S^0$

$D_1E(0) = Q(\mathbb{R}P^\infty_+)$

$O \longrightarrow Q(\mathbb{R}P^\infty_+)$

$H^k(\mathbb{R}P^\infty, \mathbb{Z}/2) \longrightarrow H^k(O)$

gen $\longmapsto \Omega w_k$.

2nd layer: get Pontryagin classes

② $E : V \longmapsto B\mathrm{Diff}^b(X \times V)$

$$E_0(0) = \widetilde{\mathrm{Diff}}^b(X \times \mathbb{R}^\infty) / \mathrm{Diff}(X)$$

$$\Theta E^{(1)} = \mathbb{H}^{\mathrm{diff}}(X)$$

$$\mathrm{Diff}^b(X \times \mathbb{R}^\infty) / \mathrm{Diff}(X) \longrightarrow \Omega^\infty\left[\mathbb{H}^{\mathrm{diff}}(X)_{hO(1)}\right]$$

__Rmk.:__ They are not all analytic (:= recoverable from their Taylor tower), Ex ① is, but this seems to be the exception.

$\widetilde{\mathrm{Top}}/\mathrm{Top}$ and the Whitehead Space

Jacob Lurie

Conventions:

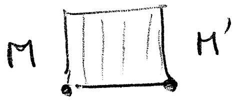
Manifolds are :
- Topological
- dim ≥ 5
- (embedded in $\mathbb{R}^\infty$)
- with boundary
- compact

$\mathrm{Man}(M)$ = "space of manifolds" having boundary ∂M

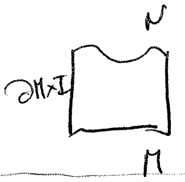
$$\boxed{\Omega\,\mathrm{Man}(M) = \mathrm{Aut}(M)}$$

$$\begin{tikzcd}\mathrm{Man}(M) \arrow[d] \\ \mathrm{Block\,Man}(M)\end{tikzcd}$$

simplicial cat.
0-simpl.: mflds w bdy ∂M
1-siplex: h-cob.



$\mathrm{H\,Cob}(M)$: "space of h-cobordisms" from M to ?



all mflds are equipped with a fixed homotopy eqn. to M!

$$\begin{tikzcd}
\mathcal{E}(M\times I) \arrow[r] \arrow[d] & \mathrm{H\,Cob}(M) \arrow[r] \arrow[d] & \mathcal{E}(M) \arrow[d] \\
\mathrm{Man}(M\times I) \arrow[r] \arrow[d] & \mathrm{H\,Cob}(M) \arrow[r] \arrow[d] & \mathrm{Man}(M) \arrow[d] \\
\mathrm{Block\,Man}(M\times I) \arrow[r] & \mathrm{Block\,HCob}(M) \arrow[r] & \mathrm{Block\,Man}(M) \\
& * \arrow[u] &
\end{tikzcd}$$

"We see a spectral sequence."

Guess:

$\exists$ filtration

$$\mathrm{BHCob}(M\times I) \qquad \mathrm{BHCob}(M\times D^n)$$

$$\underbrace{\mathrm{Man}(M) \subseteq \mathrm{Man}'(M)}_{\substack{\text{fiber}\\ \mathrm{HCob}(M)}} \subseteq \mathrm{Man}''(M) \subseteq \quad \dots \quad \mathrm{BlockMan}(M)$$

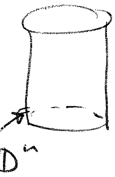
definition

classifying space for category
objects: manifolds
morphisms: h-cobordisms
of manifolds

In Man′: morphisms are h-cob, well-def. up to homeom.
In Block Man: ——— " ——— only up to blocks.

⟹ one would have to iterate the procedure

$$\mathrm{H\,Cob}(M\times D^n)$$



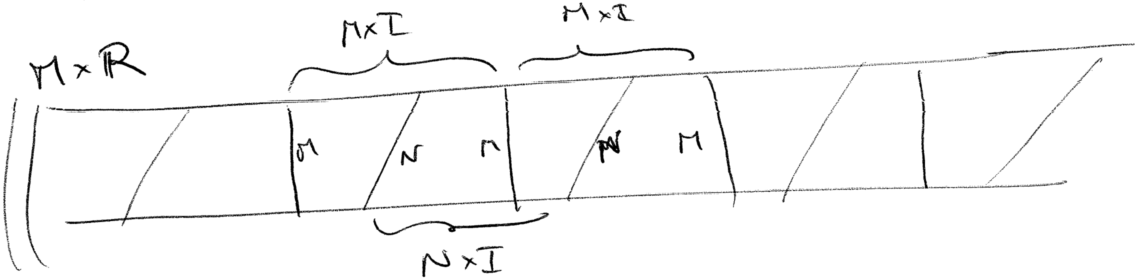
n-fold
loop space

to show this:
action of the little n-cubes operad:

plug in those cylinders, rest × M × I ~~[illegible]~~
added ⇒ OK.

---

now: start with h-cob, take another one with inverse torsion,
stick it on ⇒ M×I; infinitely many copies

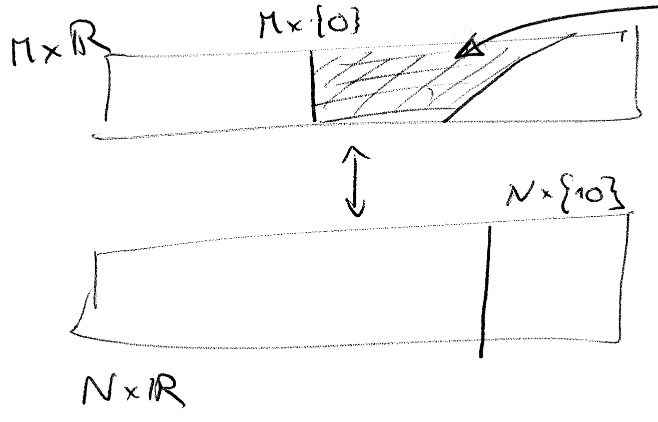


N×R

⟹ M & N cob ⇒ M×R ≅ N×R, even bounded hom.

Def: Homeom. $f: M\times\mathbb{R}^n \to N\times\mathbb{R}^n$ is bounded if it doesn't change $\mathbb{R}^n$-coordinate much

There even exists a converse:



is h-cob between M & N

(constructed from f which doesn't change the $\mathbb{R}$-coord. by more than 5)

$$\mathrm{Man}(M) \subseteq \mathrm{BCat}_{\mathbb{R}^1}(M) \subseteq \dots \subseteq \mathrm{BCat}_{\mathbb{R}^n}(M)$$

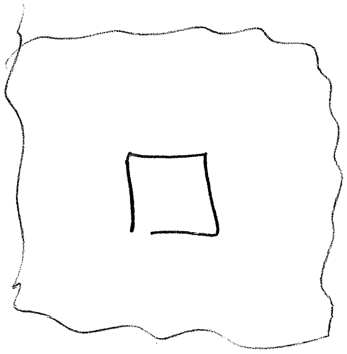
objects : manifolds
morphisms : bounded homeomorphisms $M\times\mathbb{R}^n \longrightarrow N\times\mathbb{R}^n$.

how does this fit together with the other filtration?

$$\mathrm{Top}(M\times I) \to \mathcal{C}(M) \longrightarrow \mathrm{Top}(M)$$

$$\mathrm{Man}(M\times I) \longrightarrow \mathrm{HCob}(M) \longrightarrow \mathrm{Man}(M)$$

$$\Omega\,\mathrm{BCat}_{\mathbb{R}^n}(M) = \mathrm{Top}^b(M\times\mathbb{R}^n)$$



<u>Claim:</u>

$$\mathrm{Top}(M\times I^n) \xrightarrow{\sim} \Omega^n\,\mathrm{Top}^b(M\times\mathbb{R}^n)$$

(place where the center of the space goes par. by S^n)

$$\begin{tikzcd}
 & \Omega \, H\mathrm{Cob}(M\times I^n) \arrow[d, phantom, "\wr"] & \\
\mathrm{Top}(M\times I^{n+1}) \arrow[r] & \mathcal{C}(M\times I^n) \arrow[r] \arrow[d] & \mathrm{Top}(M\times I^n) \arrow[d, phantom, "\wr"] \\
\Omega^{n+1}\mathrm{Top}^b(M\times\mathbb{R}^{n+1}) \arrow[r] & \Omega^n\mathrm{Top}^b(M\times\mathbb{R}^{n+1}) / \mathrm{Top}^b(M\times\mathbb{R}^n) \arrow[r] & \Omega^n\,\mathrm{Top}^b(M\times\mathbb{R}^n)
\end{tikzcd}$$

<u>Problem</u>: Isn't what we want?

<u>Answer 1</u>: Assume M simply connected.

<u>Answer 2</u>: Forget Block Man.

Compare $\mathrm{Man}(M)$ to $\underbrace{\varinjlim \, B\mathrm{Cat}_{\mathbb{R}^n}(M)}_{\text{this can be analyzed using } L^{-\infty}}$.

" Adding the extra $\mathbb{R}^n$-coordinate helps to get rid of the blocks ",

<u>Answer #3</u>: Fix problem.

(Problem was: connectedness)

Generally:

$$X(0) \xrightarrow{f(0)} X(1) \xrightarrow{f(1)} X(2) \longrightarrow \cdots\cdots$$

want new sequence where $f(i)$s become highly connected

$$\begin{tikzcd}
Y(0) \arrow[r] \arrow[d, "?"] & Y(1) := \bigcup_{\substack{\text{components} \\ \text{which} \\ X(0) \text{ hits}}} \arrow[r, "\text{collapse the com. comp.}"] \arrow[d, phantom, "\bigcap"] \arrow[dr] & Y(2) \arrow[r] \arrow[d] & Y(3) \arrow[r] \arrow[d] & \cdots \\
X(0) \arrow[r] & X(1) \arrow[r] & X(2) \arrow[r] & X(3) \arrow[r] & \cdots
\end{tikzcd}$$

$\{B\mathrm{Cat}_{\mathbb{R}^n} M\}$, apply construction,

$\uparrow$ has lots of symmetries, the full $O(n)$! Even after the construction has the full orth. calculus package.

27.02.06

Have constructed functor

$$\mathcal{F} : \text{inner product spaces} \longrightarrow \text{spaces}$$

$$\mathbb{R}^n \longrightarrow \mathrm{Man}(M)/_{\text{" } \wedge \text{ cobordisms"}}$$

"constant term" $\varinjlim \mathcal{F}(\mathbb{R}^n) = \text{Block Man}$

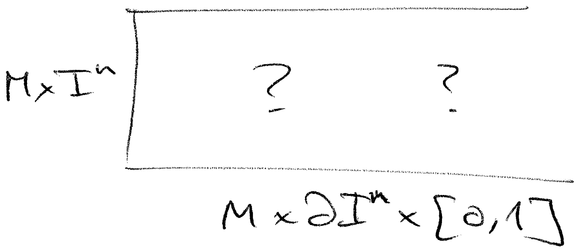
"first derivative" $\varinjlim \text{ fibers } \Omega^n\left(\mathcal{F}(\mathbb{R}^n) \longrightarrow \mathcal{F}(\mathbb{R}^{n+1})\right)$

$$\overset{\shortparallel}{\varinjlim} \; H\mathrm{Cob}(M \times I^n) = \Omega^{\pm}\mathrm{Wh}(M)$$

<u>Now: Description of Whitehead space.</u>

What is a point of $\varinjlim H\mathrm{Cob}(M \times I^n)$?

① manifold N



② $$\begin{tikzcd} M \arrow[r, phantom, "\subseteq"] \arrow[rr, bend left] \arrow[rrr, bend right] & M \times I^n \arrow[r, phantom, "\subseteq"] & \partial N \arrow[r, phantom, "\subseteq"] & N \end{tikzcd}$$

Can assume M has trivial tangent bundle

(~~choose a~~ stabilize with a disk bundle corr. to the normal bdle)

stably, not that much difference

just take $M \longrightarrow \partial N$, the incl.

$$\boxed{\text{homotopy equivalence } M \to N}$$

⇒ need

Trivialization T_N (of the tangent bdle of N)

③ Homotopy equivalence $M \to N$.

$$\begin{tikzcd} \mathrm{Wh}(M) \arrow[r] \arrow[d] & * \arrow[d, "M"] \\ \to \varinjlim_n \{\mathrm{Man}_n^{\text{framed}}\} \arrow[r] & \underline{\text{Spaces}} \end{tikzcd}$$

↑

classifying space
for n-dim. manifolds
w/ trivialized tangent
bundle

Another def. of the Whitehead Space: based on infinite-dim. mflds

$$Q = [0,1] \times [0,1] \times \ldots \times [0,1] \times \ldots$$

A Q-manifold is a (compact) space which locally looks like Q.

$$\begin{tikzcd} Q\text{-}\mathrm{Man} \arrow[d, dash] \\ \text{classifying space for } Q\text{-manifolds} \end{tikzcd}$$

(difference between ∞-dim & fin. dim. mflds: ∞-dim. ones have no tangent bdle, only trivial)

$$\begin{tikzcd} & \varinjlim_{N} \{\mathrm{Man}_n^{\text{frame}}\}^{*} \arrow[d, "\times Q", "\wr"'] \arrow[r, "\text{Waldhausen}", "\sim"'] & \text{Classifying space for Serre fibrations} \arrow[d] \\ Wh(M) \arrow[r] & Q\,\mathrm{Man} & \text{Space of polyhedra} \arrow[l, "\sim", "\times Q"'] \end{tikzcd}$$

$\mathcal{B}$

(Hatcher; built into the const., with simple maps)

triangulation (with parameters)

$*$ $P \longrightarrow \mathbb{R}^n$, take a little tube around this, do this in families.

28.02.06

⑰

Once upon a time on the train...

Michael Weiss

honest str. spaces to block str. spaces:

you can peel of layers of K-theory one by one.

we want a more integrated approach: homotopy orbit const.

Train Journey 1984

M. Weiss came from algebraic side ~~[illegible]~~, moving to geo. side (Waldh.)

Geometric problem to work on:

Group of smooth oriented homotopy spheres, 23-dim/or. diffeo,

contains an element of order 691.

$K_{22}(\mathbb{Z})$ ———————— " ———————. EXPLAIN !

Waldhausen country:

alg. K-theory, h-cobordism theory, concordances, ;

all the same thing.

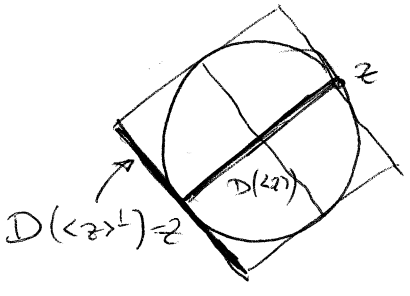
Need: mechanism for extracting concordances from exotic spheres.

Idea: Make fiber bundle $E \xrightarrow{p} S^n \subseteq \mathbb{R}^{n+1}$

$$\overset{\cup\!\!\shortmid}{z}$$

$$E_z = C\left(D(\langle z\rangle^\perp)\right)$$

$$= \left\{ f : D(\langle z\rangle^\perp) \times D(\langle z\rangle) \circlearrowleft^{\cong}_{\text{smooth}} \;\Big|\; f = id \text{ on } D(\langle z\rangle^\perp) - z \cup \partial D(\langle z\rangle^\perp) \times D(\langle z\rangle) \right\}$$



$S^n \subseteq \mathbb{R}^{n+1}$

(use $\perp$-notation instead of $(-,-)$)

$\alpha : E \to E$, $E_z \to E_{-z}$

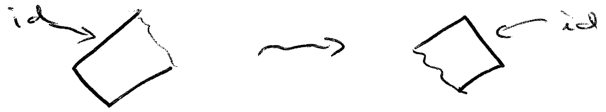
involution (which covers the standard involution on S^n)
(somehow the usual involution on concordance spaces)

$$f \in E_z \longmapsto \alpha(f) \in E_{-z}$$

$$\alpha(f)(v+tz) = f(\partial f^{-1}(v) + tz)$$

$$\partial f(v) + z = f(v+z), \ v \in D(\langle z \rangle^{\perp})$$

∂f : restriction of f to the top end of the cylinder
use ~~this~~ its inverse to change f



give me $\varphi : D^{n+1} \longrightarrow D^{n+1}$ ~~rel~~ S^n
smooth, $=$ id near ∂D^{n+1}.
(can be extended by id to the whole cylinder)

φ "$\in$" E_z $\forall z$!

$\Rightarrow$ φ determines a section
$s_\varphi : S^n \longrightarrow E$
of $p : E \longrightarrow S^n$, clearly $\mathbb{Z}/2$ - inv.,
(because $\equiv$ id on body, so no correction)

$n+1 = 22$
$n = 21$
φ gives htpy sphere of dim $n+2$,
extend to S^{n+1}
glue two D^{n+2},

"Hyperplane test"

(like a map from $\mathbb{R}P^n$ to S^n, a little twisted)
(family of concordances, parametrized as. by $\mathbb{R}P^n$)

---

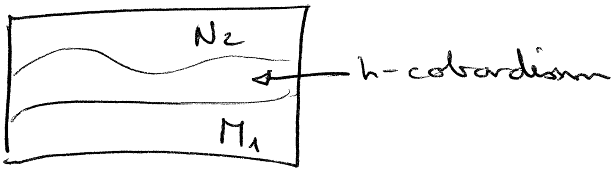
take Milnor - Kervaire htpy sphere, see what conc we get,
then go on to alg. K-theory (via h-cob.)
h-cob $\rightsquigarrow$ alg. K-theory:
put a Morse function on it, cell dec., read off Whitehead torsion etc.

$hP(X) \subset$ ~~B~~$P(X)$

subcat., morphisms those with $M_1 \overset{\simeq}{\hookrightarrow} M_2$.

$F_1 \overset{\simeq}{\hookrightarrow} M_2 \setminus (M_1 \setminus F_1) \hookleftarrow F_2$

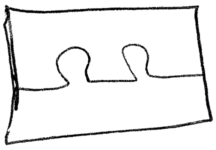
If $F_1 \cap F_2 = \emptyset$:



$hP^m_k(X) \subset hP(X)$

connected component containing

$M = X \times [a,a'] \cup$ k trivial m-handles



$P^m_k(X) \subset hP^m_k(X)$ objects.

<u>Thm 1)</u>

i) The htpy fiber of $P^m_k(X) \longrightarrow hP^m_k(X)$ approximates $\mathcal{H}(X)$.

ii) $hP^m_k(X)$ approximates $A(X)$

iii) $P^m_k(X)$ approximates a homology theory $(X \mapsto Q(X_+))$

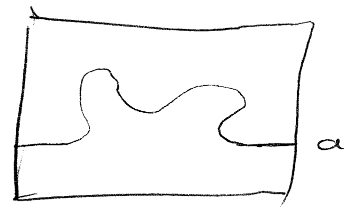
$I' = [a',b'] \in Int(I)$

$X' \subset Int(X)$ s.th. $\overline{X-X'}$ is a collar on ∂X.

$\underline{P}(X)$ partitions with $F \subset X \times I'$, $F \cap ((X-X') \times I) = (X-X') \times a$

$\simeq$ /

$P(X)$



$J' \subset Int(J)$

$\sigma : P(X) \longrightarrow P(X \times J)$

$M \longmapsto X \times [a,a'] \times J \cup M \times J'$

"You want to keep the handle ends in the middle."

You chop it off before you read the boundary.

28.02.06

Reformulate thm 1:

$$\begin{tikzcd}
\Omega Wh(X) \simeq \varinjlim_n \underline{\mathcal{H}}(X \times J^n) \arrow[r] \arrow[d] & \varinjlim_{k,n} \underline{P}^m_k(X \times J^n) \arrow[d] & : h(X) \\
* \simeq \varinjlim_n h\underline{\mathcal{H}}(X \times J^n) \arrow[r] & \varinjlim_{k,n} h\underline{P}^m_k(X \times J^n) & A(X)
\end{tikzcd}$$

(if you increase m)

is homotopy cartesian in a certain range. (Up to $m-3$?)

Pf: $j : \underline{P}^m_k(X) \longrightarrow h\underline{P}^m_k(X)$

$(M,F,N)/j \simeq \mathcal{H}(F)$ ☒

F: (based) spaces $\longrightarrow$ based spaces

F^s first derivative

$\| \varinjlim \Omega^n \operatorname{fib}(F(\Sigma^n X) \longrightarrow F(*))$

BLR $\Rightarrow$ Wh^s trivial ($2n$-conn, Ω^n still n-conn, $n \to \infty \Rightarrow$ contr.)

$$\begin{tikzcd}
h(X) \arrow[r] \arrow[d, "="'] & A(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Wh(X) \arrow[d] \\
h^s(X) \arrow[r, "\simeq"] & A^s(X) \arrow[r] & Wh^s(X) \simeq *
\end{tikzcd}$$

$\Longrightarrow A(X) \simeq h(X) \times Wh(X)$.

This $h(X)$ is $Q(X_+)$ + mystery factor, & this mystery factor is zero.

$tr : K(A) \longrightarrow THH(A)$ now: A: anything that K-theory eats

Thm (Dundas):

$K^s(A;P) = THH(A,P)$ P: bimodule over the ring A.

$THH(\Sigma^\infty \Omega X_+) = \Lambda X_+$

$THH(\Sigma^\infty \Omega X_+, S^0) = \Sigma^\infty X_+$.

1.3.06

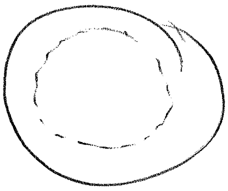
Diffeomorphisms of Disks and Spheres

Mike Hill

$$\pi_* \left(\mathrm{Diff} \left(\begin{matrix} D^n \\ D^n, S^{n-1} \\ S^{n-1} \end{matrix} \right) \right)_{\mathbb{Q}}$$

I. ① $\mathrm{Diff}(D^n) \hookrightarrow \mathrm{Diff}(D^n, S^{n-1}) \longrightarrow \mathrm{Diff}(S^{n-1})$

② $C(S^{n-1}) \longrightarrow \mathrm{Diff}(D^n, S^{n-1}) \longrightarrow \mathrm{Diff}(\mathring{D}^n)$



Prop.: a) $\mathrm{Diff}(\mathring{D}^n) = \mathrm{Diff}(\mathbb{R}^n) \simeq O(n)$.

b) Sequence ② splits

$$\begin{tikzcd}
C(S^{n-1}) \arrow[r] & \mathrm{Diff}(D^n, S^{n-1}) \arrow[r] & \mathrm{Diff}(\mathring{D}^n) \\
& O(n) \arrow[u] \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

$$\Longrightarrow \boxed{\mathrm{Diff}(D^n, S^{n-1}) = O(n) \times C(S^{n-1})}$$

here, htpy groups ess. known

try to compute this

$$\mathcal{C}(S^{n-1}) = \varinjlim C(S^{n-1} \times I^d)$$

$$\begin{tikzcd}
\mathcal{C}(S^{n-1}) \\
C(S^{n-1}) \arrow[u, hook, "\sim n/3\text{-connected}"']
\end{tikzcd}$$

$$\mathcal{C}(S^{n-1}) = \Omega^2 \S \, Wh(S^{n-1})$$

$* \longrightarrow S^{n-1}$ (bigger than $n/3$) - conn.

Reduced to computing $\pi_* Wh(*)$.

V) $A(*) = QS^0 \times Wh(*)$

Q-ly:

$$\begin{tikzcd} A(*) & QS^0 \\ \mathbb{Q} \oplus \bigoplus_{k\geq 1} \Sigma^{4k+1}\mathbb{Q} \arrow[u] & \mathbb{Q} \text{ in dim } 0 \arrow[u] \end{tikzcd}$$

$\Rightarrow \pi_*(Wh(*))_\mathbb{Q} = \bigoplus_{k\geq 1} \Sigma^{4k+1}\mathbb{Q}$

$$\pi_*(Diff(D^n, S^{n-1}))_\mathbb{Q} = \pi_* \underset{\substack{\uparrow \\ \text{product of spheres} \\ \mathbb{Q}\text{-ly known}}}{O(n)_\mathbb{Q}} \times \pi_{*+2}(Wh(*))_\mathbb{Q}$$

$*$ in the right range

$$= \begin{cases} \mathbb{Q} \oplus \mathbb{Q} & * = 4k-1 \\ 0 & \text{otherwise} \end{cases}$$

II. $\pi_*(Diff(D^n))_\mathbb{Q}$

H.S.S.: $H_*(\mathbb{Z}/2; \pi_*(\Omega Wh(D^n)))$

assume that we've inverted 2 $\Rightarrow$

$H_{*>0}(\mathbb{Z}/2; M[1/2]) = 0$

$H_0(\mathbb{Z}/2; \pi_*(\Omega Wh(D^n))) = \pi_*\left(\widetilde{Diff}(D^n)/Diff(D^n)\right)$

(this is to what the ~~stable~~ sequence abuts in the stable range)

Q-ly: $\pi_*(\Omega Wh(D^n)) = \begin{cases} \mathbb{Q} & * = 0\ (4) \quad > 0 \\ 0 & \text{otherwise} \end{cases}$

How does $\mathbb{Z}/2$ act on $\pi_*(\Omega Wh(D^n))$?

Thm (Farrell-Hsiang):

$\mathbb{Z}/2$ acts by $(-1)^{n-1}$. (still rationally)

($\Rightarrow$ all rational htpy groups are in the (+1) - Eigenspace or
—"— (-1) - Eigenspace)

Cor. (F.H.):
$\pi_*\left(\widetilde{Diff}(D^n)/Diff(D^n)\right) = \begin{cases} \mathbb{Q} & * = 0\ (4)\ \&\ n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$

1.3.06

③d

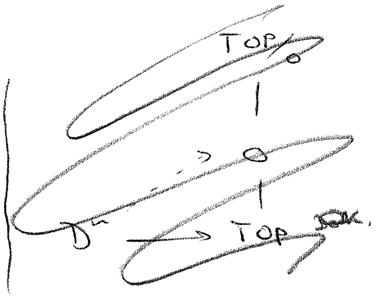
Figure out $\widetilde{\mathrm{Diff}}(D^n) / \mathrm{Diff}(D^n) \underset{\mathbb{Q}}{\simeq} B\,\mathrm{Diff}(D^n)$.

Q-ly:

$$\mathrm{Top}(D^n) \xrightarrow{\sim} \widetilde{\mathrm{Top}}(D^n) \simeq *$$

(Alexander-Trick)

$$\widetilde{\mathrm{Top}}(D^n) / \widetilde{\mathrm{Diff}}(D^n) \overset{\text{"Smoothing"}}{\simeq} \Omega^n \, TOP/O \underset{\mathbb{Q}}{\simeq} *$$



$$\Rightarrow \widetilde{\mathrm{Diff}}(D^n) \underset{\mathbb{Q}}{\simeq} * .$$

<u>Cor.:</u> $\pi_* \mathrm{Diff}(D^n)_{\mathbb{Q}} = \begin{cases} \mathbb{Q} & * = -1 (4), \ n \text{ is odd} \\ 0 & \text{otherwise} \end{cases}$

$$\mathrm{Diff}(D^n) \longrightarrow \mathrm{Diff}(D^n, S^{n-1}) \longrightarrow \mathrm{Diff}(S^{n-1})$$
$$\mathrm{Diff}(D^n, S^{n-1}) \cong O(n) \times C(S^{n-1})$$

n even:	3	0		$\mathbb{Q}^2$	$\mathbb{Q}^2$
	7	0		$\mathbb{Q}^2$	$\mathbb{Q}^2$
	11	0		$\mathbb{Q}^2$	$\mathbb{Q}^2$
	15	0		$\mathbb{Q}^2$	$\mathbb{Q}^2$
n odd:	3	$\mathbb{Q}$	$\hookrightarrow$	$\mathbb{Q}^2$	$\mathbb{Q}$
	7	$\mathbb{Q}$	$\hookrightarrow$	$\mathbb{Q}^2$	$\mathbb{Q}$
	11	$\mathbb{Q}$	$\hookrightarrow$	$\mathbb{Q}^2$	$\mathbb{Q}$
	15	$\mathbb{Q}$	$\hookrightarrow$	$\mathbb{Q}^2$	$\mathbb{Q}$

integral statements:

"**Thm:**" $\pi_*(\mathrm{Diff}(M))$ is finite type $/\mathbb{Z}$.

Guess (Rognes says that it is true for disks & spheres.)

$\rightsquigarrow$ reconstruct $\pi_*(\ )$ from $\mathbb{Q}$ stuff & $\hat{p}$ stuff for all primes.

$\pi_*(\mathrm{Diff}(D^n, S^{n-1}))^\wedge_p$

$p>2$: Again boils down to computing $\pi_* \, Wh(*)^\wedge_p$.

Thm (Rognes):

image of J spectrum

cofiber of j

- $Wh(*) \simeq \Sigma C \times \Sigma \mathbb{H}P^\infty \times \left(Wh(*) / \Sigma C, \Sigma \mathbb{H}P^\infty \right)$

spectrum

comm. cover

- $\Sigma^3 ko \to \left(Wh(*) / \Sigma C, \Sigma \mathbb{H}P^\infty \right) \to P_0 \Sigma \mathbb{C}P^\infty_{-1} / \Sigma \mathbb{H}P^\infty$

statement is p-completed.

cofiber of j: $S^0 \to j \to \Sigma C$

$bspin \xrightarrow{\psi^\ell - 1} bo \to j$ | $ko \simeq bo$

$\langle \ell \rangle = \mathbb{Z}_p^\times$

$\mathbb{C}P^\infty_{-1} = \mathbb{C}P^\infty$ together with cell in dim -2 (all taking place in spectra)

$H^*(\mathbb{C}P^\infty_{-1}) = \mathbb{Z}[y_2]\{y_2^{-1}\}$.

"regular"

Cor.: If $p \geq 5$, $n \geq 12p-5$, then $C(S^n)$ has p-torsion first in π_{4p-4} & $\pi_{4p-4}(C(S^n))^\wedge_p = \mathbb{Z}/p$.

If $p=3$, $n \geq 34$

$\pi_q(C(S^n))^\wedge_3 = \mathbb{Z}/3$.

Smoothing theory

Michael Weiss

M top. n-mfld

space of diff. structures

$$\simeq \text{Lifts}\left(\begin{tikzcd} & BO(n) \arrow[d] \\ M \arrow[r, "\tau"'] \arrow[ur, dashed] & BTOP(n) \end{tikzcd}\right)$$

Morlet
(Kirby-
Siebenmann)

those lifts that make the
map commutative up to a
given htpy.

$\Longrightarrow$ we should not think of diff. mflds,
we should think of top. mflds with such a lift.

similarly:

M smooth, closed

$$\begin{tikzcd} \mathrm{Diff}(M) \arrow[r] & TOP(M) \\ \wr\wr & \end{tikzcd}$$

$\{ f : M \xrightarrow{\text{homeom.}} M$
$+$ homotopy from $T_f : TM \to TM$
to a vector bundle autom. $\}$

start with a homeomorphism,
try to see that the map $\overset{\text{on } \tau}{}$ can
be lifted to a vector bundle
autom.

$$\mathrm{Diff}(D^n) \simeq \Omega^{n+1}\left(TOP(n)/O(n)\right)$$

similar formula for

$$\mathrm{Conc}^{\mathrm{DIFF}}(D^{n-1}) \simeq \Omega^n \,\mathrm{hofiber}\left(\begin{tikzcd} \frac{TOP(n-1)}{O(n-1)} \arrow[d] \\ \frac{TOP(n)}{O(n)} \end{tikzcd}\right)$$

$\Longrightarrow$ $A(*)$ is "like" sphere spectrum

$$A(*) = \left\{ \frac{TOP(n+1)}{TOP(n)} \right\}$$

$$\left(\text{sphere spectrum:} \quad \left\{ \frac{O(n+1)}{O(n)} \right\} \right)$$
$$\overset{\shortparallel}{S^{n+1}}$$

1.3.06

(32)

LA - theory / L - theory

Marcus Meyer

Recall: M n-dim.

Thacy: • $\tilde{S}(M) \simeq \Omega^{\infty+n}((\underline{L}_\bullet)_\%(M))$.

$\underline{L}$: 0-conn cover of L

$F_\% \to F^\% \to F$.

$A_\% = \Omega\, Wh^{TOP}$

• $\mathcal{H}^s(M) \simeq \Omega^\infty((A_\%)(M))$

stable h-cob space

$$\begin{tikzcd} w^\# : S(M) \arrow[r] & \Omega^\infty(H(M)^{h\mathbb{Z}/2}) \arrow[d, phantom, "\wr"] \\ & \Omega^\infty((A_\%)(M)^{h\mathbb{Z}/2}) \end{tikzcd}$$

Whitehead torsion

there is $\mathbb{Z}/2$ - action on A - theory (will be treated in this talk)

$$\left(\begin{aligned} X^{h\mathbb{Z}/2} &= map_{\mathbb{Z}/2}(E\mathbb{Z}/2, X) \\ X_{h\mathbb{Z}/2} &= E\mathbb{Z}/2 \wedge_{\mathbb{Z}/2} M \end{aligned} \right)$$

$(N \simeq M) \longmapsto N \hookrightarrow M \times D$, take tubular neighborhood & remove it $\Rightarrow$ h-cobordism on M.

$$\begin{tikzcd} \widetilde{TOP}(M)/TOP(M) \arrow[d, "\text{highly conn.}"', "\text{(Tibor)}"] \\ \Omega^\infty(H(M)_{h\mathbb{Z}/2}) \end{tikzcd}$$

$$\begin{tikzcd} \widetilde{TOP}(M)/TOP(M) \arrow[r] \arrow[d] & S(M) \arrow[r] \arrow[d, "w^\#"] & \tilde{S}(M) \arrow[d] \\ \Omega^\infty(H(M)_{h\mathbb{Z}/2}) \arrow[r] \arrow[d, phantom, "\wr"] & \Omega^\infty(A_\%(M)^{h\mathbb{Z}/2}) \arrow[r] & ? \\ \Omega^\infty((A_\%)(M)_{h\mathbb{Z}/2}) \arrow[ur, bend right, "N \text{ (norm map)}"'] & & \end{tikzcd}$$

(use decorations everywhere)

left hand column: highly conn.

right hand column: if you want to make it highly conn., use L-theory

so: change the middle, add some L-theory info to A-thy $\Rightarrow$ LA-theory.

Def.: $A(X,m) = S^m_! \wedge A(X)$. $! : \exists\ \mathbb{Z}/_2$-action

$S^m = (\mathbb{R}^m)^c$, $\mathbb{Z}/_2$-action given by $(-id)^c$.

$$\begin{tikzcd} LA_\bullet(X) \arrow[r] \arrow[d] & A(X,m)^{h\mathbb{Z}/_2} \arrow[d] \\ L_\bullet(X) \arrow[r, "\varphi"] & A(X,m)^{th\mathbb{Z}/_2} \end{tikzcd}$$

> This is the definition of LA!
> We will now understand the rest of the terms:

tate construction of A-theory

how to do:
(how to define the Tate construction)

- $\mathbb{Z}/_2$-action on $A(X,m)$
- $L_\bullet(X)_{\text{quadr.}}$ $\quad (L^\bullet(X))_{\text{sym.}}$
- φ
- $A(X,m)^{th\mathbb{Z}/_2}$

Thm.: $\mathcal{S}(M) \longrightarrow \Omega^{\infty+m}((LA_\bullet)_\%(M)^{h\mathbb{Z}/_2})$

is highly connected

~~Def.~~ Symmetric L-theory $L^\bullet(X)$

$\mathcal{C}$ is Waldhausen category

Def.:

An SW-product is a functor

$\odot : \mathcal{C} \times \mathcal{C} \longrightarrow$ spaces

(1) w-invariant: (pair of weak htpy equ.) $\longrightarrow$ h.e. ← OK because lands in nice (CW) spaces

(2) symmetric: $\tau : A \odot B \simeq B \odot A$, $\tau^2 = id$

(3) bilinear: cofibration sequences $\longrightarrow$ homotopy pullbacks

Examples:

a) $\mathcal{C}$ = bounded chain complexes of f.g. free modules

$$A \odot B := \Big((A,B) \longrightarrow A \oplus B \xrightarrow{DK} \underset{\substack{\text{simplicial}\\ \text{group}}}{(A\otimes B)^{DK}} \longrightarrow |(A \otimes B)^{DK}| \Big)$$

b) $R_f(X)$.

[~~Axioms~~ Conditions]

to have a notion of nondegeneracy

Example: a) ✓

b) $R_f(X)$ does not satisfy the stability axiom, you cannot desuspend, but you pass to the stabilized category $R_\infty(X)$, and this works.

$R_f(X) \hookrightarrow R_\infty(X)$ induces htpy eqv. in A-thy.

Def.: A 0-dim. sym. Poincaré object (C, ϕ) is a pair consisting of $C \in \mathcal{C}$ and $\phi \in (C \odot C)^{h\mathbb{Z}/2}$ nondegenerate. (ϕ nondeg ⇒ duality betw. C & C)

We denote ~~it~~ the set of all these by $sp_0(\mathcal{C})$.

Def.: Δ-set is a simpl. set without deg.

Def.: $L^\bullet(\mathcal{C}) = \big| [m] \longmapsto sp_0(\mathcal{C}^*(m)) \big|$

$\Big($ $\mathcal{C}^*(m) := \mathcal{C}^*(\Delta^m)$ ← functors from the poset of faces of Δ^m to $\mathcal{C}$

X simpl. complex: sub(X) subcomplexes (as a category)

$\mathcal{C}^*(X) :=$ functors sub$(X) \longrightarrow \mathcal{C}$

such that morph ⟼ cofibrations, $\emptyset \longmapsto *$,

unions ⟼ pushout $\Big)$

336/

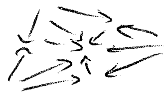
~~Example:~~

~~$\mathcal{C}$ = bounded chain complexes~~

Duality:

$m = 1$

m=2 :



$$\begin{tikzcd} C \arrow[r] & D & E \arrow[l] \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} C^* & D^* \arrow[l] \arrow[r] & E^* \end{tikzcd}$$

$$\Downarrow$$

$$\begin{tikzcd} C^* \arrow[r, hook] & Z & E^* \arrow[l, hook'] \end{tikzcd}$$

Z pushout of)

$\Rightarrow$ diagram roughly the same shape as the first,

this is to be considered as the dual.

Example:

$\mathcal{C}$ = bounded chain complexes (f. g. free) over R (with involution)

(C_*, ϕ)

$\phi \in 0$- cycle in $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}/2]}(\underset{\uparrow}{W}, C_* \otimes C_*)$ (chain complexes)

standard $\mathbb{Z}[\mathbb{Z}/2]$ - res. of $\mathbb{Z}$.

$sp_n(\mathcal{C}) = (C_*, \phi)$

$\phi \in n$- cycle . . . .

above construction : like bordism

you need notion of chain complex with boundary

Poincaré pair : $(f_* : C_* \longrightarrow D_*, \phi)$ C_* bdy of D_*, ϕ like fund. class

$(\partial M \rightarrow M)$

$\phi \in n$- cycle in $\mathrm{Hom}(W_*, \underset{\uparrow}{f_*} \otimes f_*)$

means : chain cpl. which is the mapping cone

$\rightarrow$ Poincaré obj : obj. which come with symmetric self-duality

bordism : obj. in Waldh. ~~[illegible]~~. cat $\mathcal{C}^*(1)$

nullbordism : ——— " ——————— $\mathcal{C}^*(1)$, one end is zero.

get group $\boxed{sp_n(\mathcal{C})/_{\sim \text{bordism}} = \pi_n(L^\cdot(\mathcal{C})) \, ,}$

The involution on A-theory

$$\mathcal{C} \rightsquigarrow x\mathcal{C}$$

$$x\mathcal{C} \begin{cases} (A,B,\phi) \;, & A,B\in\mathcal{C},\ \phi\in A\odot B \text{ nondeg.} \\ \text{Mor} & \end{cases}$$

$$x\mathcal{C} \longrightarrow \mathcal{C}$$

$$(A,B,\phi) \longmapsto A$$

induces a h-equ. on A-theory.

advantage of $x\mathcal{C}$: $\exists$ involution $(A,B,\phi) \longrightarrow (B,A,\tau(\phi))$,

induces $\mathbb{Z}/2$-action.

Map between L-theory and A-theory

$$\begin{tikzcd}
L^{\cdot}(\mathcal{C}) = \left| [m] \to sp_0(\mathcal{C}^*(m)) \right| \arrow[d, "\text{incl}"] \\
\left| [m] \to (\underset{\uparrow \text{ weak equ.}}{w}\, x\mathcal{C}^*(m))^{h\mathbb{Z}/2} \right| \arrow[d] \\
\left| [m] \to \underbrace{(wS_{\cdot}\, x\mathcal{C}^*(m))}_{K(\mathcal{C}^*(m))}{}^{h\mathbb{Z}/2} \right| \arrow[r, "\sim"] & K(\mathcal{C})^{th\mathbb{Z}/2}
\end{tikzcd}$$

1.3.06

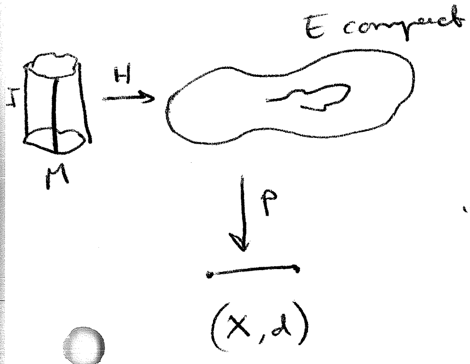
(35)

Controlled Topology I

(Qayum)

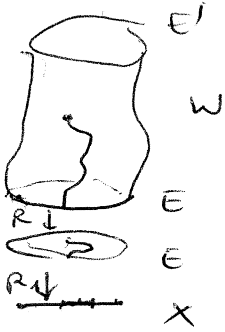
$\mathcal{P} = \mathcal{C}$ stable concordance space (pseudo-isotopy)

geometric model for $\mathcal{B}\mathcal{P}^{\%}$.



finite simplicial complex w/ metric

"we don't want the images of the tracks to become too big"



Def'n: A homotopy $H : M \times I \longrightarrow E$ is ε-controlled over (X,d) if $\forall m \in M$ $\operatorname{diam}(pH(m,I)) < \varepsilon$.

Def: An h-cobordism (W, E, E') is ε-controlled if $\exists$ choice of strong deformation retraction $R : W \times I \to W$,

$R(-,0) = id_W$
$R(-,1) \subseteq E$
$(R(E,t) = id_E)$

and if $\overset{R(-,1)\nearrow}{R_1} \circ R : W \times I \to E$ is ε-controlled. ~~[illegible]~~.

Def: A homotopy equivalence $f : M \longrightarrow E$ is ε-controlled if $\exists$ choices

$g : E \longrightarrow M$ h. inverse and

$\underset{g\circ f,\, id_M}{H : M \times I \longrightarrow M}$, $\underset{f\circ g,\, id_E}{H' : E \times I \longrightarrow E}$.

such that $f \circ H$ and H' are ε-controlled.

Def: Any of the above is infinitesimally controlled if it fits into a $[0,\infty)$-parametrised family of maps $\varepsilon_t \longrightarrow 0$ as $t \to \infty$.

Shrinking lemma (Quinn):

$\forall$ control map $p : E \to (X,d)$

$\exists\ \delta > 0$:

any δ-controlled h-cobordism is infinitesimally controlled over (X,d).

Thm [Chapman-Ferry]:

E closed top. mfld, dim > 4, metric d

$\exists\ \alpha > 0$: each α-controlled h.e. $f : M \to E$ is homotopic to a homeomorphism.

Corollary. Let $n > 4$.

If $f : M^n \to T^n$ h.e.

then

$\exists$ self-cover p s.th.

$$\begin{tikzcd} \hat{M} \arrow[r, "\hat{f}"] \arrow[d] & \hat{T} \simeq T \arrow[d, "\exists p"] \\ M \arrow[r, "f"] & T \end{tikzcd}$$

, $\hat{f}$ is homotopic to a homeo.

$T^n = \mathbb{R}^n / \mathbb{Z}^n$ $\quad s \in \mathbb{Z}_{>0}$:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "s"] \arrow[d] \arrow[dr, phantom, "\equiv"] & \mathbb{R}^n \arrow[d] \\ T^n \arrow[r, "p"] & T^n \end{tikzcd}$$

Thin h-cobordism theorem [Quinn]:

E closed top. mfld., dim > 4, metric d

Then $\exists\ \delta > 0$:

any $(W; E, E')$ δ-controlled

h-cobordism is trivial:

$$(W; E, E') \xrightarrow[\approx]{} (E \times I, E, E')$$

identity on E.

1.3.06

Fibered version:

Definition: A connected space F is $\underline{K\text{-flat}}$ if

$$Wh_1(\pi_1 F \times \mathbb{Z}^n) = 0 \quad \forall n \geq 0 .$$

K-theory (⇒ all lower K-gr. vanish)

Theorem:

Let $F \to E \xrightarrow{p} X$ be a fiber bundle of connected closed top mflds

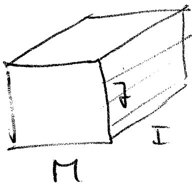
Suppose (X, d) metric space, $\dim(E) > 4$, F is K-flat.

Then $\exists\, \delta > 0$:

any h-cobordism $(W; E, E')$ δ-controlled over (X,d) is trivial.

Let M be a connected compact top. mfld.

Def.: $P(M) := \operatorname{colim}_{j \in \mathbb{N}} TOP\big((M \times I^j) \times I \text{ rel } (M \times I^j) \times 0 \cup \partial(M \times I^j) \times I\big)$



σ_j stabilization map $\left(\begin{matrix}\text{smear everything } \overbrace{\neq \text{id}}^{\text{on body}} \\ \text{into the back face}\end{matrix}\right)$

$$P_i(M) := \begin{cases} P(M \times \mathbb{R}^i) & \text{if } i \geq 0 \\ \Omega^n P_0(M) & \text{if } i < 0 . \end{cases}$$

$$P_i(M) \longrightarrow \Omega P_{i+1}(M)$$

Thm (Hatcher):

The structure map is a w.h.e.,

P: Spaces $\longrightarrow$ Ω-spectra

is homotopy invariant.

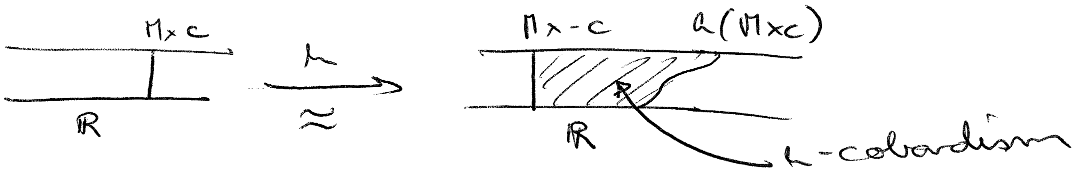
$$\begin{tikzcd} M \arrow[r, hook, "f"] \arrow[dr, hook] & N \arrow[d, hook] \\ & M \times I^j \end{tikzcd}$$

Hatcher: geometrically; 1-dim foliation, the image of the tracks, extend that to a neighborhood of M, then to all of N.

Theorem (Anderson - Hsiang):

$$\begin{cases} \pi_{-1} P(M) \cong Wh_1(\pi_1(M)) \\ \pi_{-2} P(M) \cong \widetilde{K}_0(\mathbb{Z}[\pi_1 M]) \\ \pi_{i-2} P(M) \cong K_i(\mathbb{Z}[\pi_1 M]) \quad \forall i < 0 \end{cases}$$

$$\pi_{-1} P(M) \cong \pi_0 P^b(M \times \mathbb{R})$$



category $\mathcal{C}$

objects: fiber bundles $F \to E \xrightarrow{p} |X|$ finite simplicial complex

morphisms:
$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ |X| \arrow[r, "|f|"] & |X'| \end{tikzcd}$$

Definition (James):

An ex-spectrum $\mathcal{E}$ over X:

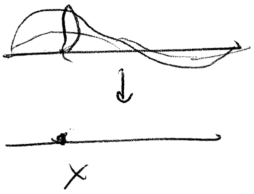
$\mathbb{Z}$-indexed sequence of $\begin{matrix} \mathcal{E} \\ \downarrow \\ |X| \end{matrix}$ $\uparrow$ 0-section

and structure maps
$$\begin{tikzcd} \mathcal{E}_i \arrow[rr] \arrow[dr] & \simeq & \Omega_X \mathcal{E}_{i+1} \arrow[dl] \\ & X & \end{tikzcd}$$

loop space over X: $\coprod_{x \in |X|} \Omega_{\theta(x)}((\mathcal{E}_{i+1})_x)$.

$$\|$$

$\Omega_X \mathcal{E}_{i+1} := \Omega_0$ Sections $(|X|, \mathcal{E}_{i+1})$



Remark:

If $X = *$, ~~then~~ $\mathcal{E}$ is a spectrum of based spaces.
In general, obtain spectrum ~~[illegible]~~ $\mathcal{E}/|X|$.

1. 3. 06

38

Def: $\mathcal{P} : \mathcal{C} \longrightarrow \Omega\text{-Spectra}$

$$\mathcal{P}(0) := \coprod_{\substack{\sigma \text{ simplex} \\ \text{of } X}} \mathcal{P}(p^{-1}\sigma) \times |\sigma| \Big/ \left(\mathcal{P}(p^{-1} \cancel{\sigma}\, \partial_i \sigma) \times |\partial_i \sigma| \xrightarrow{\partial_i} \mathcal{P}(p^{-1}\sigma) \times |\sigma| \right)$$

blocked pseudo-isotopy: is some Ω-spectrum

$$H(X; \mathcal{P}(p))_{\xi} = \underbrace{\operatorname*{colim}_{j \in \mathbb{Z}}}_{\substack{\operatorname{hocolim} \\ r \in \mathbb{N}}} \Omega^{j-n} \mathcal{P}_i(p) / |\mathbb{B}_r X|$$

B_r : rth barycentric subdivision

Def: controlled pseudo-isotopy

(X,d)

$$\mathcal{P}(X; p)[k] := \text{Controlled}_{(X,d)}(\Delta^k \times [0,\infty), \mathcal{P}(p)\cancel{(}|X|)$$

Remark: $X = *$. no control exists. (everything is controlled).

$$\mathcal{P}(E \longrightarrow *) = \mathcal{P}(E) \simeq \mathcal{P}(*, E \to *)$$

~~T~~ **Theorem** [Quinn]:

There is a functorial h.e.

$$A : H(X; \mathcal{P}(p)) \xrightarrow{\simeq} \mathcal{P}(X; p).$$

Proof of fibered version:

$$\begin{tikzcd}
H(X; \mathcal{P}(p)) \arrow[r, "\text{assembly}"] \arrow[d, "A"', "\simeq"] & \mathcal{P}(E) \arrow[d, "A"', "\simeq"] \\
\mathcal{P}(X; p) \arrow[r, "\text{forget}", "\text{control}"'] & \mathcal{P}(*; E \to *)
\end{tikzcd}$$

Take $\varepsilon > 0$ so small that any ε-controlled h-cob of E over X is infinitesimally controlled, by shrinking lemma.

Whitehead torsion in

$\pi_{-1}\mathcal{P}(E) \cong Wh_1(\pi_1 E)$

lies in image of $\pi_{-1}\mathcal{P}(X;p)$

hence $H_{-1}(X;\mathcal{P}(p))$.

$$E^2_{ij} = H_i(X;\pi_j\mathcal{P}(F)) \Rightarrow H_{i+j}(X;\mathcal{P}(p)).$$

∎

1.3.06

Controlled K-theory & L-theory

Michael Weiss

Thm (Chapman):

X, Y are cpct CW. $f : X \xrightarrow{\simeq} Y$.

If $f \simeq$ homeom., then $\tau(f) = 0$.

Thm (West):

If X cpct ENR, then $X \simeq$ cpct CW. (Borsuk Conj.)

General nonsense:

$\mathcal{C}, \mathcal{D}$ two categories of top. spaces, $\mathcal{C} \subset \mathcal{D}$.

For X in $\mathcal{D}$, let $S(X) = \text{hofiber}_X [B\mathcal{C} \to B\mathcal{D}]$.

Suppose given $F : \mathcal{D} \to$ Spaces, $\hat{F} : \mathcal{D} \to$ Spaces

taking all morphisms to homotopy equivalences,

and natural transformation $\alpha : \hat{F} \to F$.

Suppose we have a rule selecting for every X in $\mathcal{D}$ a "charact. elt."

$\lambda(X) \in F(X)$, also ... for X in $\mathcal{C}$ $\hat{\lambda}(X) \in \hat{F}(X)$,

assume $\alpha(\hat{\lambda}(X))$ "=" $\lambda(X)$. (X in $\mathcal{C}$)

"=" : $\exists$ path between them (extra data)

+ "universality" of λ and $\hat{\lambda}$.

Then $\exists$ map

$$S(X) \longrightarrow \text{hofib}_{\lambda(X)} \left[\hat{F}(X) \xrightarrow{\alpha} F(X) \right].$$

$\boxed{\text{Idea}}$ λ amounts to natural transformation

$$\begin{tikzcd} X \arrow[d] & & X \arrow[d] \\ {* \simeq \left| \mathcal{D}/X \right|} \arrow[rr, Rightarrow, "\lambda"] & & F(X) \end{tikzcd}$$

because over categories have terminal objects

So get

$$\begin{tikzcd}
{|\mathcal{D}| \simeq \underset{X}{\operatorname{hocolim}} |\mathcal{D}/X|} \arrow[r, "\lambda_*"] & \underset{X}{\operatorname{hocolim}} F(X) \arrow[r] & {|\mathcal{D}|} \\
\underset{X \text{ in } \mathcal{C}}{\operatorname{hocolim}} |\mathcal{C}/X| \arrow[u] \arrow[r, "\hat{\lambda}_*"] & \underset{X \text{ in } \mathcal{D}}{\operatorname{hocolim}} \hat{F}(X) \arrow[u] \arrow[r] & {|\mathcal{D}|} \arrow[u] \\
\mathcal{S}(X) \arrow[u] & \operatorname{hfiber}_{\chi(X)}[\hat{F}(X) \to F(X)] \arrow[u] &
\end{tikzcd}$$

$\Rightarrow$ λ_* is like a section of $\underset{x}{\operatorname{hocolim}} F(X) \downarrow |\mathcal{D}|$ (think of as a fibration)

$\Rightarrow$ get induced map, that is how you get the map from $\mathcal{S}(X)$ to this homotopy fiber.

One hopes that this is a homology equivalence, since one wants to calculate the structure space.

Lemma:

$\mathcal{A}$ additive category, idempotent complete.

Let $\mathcal{B}$ be the category of bounded chain complexes in $\mathcal{A}$. Then $\mathcal{B}$ is also "idempotent complete,"

I.e.: Given C in $\mathcal{B}$ and $p : C \to C$, $p \circ p \simeq p$,

then $\exists$ $f : C' \to C$ in $\mathcal{B}$ such that

want to say: C' rep. the same as $\operatorname{im}(p)$

$$\operatorname{im} f_* : [B, C'] \underset{\substack{\uparrow \\ \text{injective}}}{\longrightarrow} [B, C] = \operatorname{im} p_* : [B, C] \circlearrowleft$$

suitably unique. (Given another $g : C'' \to C$ with these properties, then $\exists$ $C'' \xrightarrow[e]{\simeq} C'$ and homotopy h from $p^n f e \simeq p^n g$, $n \gg 0$.

1.3.06

__Examples:__ Fix space X cpct.

$\mathcal{A}(X)$ = {idempotent completion of} category of geometric modules over X.

Objects: finite sets over X.

morphism $S \longrightarrow T$: matrix $(a_{ij})_{(i,j) \in T \times S}$,

a_{ij} = finite linear combinations of path classes in X from label(j) to label(i).

$[j \in S \Rightarrow \text{label}(j) \in X, \; S \xrightarrow{\text{label}} X]$

composition is matrix multiplication + concatenation of paths.

[nice: you can talk about it without fixing a basepoint in X; good when you want to do assembly, because then you never want a basepoint.]

$\mathcal{B}(X)$ = bounded ch. cpl. in $\mathcal{A}(X)$.

__Def.:__

A CW-space Y is "generic" if:

it comes with specified char. maps $D^n \xrightarrow[f]{} Y$ for each n-cell s.th.

$f|\partial D^n \pitchfork$ midpoints of all $(n-1)$-cells.

A cellular map $g : Y_1 \longrightarrow Y_2$ (generic)

__is generic__ if $g|_{n\text{-cells}} \pitchfork$ midpoints of all n-cells.

Suppose we have $g : Y \longrightarrow X$ (finitely dim. space X)

Y generic cpct CW-space. Then $C_*(Y)$ promoted to $\mathcal{B}(X)$.

(has to do with univ. covering?)

Suppose $\exists\, i : X \longrightarrow Y$ such that $g i \simeq id_X$,

Then we get $p = i \circ g : Y \longrightarrow Y$ $p \circ p \sim p$ (over X),

leads to $p_* : C_*(Y) \longrightarrow C_*(Y)$ in $\mathcal{B}(X)$, $p_* \circ p_* \simeq p_*$.

Get summand of $C_*(Y)$, call it $C_*(X)$.

was ex. of how to make certain characteristics

$X \mapsto \lambda(X) = C_*(X)$

$\cap$

$K(\mathcal{B}(X))$

$\|$

$F(X)$

2nd example:

Consider $(\bar{X}, X)$ "control space", $\bar{X}$ cpct.

X open in $\bar{X}$. Make add. category $\mathcal{A}(\bar{X}, X)$.

Objects are: locally finite sets over X.

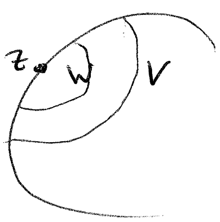
Morphisms: $S \to T$, matrices $(a_{ij})_{(ij) \in T \times S}$

a_{ij}: linear combination of path classes (locally finite) from label (j) to label (i).

Control condition: $\forall z \in \bar{X} \setminus X$ and neighborhood V of z in $\bar{X}$

$\exists$ smaller neighborhood W of z in $\bar{X}$ such that each of these paths which meets W is contained in V

(Take idempotent completion.)



Suppose given another control space $(\bar{Y}, Y)$, where Y has a controlled CW-structure.

Suppose we have maps of control spaces

$$\begin{tikzcd}(\bar{Y}, \cancel{X}Y) \arrow[r, shift left, "j"] & (\bar{X}, X) \arrow[l, shift left, "i"]\end{tikzcd}$$

$gi \simeq id$ controlled rel $\bar{X} \setminus X = \bar{Y} \setminus Y$.

(maps taking interior to interior, on bdy $\equiv$ id.)

Get $C_*(X)$ in $\mathcal{B}(\bar{X}, X)$.

Example: Fix Z compact ENR.

Let $X = Z \times \mathbb{N}$, $\bar{X} = Z \times \bar{\mathbb{N}}$ — $\mathbb{N} \cup \infty$.

Find $(\bar{Y}, Y)$ as above.

Get $C_*(X)$ ~~[illegible]~~ $\in K(\mathcal{B}(\bar{X}, X))$.

1.3.06

Note:

$$C_*(\mathbb{Z}\times 0)\oplus C_*(X)\simeq C_*(X)\quad \text{(Hilbert hotel trick)}$$

parovesz: $C_*(\mathbb{Z})\underset{\text{path}}{\rightsquigarrow} 0$ in $K(\mathcal{B}(\overline{X},X))$.

so get

$$\hat\lambda(Z)\in \underbrace{\text{hofiber}\left[K(\mathcal{B}(Z))\longrightarrow K(\mathcal{B}(Z\times\overline{N},Z\times N))\right]}_{\text{is } K^{\%}(\mathcal{B}(Z))}.$$

$\mathcal{C}$ = cat. of compact ENRs + homeom. (better: simple maps)

$\mathcal{D}$ = cat. of fin. dom. spaces with hom. equ.

two characteristics as above.

$S(X)\longrightarrow$ hofiber of some assembly.

---

for X in $\mathcal{D}$: $\lambda(X)\in K(\mathcal{B}(X))=K(\mathbb{Z}\pi_1 X)$

X in $\mathcal{C}$: $\hat\lambda(X)\in K^{\%}(\mathcal{B}(X))\simeq\Omega^\infty(X_+\wedge \underline{\underline{K}}(\mathbb{Z}))$

$\underset{\text{for } X \text{ in } \mathcal{D}}{\rightsquigarrow}$ $\mathcal{S}(X)\longrightarrow \text{hofiber}_{\lambda(X)}\left[K^{\%}(\mathcal{B}(X))\xrightarrow{\text{assembly}}K(\mathcal{B}(X))\right]$.

Apply this to X in $\mathcal{C}$, identity structure $X\xrightarrow{\text{id}}X$.

proves West's theorem.

If element lifts, then it has no interesting finiteness obstruction.

Euler char in here: sum of usual Euler char + Wall finiteness obstr.

Apply to any $Z'\xrightarrow{\simeq}Z$ $\quad(Z, Z'$ in $\mathcal{C})$

π_0 (hofiber) = Whitehead group.

tells me how I can ass. an elt in the Whitehead gr. to these data

Alt. def of the Whitehead group which does not use cells.

2.3.06

On Dwyer - Weiss - Williams (DWW)

Jeff Giansiracusa

χ Euler characteristic $\quad \chi(X,*) = \Sigma(-1)^i \dim \tilde{H}_i(X)$.

$A \hookrightarrow B \to B/A \qquad \chi(B) = \chi(A) + \chi(B/A)$

Waldhausen:

Apply the exact sequence K-theory machine to spaces.

$\Rightarrow$ $S_\bullet$ construction $\Rightarrow$ A-theory.

A-theory = universal home for Euler char.

$$A(Y) = \Omega\,|wS_\bullet R_f(Y)|$$

$$\chi_{abs}(Y) = \chi_{rel}\left(\underbrace{\begin{matrix} Y \sqcup Y \\ \downarrow\uparrow \\ Y \end{matrix}}\right) \qquad \chi(Y) \in A(Y)$$

Depends naturally on Y:

$$\begin{tikzcd} F \arrow[r] & E \arrow[d] & {} \arrow[r, Rightarrow, "\text{apply } A", "\text{fibrewise}"'] & {} & A(F) \arrow[r] & A_B(E) \arrow[d] \\ & B & & & & \cancel{A(B)}\ B \arrow[u, bend right, "\chi"'] \end{tikzcd}$$

$b \in B \longmapsto \chi(E_b) \in A(E_b)$ gives a section.

Note: This section is fibre - homotopy invariant.

Thm:

$$\begin{tikzcd} A^{\%}_B(E) \arrow[r, "\text{fibrewise}", "\text{assembly}"'] \arrow[dr] & A_B(E) \arrow[d] \\ & B \arrow[ul, bend left, "\chi_{ex}"] \arrow[u, bend right, "\chi"'] \end{tikzcd}$$

If this is a bundle of topological compact manifolds, then there is a factorization now: construct a model for $A_R^{\%}(E)$:

423) $Y \longrightarrow$ control space $\mathbb{J}Y = (Y\times[0,\infty], Y\times[0,\infty))$

$R\mathcal{G}^{ld}(\mathbb{J}Y)$ = category of (locally homotopy fin. dom.) retractive spaces over $Y\times[0,\infty)$.

and germs at ∞ of retractive maps

$R_f(Y)$: retractive spaces over Y

maps are over and rel. Y

Waldhausen structure comes from just maps rel Y

$C\mathcal{G}^{ld}(\mathbb{J}Y)$ = same objects

morphisms are (germs at ∞) of controlled maps rel $Y\times[0,\infty)$

[htpy can pass through maps which aren't necessarily respecting the retraction]

$C\mathcal{G}^{ld}(\bar{Q}, Q)$

$$\begin{tikzcd}
X \arrow[r] & X \lambda I \\
Q\times I \arrow[u, "s"] \arrow[r] \arrow[ur, phantom, "\lrcorner\ \text{pushout}"] & Q \arrow[u]
\end{tikzcd}$$

← use this to def. homotopy

lift cofibs, w.e. to $R\mathcal{G}^{ld}(\mathbb{J}Y)$

Prop: This makes $R\mathcal{G}^{ld}(\mathbb{J}Y)$ a Waldhausen category.

$Y \longmapsto K(R\mathcal{G}^{ld}(\mathbb{J}Y))$

Thm: This guy is excisive, the coefficient spectrum is $S^1\wedge \underline{A}(*)$.

[Construction due to Anderson - Connolly - Ferry - Pedersen]

Want to produce excisive characteristic χ_{ex}

for Ω a that guy.

Sequence of Waldhausen categories:

"vanishing" K-theory is $\simeq *$

$$\begin{tikzcd} \mathcal{U}(Y) \arrow[r] & R\mathcal{C}^{ld}(\rrbracket Y) \\ \chi_{ex}(Y) \arrow[r, mapsto] & 0 \end{tikzcd}$$

$\Rightarrow \chi_{ex}$ lifts to $\underbrace{\text{hofib}\left(K(\mathcal{U}(Y)) \xrightarrow{\simeq *} K(R\mathcal{C}^{ld}(\rrbracket Y))\right)}$

$$\cong \Omega\,\Omega^\infty(S^1 \wedge \underline{\underline{A}}(*) \wedge Y_+)$$

$$\cong A^{\%}(Y).$$

$\mathcal{U}(Y)$ = category of proper retractive spaces over $Y \times [0,\infty)$

i.e. $\begin{tikzcd} X \arrow[d, "\text{proper}"] \\ Y \times [0,\infty) \arrow[u, bend left] \end{tikzcd}$

spaces : ENRs

morphisms = retractive maps.

Weak equivalences and cofibrations are cooked up to imply that $K(\mathcal{U}(Y)) \simeq *$.

(uses Eilenberg swindle, you push things to ∞)

χ_{ex} :

$$\chi_{ex}(Y) = \begin{tikzcd} Y\times 0 \amalg Y \times [0,\infty) \arrow[d, shift right, "i \amalg id = r"'] \\ Y \times [0,\infty) \arrow[u, shift right, "s"'] \end{tikzcd} \in \mathcal{U}(Y)$$

$$\begin{tikzcd} \chi_{ex}(Y) \arrow[d, mapsto] & \mathcal{U}(Y) \arrow[d] \\ ? & R\mathcal{C}^{ld}(\rrbracket Y) \end{tikzcd}$$

$id_{\chi_{ex}(Y)}$ represents the same germ as $Y \times 0 \amalg Y \times [0,\infty) \circlearrowleft^{0 \amalg id}$

$\Rightarrow id$ on $\chi_{ex}(Y) = 0$ on it.

So $\chi_{ex}(Y)$ is zero in $R\mathcal{C}^{ld}(\rrbracket Y)$

$\Rightarrow \chi_{ex}(Y)$ lifts to the homotopy fibre $\in A^{\%}(Y)$.

[elt. in $R\mathcal{C}^{ld}(\rrbracket Y)$, is zero in 2 diff ways: - as a germ
- Eilenberg swindle

actually, U might be too complicated]

χ for A-theory is homotopy invariant.

χ_{ex} for $A^{\%}$ is only ~~simple homotopy equivalences~~ natural on the subcategory of simple maps (between ENRs).
proper

(what they are actually saying: ~~bas~~ cell-like maps, i.e., ~~the~~ hereditary homotopy equivalences)

Check that χ_{ex} maps to χ under fiberwise assembly.

(should also be true for ENRs, but technical problem, their construction of the excisive characteristic is very rigid, only discrete, not continuous (fiber bundles); could set it up not with fiber bundles, but with the classifying space of cell-like maps) can be repaired for mflds

---

$$\begin{tikzcd}
A_B^{\%}(E) \arrow[rr] & & A_B(E) \\
& B \arrow[ul, "\chi_{ex}"] \arrow[ur, "\chi"'] &
\end{tikzcd}$$

Suppose χ is a characteristic for an excisive functor F.

$\chi(Y) \in F(Y)$, naturality properties

a homology class ↗ ↖ F-homology of Y

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r] & \gamma E(n) \arrow[d] \\
& BTOP(n)
\end{tikzcd}
\qquad
\begin{tikzcd}
S^n \arrow[r] & \gamma(n) \arrow[d] \\
& BTOP(n)
\end{tikzcd}$$
$\gamma(n)$ = fibrewise 1-pt compactification

χ gives a section of $F_{BTOP(n)}(\gamma(n)^{\circ})$,

a twisted cohomology class.

think of this as a characteristic class $[e_n]$ for R^n-bundles.

2.3.06

For M^n a manifold w/ tangent bundle τ

$$[e_n](\tau) \in H^{\tau}(M, \underline{\underline{F}}(*)), \longleftarrow \text{cohomology}$$

$$\chi(M) \in H(M, \underline{\underline{F}}(*)) \longleftarrow \text{homology}$$

Thm: $\quad PD\,[e_n](\tau) = \chi(M)$.

Sketch of proof:

choose a family of exponentials

$\forall x \in M \quad \exp_x : \tau_x \hookrightarrow M \quad$ varying continuously w.r.t. x.

$M \longrightarrow \tau_x^{\bullet}$ ← 1-pt-comp. collaps map.

$\Longrightarrow$ induces $F(M) \longrightarrow F(\tau_x^{\bullet})$

$$PD : F(M) \longrightarrow \Gamma(M, F_M(\tau^{\bullet}))$$

$$p \longmapsto x \mapsto f_x(p).$$

see that $\chi \mapsto [e_n](\tau)$.

[That's what DWW call the universal index theorem.]

Thm: $\quad PD\,[e_n](\tau) = \chi(M)$, also works in families.

Thm: For $M \longrightarrow E \downarrow B$ a bundle of regular manifolds,

the ~~[illegible]~~ tangent $\mathbb{R}^n$-bundle comes from ~~a~~ disk bundle by deleting ∂.

$\cup$

smooth.

$$\begin{tikzcd}
Q^{\bullet}_B(E) \arrow[rr, "\eta"] & & F_B(E) \\
& B \arrow[ul, bend left, "trf"] \arrow[ur, bend right, "\chi"'] \arrow[u, phantom, "\equiv"] &
\end{tikzcd}$$

Becker-Gottlieb transfer → trf

$\Longrightarrow$

$$\begin{tikzcd}
& A_B^{\%} \arrow[r] & A_B \\
Q_B \arrow[ur, "\eta"] & & B \arrow[ll, "trf"] \arrow[ul, "\chi_{ex}"'] \arrow[u, "\chi"']
\end{tikzcd}$$

unit map

see it on the Poincaré dual side

By construction

$e_n = \eta_* b_n$
apply PD to both sides.

~~From~~ WW II to WW III ~~Part 1~~

2.3.07
~~TALK 7~~
(45)

Julia

Aim: Study $\mathcal{S}(M)$.

Goal of WW III: Highly conn. map

stable classes

$$\mathcal{S}(M) \longrightarrow \ker\left[\Omega^{\infty+m} LA_\bullet^{\%}(M,\nu,m) \xrightarrow{\text{loc. deg.}} \mathbb{Z}\right]$$

$\uparrow$ hofib of assembly map $(LA_\bullet^{\%} \to LA_\bullet)$

$\cong$ VLA$^\bullet$

remember (Markus):

$$\begin{tikzcd}
LA_\bullet(M) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A(M,m)^{h\mathbb{Z}/2} \arrow[d] \\
L_\bullet(M) \arrow[r] & A(M,m)^{th\mathbb{Z}/2}
\end{tikzcd}$$

Tate construction:
$X \circlearrowleft G$

$$X_{hG} \xrightarrow{N} X^{hG} \longrightarrow X^{thG}$$

$\uparrow$ norm map

[$\mathcal{C}$ Waldhausen cat. "with duality"]

$$\Xi : L^\bullet(\mathcal{C}) \longrightarrow \Omega^\infty\left(\underline{\underline{K}}(\mathcal{C})^{th\mathbb{Z}/2}\right)$$

$\|$

$|m \mapsto sp_0\, \mathcal{C}^*(m)|$

$\downarrow$

$|m \mapsto K(\mathcal{C}^*(m))^{h\mathbb{Z}/2}|$ $\sim$

§1. Sketch ~~this~~ $\simeq$:

main point:

$m \mapsto \underline{\underline{K}}(\mathcal{C}^*(m))$ is free augmented resolution of $\underline{\underline{K}}(\mathcal{C})$,

i.e. $\ldots \longleftarrow N\underline{\underline{K}}(\mathcal{C}^*(m-1)) \longleftarrow N\underline{\underline{K}}(\mathcal{C}^*(m)) \longleftarrow \ldots$

(Dold-Kan constr.) (htpy cat. of spectra)

- exact
- free over $\mathbb{Z}/2$ for $m>0$.

simpl. ~~obj~~ ab. gr.: take kernels of all d except d_0, use d_0 as bdry map.

Waldhausen additivity theorem:

$$F' \rightarrowtail F \twoheadrightarrow F''$$

exact sequence of functors $F : \mathcal{C}^{?} \to \mathcal{C}'$

(i.e. $F'(A) \rightarrowtail F(A) \twoheadrightarrow F''(A) \quad \forall A \in \mathcal{C}$)

$$\Rightarrow K(F') \vee K(F'') \simeq K(F) .$$ (use this to split off modules as in [illegible])

Here: $\underline{\underline{K}}(\mathcal{C}^*(m)) \simeq \bigvee_{\text{faces of } \Delta^m} \underline{\underline{K}}(\mathcal{C})$ ($2^{n+1}-1$ copies)

$\S\, m=1$: $\mathcal{C}^*(1)$

$$\begin{tikzcd} C \arrow[r, tail] & D \arrow[d] & E \arrow[l, tail] \\ & (C, D, E) & \end{tikzcd}$$

$$\Rightarrow K(\mathcal{C}^*(1)) \simeq K(\mathcal{C}) \vee K(\mathcal{C}) \vee K(\mathcal{C})$$

$$\Rightarrow NK(\mathcal{C}^*(\underset{\substack{\uparrow \\ m}}{1})) \simeq \underset{\substack{\uparrow \\ \Delta^m}}{K(\mathcal{C})} \vee \underset{\substack{\uparrow \\ \partial_0\Delta^m}}{K(\mathcal{C})} .$$

calculate: action free, $\begin{pmatrix} -\tau & \tau \\ 0 & \tau \end{pmatrix}$.

exactness: ✓ $\partial_0\Delta^m \mapsto \Delta^{m-1}$ ~~[illegible]~~, ∂_0 was bdry map.

exactness $\Rightarrow |m \mapsto \underline{\underline{K}}(\mathcal{C}^*(m))| \simeq *$

$$\Rightarrow |m \mapsto \underline{\underline{K}}(\mathcal{C}^*(m))_{h\mathbb{Z}/2}| \simeq * .$$

freeness $\Rightarrow$

$$\begin{tikzcd} & {|m \mapsto \underline{\underline{K}}(\mathcal{C}^*(0))^{th\mathbb{Z}/2}|} \arrow[d, "\simeq"] \\ \underline{\underline{K}}(\mathcal{C})^{th\mathbb{Z}/2} \arrow[ur, phantom, "\simeq"] & {|m \mapsto \underline{\underline{K}}(\mathcal{C}^*(m))^{th\mathbb{Z}/2}|} \end{tikzcd}$$

← because degreewise htpy eq., because free,

htpy ~~co~~fiber sequence

$$\begin{tikzcd} {|m \mapsto \underline{K}(\mathcal{C}^*(m))_{h\mathbb{Z}/2}|} \arrow[r] \arrow[d, phantom, "\wr"] & {|m \mapsto \underline{K}(\mathcal{C}^*(m))^{h\mathbb{Z}/2}|} \arrow[r, "\sim"] & {|m \mapsto \underline{\underline{K}}(\mathcal{C}^*(m))^{th\mathbb{Z}/2}|} \arrow[d, phantom, "\wr"] \\ * & & \underline{\underline{K}}(\mathcal{C})^{th\mathbb{Z}/2} \\ & L^\bullet(\mathcal{C}) \arrow[uu, bend left] \arrow[ur, "\equiv"] & \end{tikzcd}$$

TALK ②

2.3.06

(46)

§2. Visible L-theory

$$\begin{tikzcd}
\text{quadratic L-theory} \arrow[r] & L_\bullet \arrow[rr, "~~norm~~\ w"] \arrow[dr] & & L^\bullet & \text{symmetric L-theory} \arrow[l] \\
& & VL^\bullet \arrow[ur] & & \text{used in WW III instead of } L_\bullet \arrow[ll] \\
& & \text{visible sym. L-theory} \arrow[u] & &
\end{tikzcd}$$

(for chain cpl.: introduced by Ranicki;
for retractive spaces: ~~WW~~ → WW)

<u>chain complexes (over R with inv):</u>

Def: symmetric structure of dim m on C

$\varphi : \Sigma^m W \longrightarrow C^t \otimes_R C$, i.e. $\varphi \in \Omega^m(C^t \otimes_R C)^{h\mathbb{Z}/2}$

W ↑ free $\mathbb{Z}/2$-res. of $\mathbb{Z}$

chain map of $\mathbb{Z}[\mathbb{Z}/2]$-mod. chain cpl.

$\varphi(1_{w_0}) \in (C^t \otimes_R C)_m$, i.e. deg. m-map $C^{-*} \longrightarrow C$.

φ nondeg. ⟺ ↑ ch. htpy equ.

that is why we speak of duality !

---

sym. L-gr.: $L^m(R) := \{(C,\varphi) \mid \underset{\text{nondeg}}{\varphi \in} \Omega^m(C^t \otimes_R C)^{h\mathbb{Z}/2}\} /_{\text{bordism}}$

quadratic L-gr.: $L_m(R) = \{(C,\varphi) \mid \underset{\text{nondeg}}{\varphi} \in \Omega^m(C^t \otimes_R C)_{h\mathbb{Z}/2}\} /_{\text{bordism}}$

- conn. Poincaré duality spaces X
  def. elt $\sigma^*(X) \in L^m(R)$, symmetric signature.

- ∃ LES
  $\ldots \longrightarrow L_n(R) \xrightarrow{w?} L^n(R) \longrightarrow \hat{L}^n(R) \longrightarrow L_{n-1}(R) \to \ldots$
  $\hat{L}^n$ ↑ no alg. info
  ~~make it easier~~ by using vis. L-gr.

$R = \mathbb{Z}\pi$: $(C^t \otimes_{\mathbb{Z}\pi} C)^{h\mathbb{Z}/2} = ((C^t \otimes C)_\pi)^{h\mathbb{Z}/2}$

vis. sym. L-gr.: $VL^n(\mathbb{Z}\pi) = \{(C,\varphi) \mid \underset{\text{nondeg}}{\varphi} \in \Omega^n((C^t \otimes C)^{h\mathbb{Z}/2})_{h\pi}\} /_{\text{bord.}}$

- ∃ $VL^m(\mathbb{Z}\pi) \longrightarrow L^m(\mathbb{Z}\pi)$, ~~[illegible]~~ ↑ when not nondeg.

- ∃ visible sym. sign. $\sigma^*(X) \in VL^m(\mathbb{Z}\pi)$.

- ∃ LES $\ldots \to L_n(\mathbb{Z}\pi) \to VL^n(\mathbb{Z}\pi) \to \bigoplus_{i+j=n} H_i(B\pi; \hat{L}^j(\mathbb{Z})) \to L_{n-1}(\mathbb{Z}\pi) \to$

Ranicki also spectra version.

(classifies) closed mfld or / an or. PD space of / formal dim n } ⟷ $\pi_0(\text{hofib}[\Omega^{\infty-n}(X_+ \wedge VL^\cdot(\mathbb{Z})) \to \Omega^{\infty-n} VL^\cdot(\mathbb{Z}[\Omega X_+])])$
an pt. det by $\sigma^*(X)$.
(surjection),

§3. Vis sym. for retr. spaces

Y_1, Y_2 fin dom retr. sp. $/X$.

Def.:

"unstable" Spanier - Whitehead product $Y_1 \wedge Y_2$:

$$Y_1 \underset{X\times X}{\wedge} Y_2 := Y_1 \times Y_2 /\sim \qquad (y_1,x)\sim(r_1(y_1),x) \quad (x,y_2)\sim(x,r_2(y_2))$$

$$\begin{tikzcd} Y_1 \underset{X}{\wedge} Y_2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y_1 \underset{X\times X}{\wedge} Y_2 \arrow[d] \\ X \arrow[r, "\Delta"] & X\times X \end{tikzcd}$$

, then divide out X.

so: only keep pts if both are not in X

stable cat. $sR(X)$

Ob: (Y,k), $Y\in R(X)$, $k\in\mathbb{Z}$

$$\mathrm{Mor}\,(~~sR(X)~~ (Y,k),(Y',k')) := \mathrm{colim}_i\, \mathrm{mor}\,(\overset{\Sigma_X}{\Sigma^{i-k}} Y, \Sigma^{i-l} Y')$$

$[(\Sigma^k Y, k) \cong (Y,0)$, so (Y,k) formal k-fold desuspension$]$

Def.: $$(Y_1,k)\underset{j}{\odot}(Y_2,l) := \mathrm{colim}_n\, \Omega^{2n}\Sigma^j(\Sigma^{n-k}Y_1 \underset{X}{\wedge} \Sigma^{n-l} Y_2)$$

Ω - spectrum

Def.: m-dim <u>sym.</u> str. on (Y,k): elt. of $\Omega^m(((Y,k)\odot(Y,k))^{h\mathbb{Z}/2}$

— " — <u>vis. sym.</u> — " — $\Omega^m((Y,k)\odot(Y,k))^{\mathbb{Z}/2}$

— " — <u>quadr</u> — " — $\Omega^{\infty+m}(((Y,k)\odot(Y,k))_{h\mathbb{Z}/2}$.

Rmk.:

$(Y,k)\odot_\cdot(Y,k)$: underlying spectrum of $\mathbb{Z}/2$-spectrum $\overset{\uparrow}{S^{-kW}_{\mathbb{Z}/2}} \wedge Y^{\wedge 2}$.

$nW \mapsto S^{(n-k)W}$ shifted $\mathbb{Z}/2$-sph. sp.

Conv.: G fin gr. W reg. rep. of G

<u>G-spectrum</u> C: $\{C_{nW}\}$, $S^W \wedge C_{nW} \longrightarrow C_{(n+1)W}$

diag W-action

underlying sp.
↳ <u>uC</u> : Ω-sp., j-th sp. $\mathrm{colim}_n\, \Omega^{nW}\Sigma^j C_{nW}$, degreewise action of G

$(uC)^G$: fixed pt sp. (also Ω-sp.)

Prop.: G fin., Z G-space free away from $*$, C G-sp. $\Rightarrow$

$$\begin{tikzcd} (u(C\wedge Z))^G \arrow[r, phantom, "\simeq"] \arrow[dr, hook, bend right] & (u(C\wedge Z))_{hG} \arrow[d, "W"] \\ & (u(C\wedge Z))^{hG} \end{tikzcd}$$

idea: char. norm map as assembly map:

⇒ map L. → L. factors over L. ...

Cor.: ∃ nat. htpy fibr seqn.

$$((Y,k)\odot_\cdot(Y,k))_{h\mathbb{Z}/2} \longrightarrow ((Y,k)\odot_\cdot(Y,k))^{\mathbb{Z}/2} \longrightarrow \Sigma^{\infty-k}(Y/X)$$

Problem Session

2.3.06

(47)

Igusa's Stability Theorem: (Chris)

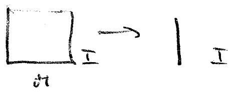
$$C(M) \xrightarrow{\sigma} C(M\times I)$$

σ is $\frac{\dim(M)}{3}$ -connected

k-connected for $\max(2k+7, 3k+4) < \dim(M)$.

Idea: $C(M) \simeq \boxed{\mathcal{E}(M)}$

$$\overset{\shortparallel}{\left\{ f: M\times I \longrightarrow I \;\middle|\; \text{no crit. pts} \right\}}$$



$$PDiff(M) \longrightarrow C(M) \longrightarrow \mathcal{E}(M)$$

$$\begin{tikzcd} PDiff(M) \arrow[d, phantom, "\wr"] \\ * \end{tikzcd}$$

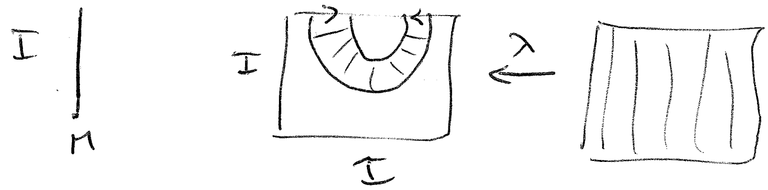
want suspension map

$\mathcal{E}(M) \longrightarrow \mathcal{E}(M\times I)$ which is compatible with the suspension on the concordance space

we do not conjugate the isomorphism to get the suspension;

$C(M) \longrightarrow C(M\times I)$

we do λ, then do our function (roughly)



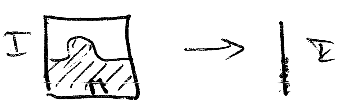
$W = M\times I$

$\mathcal{H}(W) = \{$ generalised Morse functions – Morse singularities – birth-death singularities $\}$

$\mathcal{H}^0(W) = \left\{ GMF \;\middle|\; \begin{matrix}\text{no crit}\\ \text{vals in } [1/2, 1]\end{matrix} \right\}$

$\mathcal{H}$ because it's similar to the space of handlebody decompositions

$\widetilde{\mathcal{H}}^0(W) = \{ (V, h) \mid V = f^{-1}([0, 1/2]),\ f \in \mathcal{H}^0(W),\ h = f|_V \}$



Prop.:

$$\begin{tikzcd}
\mathcal{E}(M) \arrow[r] \arrow[d] & \mathcal{H}^0(M\times I) \arrow[r] \arrow[d] & \tilde{\mathcal{H}}^0(M\times I) \arrow[d] \\
\mathcal{E}(M\times I) \arrow[r] & \mathcal{H}^0(M\times I^2) \arrow[r] & \tilde{\mathcal{H}}^0(M\times I^2)
\end{tikzcd}$$

fibre sequ.
& htpy cart.

Thm: $\mathcal{H}^0(W) \longrightarrow Q(BO\wedge(W_+))$

$\uparrow$
(k+1)-conn.

Note:

$$\begin{tikzcd}
\mathcal{H}(W) \arrow[r] \arrow[d] & \mathcal{H}^0(W) \arrow[r] & \tilde{\mathcal{E}}(W) \arrow[d, phantom, "\simeq" rotate=90] \\
Q(BO\wedge(W_+)) & & *
\end{tikzcd}$$

$\tilde{\mathcal{H}}^0_{i,i+1}(W) \subset \tilde{\mathcal{H}}^0(W)$

MCP $i,i+1$
BDP $i+1/2$

Thm (Hatcher's Two Index Theorem):

(k, k+1 conn. if

a) $k+3 \leq i \leq n-k-3$

b) $3k+4 \leq n$.

Proof: by cancelling until you hit the middle

Lemma:

$\tilde{\mathcal{H}}^0_{i,i+1}(W) \longrightarrow \tilde{\mathcal{H}}^0_{i,i+1}(W\times I)$ is $(k+1)$-connected

when c) $3 \leq i < n-3$
d) $k+1 \leq n-2i-3+\min(i, n-i-1)$

Take $i = k+3 \Longrightarrow$ Thm.

2.3.06

78

Pf Lemma:

stratify $\widetilde{\mathcal{H}}^{\circ}_{i,i+1}(W)$ into $\widetilde{\mathcal{H}}^{\circ}_{x}(W)$.

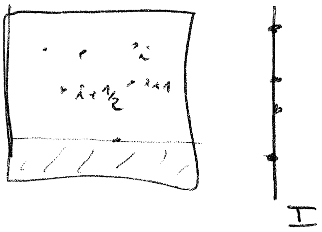
elements: functions

functions is in the stratum given by:

$\llcorner (x_1, \dots, x_p)$

$\llcorner (\cdot, \cdot, \cdot)$

# of crit values for every crit val, # of those of index i, $i+1$, $i+\frac{1}{2}$ resp.



$$\widetilde{\mathcal{H}}^{\circ}_{x}(W) \xrightarrow{\sigma_x} \widetilde{\mathcal{H}}^{\circ}_{x}(W \times I)$$

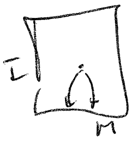
If σ_x is $(k+1)$-conn $\forall\, x$ then $\widetilde{\mathcal{H}}^{\circ}_{i,i+1}(W) \longrightarrow \widetilde{\mathcal{H}}^{\circ}_{i,i+1}(W \times I)$ $(k+1)$-conn

Rmk: stratification related to localization-dec.

Reduce to one critical value:

$$\widetilde{\mathcal{H}}^{\circ}_{(x_2,\dots,x_p)}(W) \longrightarrow \widetilde{\mathcal{H}}^{\circ}_{x}(W) \longrightarrow \widetilde{\mathcal{H}}_{x_1}(W)$$

$$Y = \widetilde{\mathcal{H}}_{(\{i\},\emptyset)}(W) \simeq \mathrm{Emb}(D^i, S^{i-1}; W, M \times \{0\}) \Big/ \mathrm{Iso}(D^i)$$



downward flow from one crit pt.

$$\mathrm{Emb}(D^i, S^{~~k}i-1}; W, M \times 0) \longrightarrow \mathrm{Map}(D^i, S^{i-1}; W, M \times 0)$$

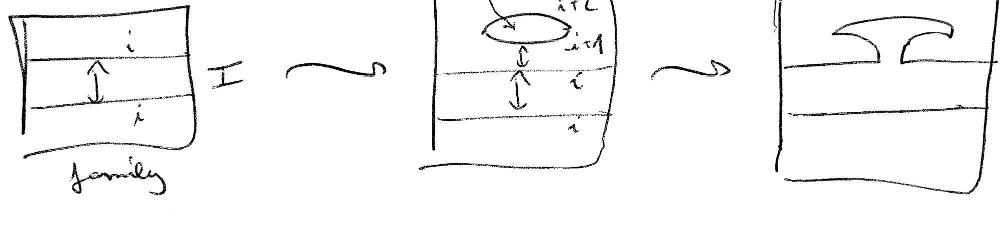
$\uparrow$ is $(n-2i)$-conn.

Susp. is h.c. on Map

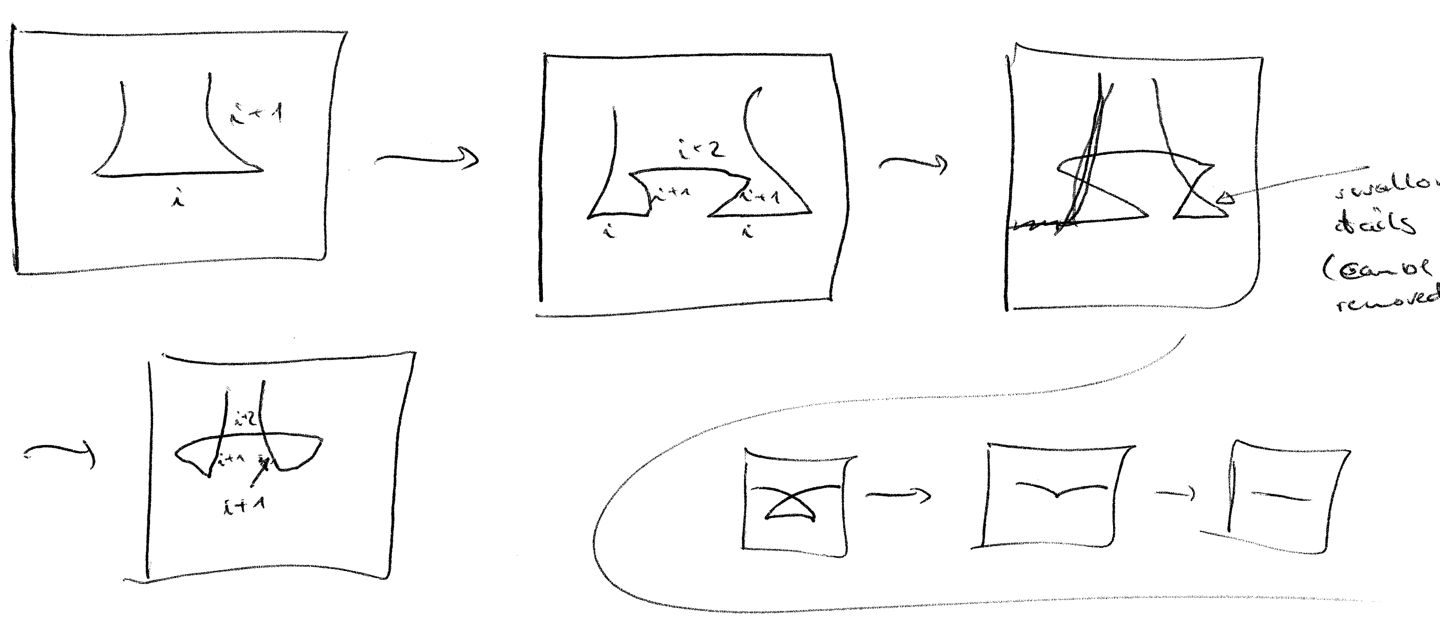
$\Longrightarrow$ ~~S~~ suspension on Y is $(n-2i)$-connected

Working harder: $(n-2i) + \min(i, n-i-1)$-conn.

re Thm:



in families, you can have flow lines from one crit pt. ~~to one of the same~~ crit value (normally not!)



<u>Rmk</u>: The bottleneck for the estimate lies in Hatcher's Two Index Thm

<u>Rmk</u>: Hatcher "proved" this in the topological case, but the proof was not accepted.

2.3.06

(49)

WWW III

Michael Weiss

$$\begin{tikzcd}
\dfrac{\widetilde{TOP}(M)}{TOP(M)} \arrow[r] \arrow[d, "\text{highly conn.}"] & \mathcal{S}^s(M) \arrow[r] \arrow[d] & \tilde{\mathcal{S}}^s(M) \arrow[d, "\simeq"] \\
\begin{matrix} \left(\begin{matrix} A_{\%}\text{-theory} \\ \text{of } M \end{matrix}\right)^{h\mathbb{Z}/2} \\ \wr \\ (\underline{Wh}^{TOP}\text{-theory}) \end{matrix} \arrow[r] & \bigcirc ? \arrow[r] & L^{\bullet}_{\%}\text{-}~~(H)~~\text{ theory of } M \\
& LA_{\bullet\%}\text{-theory} \arrow[u] &
\end{tikzcd}$$

Answer: We do already have a map

$$\begin{tikzcd}
L\text{-theory} \arrow[r, "\Xi"] \arrow[rr, bend right] & (A\text{-theory})^{th\mathbb{Z}/2} \arrow[r] & (\Sigma A\text{-th.})_{h
\end{tikzcd}$$

↑

hofiber of this : LA-theory

So we want a map $\mathcal{S}^s(M) \longrightarrow \Omega^{\infty+n} LA_{\bullet\%}(M, \nu, m)$

Idea: try to solve old ~~a~~ problems by inventing suitable characteristics and signatures.

Signatures and Characteristics

- PD space X formal dim. n, (fin. dominated)

  $\sigma(X) \in \underline{\Omega}^{\infty+n} \underline{\underline{L}}^{\bullet}(X, \nu)$

  underlying retr. space is $X \amalg X$

  $\Big[S^n \to X \xrightarrow{\text{diag}} (X \amalg X) \odot (X \amalg X)$, or the like defn.

- or : PD space X as before

  $\sigma(X) \in \Omega^{\infty+n} VL^{\bullet}(X, \nu)$

- X as before:

  gives a twisted $\mathbb{Z}/2$-action on the A-theory

  give ν a formal dim –

  $\chi^{h\mathbb{Z}/2}(X) \in \Omega^{\infty+n} \big(S^n_! \wedge \underline{\underline{A}}(X, \nu) \big)^{h\mathbb{Z}/2} \Big[\Omega^{\infty} \big(\underline{\underline{A}}(X, \nu) \big)^{h}$

- still X as before, $\sigma_{LA}(X) \in \Omega^{\infty+n} \dot{V}LA^{\bullet}(X,\nu,n)$
  (saying: compatibility between the other two elts',
  both spaces have map to Tate construction,
  elements mapped, agree there $\Rightarrow$ elt. in pullback)

- M^n closed mfld: $\sigma_{LA}^{\%}(M) \in \Omega^{\infty+n} VLA^{\bullet\%}(X,\nu,n)$.

[abstract nonsense about 2-categories, 2-characteristics $\Rightarrow$]

– get map $\mathcal{S}(X) \longrightarrow \Omega^{\infty+n} VLA^{\bullet}_{\%}(X,\nu,n)$

$$\begin{tikzcd} \mathcal{S}(X) \arrow[r] & \Omega^{\infty+n} VLA^{\bullet}_{\%}(X,\nu,n) \arrow[d, phantom, "\cong"] \\ & \Omega^{\infty+n} LA_{\bullet\%}(X,\nu,n). \end{tikzcd}$$

$$\mathcal{S}(M) \longrightarrow \Omega^{\infty+n} LA_{\bullet\%}(M,\nu,n)$$

[was constructed purely by general nonsense;
proving that it is highly conn is also general nonsense]

Why did the V drop out?

$$\begin{tikzcd} L_{\bullet}^{\%} \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{no cart.}"] & L_{\bullet} \arrow[d] \\ VL^{\bullet\%} \arrow[r] & VL^{\bullet} \end{tikzcd}$$

---

to prove that is highly conn.:

We need other maps

$$\underset{\underset{\text{con~~trol~~: bounded}}{\uparrow}}{\mathcal{S}_{ctr}}(M\times\mathbb{R}^i) \longrightarrow \Omega^{\infty+n+i} LA_{\bullet\%}(M\times\mathbb{R}^i, \nu, n+1, \underset{\underset{\text{over } \mathbb{R}}{\uparrow}}{C}$$

(~~bounded~~ $\Rightarrow$ ctr.)

(control space $(\mathbb{D}^i, \underset{\underset{\mathring{\mathbb{D}}^i}{\uparrow}}{\mathbb{R}^i}))$

(These will also be shown highly conn.)

(Relationship between all these maps by crossing with $\mathbb{R}^i$; ~~is cont~~
is controlled & everything)

Plan: downward induction. ~~Case i=∞~~.

Case $i=\infty$:

$$\begin{tikzcd}
\mathcal{S}_{ctr.}(M\times\mathbb{R}^{\infty}) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & \operatorname{colim}_{i\to\infty} \Omega^{\infty+n+i} LA_{\bullet}\%(M\times\mathbb{R}^{i}, etc.) \arrow[d, phantom, "\cong" rotate=90] \\
\mathcal{T}_{ctr}(M\times\mathbb{R}^{\infty}) \arrow[r, "\simeq", "\text{except for } \mathbb{Z}"'] & \operatorname{colim}_{i\to\infty} \Omega^{\infty+n+i} L_{\bullet}\%(M\times\mathbb{R}^{i}, etc.)
\end{tikzcd}$$

Reason: $\operatorname{colim}_{i} \underline{\underline{A}}\%(M\times\mathbb{R}^{i}, ctr) \simeq *$

Induction Step:

$$\begin{tikzcd}
\mathcal{S}_{ctr}(M\times\mathbb{R}^{i}) \arrow[r] \arrow[d] & \Omega^{\infty+n+i} LA_{\bullet}\%(M\times\mathbb{R}^{i}, ctr) \arrow[d] \\
\mathcal{S}_{ctr}(M\times\mathbb{R}^{i+1}) \arrow[r] & \Omega^{\infty+n+i+1} LA_{\bullet}\%(M\times\mathbb{R}^{i+1}, ctr')
\end{tikzcd}$$

vertical hofiber in left column: $h\mathrm{cob}_{ctr}(M\times\mathbb{R}^{i})$

right col.: $\Omega^{\infty}A_{\%}(M\times\mathbb{R}^{i}, ctr)$

map of vert hofibers is the "usual" map.

i ~~k~~-fold deloop of

$$h\mathrm{cob}(M\times D^{i}) \longrightarrow \Omega^{\infty}A_{\%}(M, ctr)$$

by hypothesis: highly conn.

---

in order to obtain a family version), the $\mathbb{R}^{i}$ plays a role, important & difficult (because characteristics ~~might be~~ are rather ~~too~~ rigid ([illegible])).

50b)

3.3.06

Higher torsion invariants I

Lott

Whitehead torsion

(X,Y) pair of connected CW-complexes

$Y \subset X$ deformation retract

$(\tilde{X}, \tilde{Y})$ = free $\mathbb{Z}\pi_1$-module whose homology groups are 0

$\uparrow$ univ. cover

To define torsion, choose basis b_i for boundaries and c_i for chains

$$\tau(X,Y) = \sum_i (-1)^i \left[b_i b_{i-1} / c_i \right]$$

$\uparrow$ elt. in $K_1(\mathbb{Z}\pi_1] / (\pm 1)$ given by the change of basis

---

s-cobordism theorem

(W, M, M') h-cobordism with $\dim W \geq 6$

then W diffeomorphic to $M \times I$ iff $\tau(W,M)$ vanishes.

Stallings existence

Fix π_1, M dim ≥ 5

$\forall \tau \in K_1(\mathbb{Z}\pi_1)/_{\{\pm\pi\}} \; \exists (W, M, M')$ h-cobord s.t. $\tau(W,M) = \tau$.

Reidemeister torsion

Start with (W, N, N'), $N \cup N' = \partial W$.

$\rho : \pi_1(W) \longrightarrow U(r)$ (unitary group)

acyclic

$H_*(W, N, \rho) :=$ homology of $\mathbb{C}^r \otimes_\rho C_*(W,N) = 0$.

Get $0 \leftarrow \tilde{C}_0 \leftarrow \tilde{C}_1 \leftarrow \ldots \leftarrow \tilde{C}_n \leftarrow 0$.

chain complex of free $\mathbb{Z}\pi_1(W)$-modules. $\left[C_*(\tilde{W}, \overset{p : \tilde{W} \to W}{p^{-1}(N)}) \right]$

$C_* := \mathbb{C}^r \otimes_\rho \tilde{C}_*$ with boundary $\partial_i = 1 \otimes \partial_i$.

Reidemeister torsion is given by $\tau = \mathrm{Re}(\log(\det \partial)) \in \mathbb{R}$.

51b/

Lens spaces

$$S^{2n-1} \subset \mathbb{C}^N$$

take $\underbrace{\text{a free}}_{\text{linear}}$ action of $\mathbb{Z}/m$ on S^{2n-1}

with choice of basis, write the quotient as $L(r_1,\dots,r_n)$ $(r_1,\dots,r_n)\in(\mathbb{Z}/$

$$\begin{pmatrix} e^{\frac{2\pi i r_1}{m}} & & \\ & \ddots & \\ & & e^{\frac{2\pi i r_n}{m}} \end{pmatrix}$$

Rmk: S^3:
$\underset{\text{usual}}{L(p,q)} = L(1,q)$, action of $\mathbb{Z}$
in this notation

orientation reversal comes from negating any r_i.

We take $\pm L(r_1,\dots,r_n)^{0<r_i<\frac{m}{2}}$

A lens space is a quotient ~~by~~ of S^{2n-1} by ~~the~~ a free action of $\mathbb{Z}/m$

Thm:

Each lens space is isomorphic to a unique one of the form

$\pm L(r_1,\dots,r_m)$ $\quad 1\leq r_1 \leq \dots \leq r_n < \frac{m}{2}$.

Idea:

Lens spaces have a cell structure with one cell in each dimension

$$\partial e_{2k-1} = (\rho^{r_k}-1)e_{2k-2}$$

$$\partial e_{2k} = (1+\rho+\dots+\rho^{n-1})e_{2k-1}$$

$$\Big\downarrow \text{ work}$$

QED.

3.3.06

(52)

Dictionary

classical	modern
$\mathbb{Z}\pi_1(X)$	$\mathbb{S}[\Omega X]$
representation ρ of $\pi_1(X)$	$\mathbb{C}$ - $\mathbb{S}[\Omega X]$ - bimodule
"singular chain complex of X with twisted coefficients in $\mathbb{C}^r$"	"singular chain complex of X with coefficients in some bimodu

Igusa Axioms

Def.: A bundle $E \to B$ is *unipotent* if $\pi_1(B)$ acts unipotently on rational homology of the fibers.

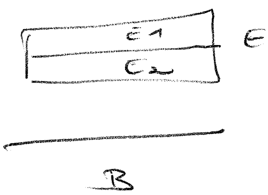
A *higher torsion invariant* is a real characteristic class $\tau_{2k} \in H^{4k}(B; \mathbb{R})$ for any unipotent smooth bundle $E \to B$ with closed fibers satisfying

1) *Additivity*

If $E = E_1 \cup E_2$ union of unip. bundles over B with vertical ~~bundles~~ boundaries $\partial E_1 = \partial E_2 = E_1 \cap E_2$. Then

$$\tau(E) = \frac{1}{2}\tau(DE_1) + \frac{1}{2}\tau(DE_2)$$

(D means fiberwise double)



2) *Transfer*

If we have an oriented S^n - bundle $V \to E$ (associated to some $SO(n+1)$ - bundle),

then $\tau_B(V) \in H^{4k}(B, \mathbb{R})$ and

$$\tau_E(V) \in H^{4k}(E; \mathbb{R})$$ satisfying

$$\tau_B(V) = \chi(S^n)\,\tau_B(E) + tr_B^E(\tau_E(V))$$

$tr_B^E : H^*(E, \mathbb{Z}) \longrightarrow H^*(B; \mathbb{Z})$ given by $tr_B^E(x) = p_*(x \cup e(T^v E$

Def.: A torsion invariant is $\left\{ \begin{array}{l} \text{even} \\ \text{odd} \end{array} \right.$ if it vanishes for $\left\{ \begin{array}{l} \text{odd} \\ \text{even} \end{array} \right.$ -dimensional fibers.

Lemma: If τ is an invariant then $(-1)^n \tau$ is also

n = dim of fiber

$$\Rightarrow \quad \tau^+ = \frac{\tau + (-1)^n \tau}{2}$$

$$\tau^- = \frac{\tau - (-1)^n \tau}{2}$$

Thm: $\forall\ k > 0\ \exists!$ even and odd torsion invariant
[Igusa] in degree $4k$. (up to scalar).

Thm [Ma]:
Bismut-Lott torsion is proportional to the odd part of Igusa's FR torsion when $\exists$ fiberwise Morse function (a

$\exists$ 3 definitions of torsion
- DWW [homotopy theoretic]
- Igusa [combinatorial]
- Bismut-Lott [analytic]

morally speaking: the sa
(both index theory)
(index thm: Euler classes are Poincaré duals of Euler classes)

↑ satisfies transfer
(in additivity Thm, $\exists$ error term)

Igusa: relationship to MMM-classes.

higher torsion inv. related to Reidemeister torsion;
$B = pt$, $k = 0 \Rightarrow$ Reidemeister class.

Miller-Morita-Mumford classes

$$M_{2k}(E) = tr_B^E\left((2k)!\ ch_{2k}(T^\vee E)\right)$$

- always even (so prop. to τ^+)

could put any other lin comb. of Pontryagin clas in instead of ch

3.3.06

(53)

Higher torsion invariants II

John

Construction of torsion, via morality.

inv. of manifold M.

$$\begin{tikzcd} M \arrow[d] & \\ * \arrow[r] & \text{space of mflds } M \simeq BDiff(M) \end{tikzcd}$$

inv. of smooth families

$$B \longrightarrow BDiff(M)$$

$$\begin{tikzcd} M \arrow[r, hook] & E \arrow[d] \arrow[r, dash] & V \text{ - local system} \\ & B & \pi_1 E \arrow[r, hook, "\rho"'] & V \quad R\text{-module} \end{tikzcd}$$

Idea: We know how to do something over a point, now we try to do it over all pts in the space: do it fiberwise.

Exc: do it, fiberwise [DWW]

$$\begin{tikzcd} A_B(E) \arrow[r, "\rho"] \arrow[d] & K(R) \\ B \arrow[u, bend left, "\chi"] & \end{tikzcd}$$

make $\rho\chi$ zero:

Consequence of index theorem,

smoothness $\Rightarrow \rho \cdot \chi \simeq 0$ - component.

| find 2 reasons, make torsion out of that

acyclicity force $H^*(p^{-1}b;V)=0$.

| smoothness: one reason
| acyclicity: another reason.

---

setup of char.

$$\begin{tikzcd} {|sp/obj|} \arrow[rr, "\chi"] & & spectra & A(E) \\ & sp \arrow[ul] \arrow[ur, "A"'] & & \\ & E & & \end{tikzcd}$$

$* \simeq |sp/t|$

Input: E, V; cond: acycl.

$\rightsquigarrow$ get 2 canonical paths to 0 from pt $\rho \circ \chi$

$\Rightarrow$ loop.

This loop is a torsion inv.

(in $\Gamma B, K($

53b

another construction using Morse functions:

$$\begin{tikzcd} M \arrow[r, hook] & E \arrow[d] \\ & B \end{tikzcd}$$

ask for $f: E \to \mathbb{R}$ which is Morse in every fiber

we cannot quite get this, we allow GMF (generalised Morse fct

[In homotopy theory: We don't mind making choices, but we want the space of these choices to be contractible

pick one
- get cell cplx on each fiber
- model of (linearised) simple htpy Wh as the target of the construction
- get universal classes $\tau_\bullet \in H^\bullet(Wh^n_\bullet; \mathbb{R})$
  $\tau_\bullet \in H^\bullet(BDiff_0; \mathbb{R})$

$$\begin{tikzcd} & M \arrow[d, hook] & & \\ V - & E \arrow[d] & & Wh^n_\bullet \\ & & & BDiff_0(M) \arrow[u, dashed] \arrow[d] \\ & B \arrow[rr] \arrow[uurr, dashed] \arrow[urr, dashed] \arrow[drr, "f^*"'] & & BDiff(M) \arrow[d] \\ & & & B[Aut(H_{\mathbb{Q}}^\bullet M)] \end{tikzcd}$$

discrete group

Question now:

When can we define these universal classes?

When can we lift?

A fGMF is a framed GMF, comes with framing of (−1)-eigenspace of $D^2 f$

Thm: $\widehat{BDiff}(M) \xrightarrow[\text{conn.}]{(n-1)-} BDiff(M)$

space of framed fGMF

birth-death sing. get rid of having too many conn. comp

M-space of fGMFs, is (n−1)-connected

h-principle involved in choosing the framing

(approximate C^0-fct by C^1-fct)

(smooth functions, but you can deform them in a cont. way, that is good enough

3.3.06

$$\begin{tikzcd} & \widehat{B\mathrm{Diff}}_o(M) \arrow[r] \arrow[d] & \mathrm{Wh}^h_o \\ B^{i<n-i} \arrow[ur] \arrow[r] & B\mathrm{Diff}_o(M) & \end{tikzcd}$$

<u>Construct:</u>

$\mathrm{Simp}(B) \longrightarrow$ Filt cplxs$(\overset{\pi_1 E}{G}, \overset{\mathbb{Z}\pi}{R})$

$\sigma \in \mathrm{Simp}(B)$ $\quad f_t : E \to \mathbb{R}$ $\quad$ GMF (fiberwise)

looking at Morse sing b (forget about birth-death sing.)

$\underset{r\in R}{f^{-1}r}$ − disjoint subspaces of σ

order $f^{-1}_r > f^{-1}r'$ if $r > r'$.

V_t in simpl$(B) \rightsquigarrow$ ch cplx.

$\sigma \longmapsto \mathrm{sing}(E|_\sigma\ V)$

filter using poset structure

<u>filtration:</u>

$Q_i \subset \mathbb{R}$ $\quad E^Q|_\sigma \subset$ subset of things on flows from crit. values of f

$E^i\{\_<i\}$

$\longrightarrow$ induces a filtration by restricting supp.

This is supposed to give a linear model of the smooth Whitehead spa

[This is a linear approximation of what Vogell defined.]

Thm:

$$\begin{tikzcd} \mathrm{Wh}^h_o(X) \arrow[r] & Q(X_+) \arrow[r] & K(\mathbb{Z}\pi) \\ \mathrm{Wh}^{Diff}(X) \arrow[u, "\text{guess}", "\text{lin.}"'] \arrow[r, hook] & Q(X_+) \arrow[u] \arrow[r] & A(X) \arrow[u] \end{tikzcd}$$

Think FC as finite sets w/ G-action linearised as R-mod

$FC^h \simeq \underbrace{\mathrm{Wh}^h_o}_{\text{linear } \mathrm{Wh}^{DIFF}}$ acyclic chains

§46/

Construction:

$Wh_\bullet$ – comes from filt cplxes

K – comes from cplxes

↗

Borel regulators explicitly constructed

$$\begin{tikzcd}
K_\bullet(\mathbb{C}) \arrow[r, "nI_{nk}"'] \arrow[d, phantom, "\wr"] & \mathbb{R} \\
\pi_\bullet(BGL(\mathbb{C}^\delta)^+ \times \mathbb{Z}) \arrow[r, "h"] & H_\bullet(\quad) \arrow[u]
\end{tikzcd}$$

$I \in H_c^\bullet(BGL\,\mathbb{C};\mathbb{R})$

flat:

$$\begin{tikzcd}
& BGL(\mathbb{C}^\delta) \arrow[d] \\
X \arrow[r] \arrow[ur, dashed] & BGL(\mathbb{C})
\end{tikzcd}$$

Kamber-Tondeur classes for flat bundles

$\updownarrow$

Borel.

pulling back: $\tau_{2k} \in H^{4k}(Wh_\circ^h;\mathbb{R})$

Then $H^{4k}(B\,Diff_0(M)) \ni \tau_{2k}$.

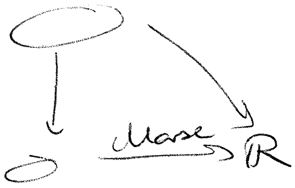
Thm (Igusa):

$$\begin{tikzcd}
S^{n-1} \arrow[r, hook] \arrow[d, phantom, "\cap"] & E \arrow[r] \arrow[d, phantom, "\cap"] & B \\
\mathbb{C}^n \arrow[r, hook] & \xi \arrow[ur] &
\end{tikzcd}$$

$$\tau_{2k}(E) = (-1)^{k+1}\zeta(2k+1)\,ch(\xi)$$

↑

forced by transfer, induction and splitting



3.3.06

Second derivatives ~~in th~~ orthogonal calculus and applications to automorphisms

Michael Weiss

$$\begin{tikzcd} DIFF(M) \arrow[r] & \widetilde{DIFF}(M) \arrow[r] & \Omega^\infty\left(hcob^{DIFF}(M)\right)_{h\mathbb{Z}/2} \\ & \frac{\widetilde{DIFF}(M)}{DIFF(M)} \arrow[ur] & \end{tikzcd}$$

homotopy fiber sequence in a stable range

like a first Taylor approximation so much is concentrated at prime 2.
(Stiefel-Whitney-classes)

2^{nd} Taylor approximations and orthogonal calculus

"The easiest problem I could handle"

Functor $V \longrightarrow BTOP(V)$ (V: f-dim real vect. space w. inner product)

Functor $V \longrightarrow BO(V)$

Derivative spectra $\Theta^{(1)}(BO)$, $\Theta^{(1)}(BTOP)$, $\Theta^{(2)}(BO)$, $\Theta^{(2)}(BTOP)$

Known:
- $\Theta^{(1)}(BO) = \underline{S}^0$ (triv. action of $O(1)$)
- $\Theta^{(2)}(BO) = \Omega\underline{S}^0 = \underline{S}^{-1}$ (action of $O(2)$ is rationally trivial)
- $\Theta^{(1)}(BTOP) = \underline{\underline{A}}(*)$ (action of $O(1)$ by SW-duality)
- $\Theta^{(2)}(BTOP) =$ Mystery

- $\Theta^{(1)}(BO) \longrightarrow \Theta^{(1)}(BTOP)$ split mono with $O(1)$-action
- $\Theta^{(2)}(BO) \longrightarrow \Theta^{(2)}(BTOP)$ split mono (integrally) without $O(2)$-action
- —— " —— <u>not</u> split mono integrally with $O(2)$-action

Reason: If it were, then the integral Pontryagin classes for BO would come from BTOP.
(But rationally, they do come from BTOP.)

55b/

<u>Hypo I:</u> $\Theta^{(2)}(BO) \longrightarrow \Theta^{(2)}(BTOP)$ rat. split mono with $O(2)$-act

Essentially equivalent formulations:

- $e_{2n}^2 = p_n \in H^{4n}(BSTOP(2n);\mathbb{Q})$ (in $H^{4n}(BSO(2n);\mathbb{Q})$ it is known ~~obvious~~)
- $\ker[H^*(BTOP;\mathbb{Q}) \longrightarrow H^*(BTOP(n);\mathbb{Q})]$

  "=" $\ker[H^*(BO;\mathbb{Q}) \longrightarrow H^*(BO(n);\mathbb{Q})]$, $\forall n$.

"i.e.: these interesting statements are essentially statements from orthogonal calculus, 2nd derivatives".

---

translate this into:

<u>Hypo II:</u> The map

(implies Hypo I)

n odd

$$\mathrm{Reg}(D^n \times D^2, D^2) \longrightarrow \Omega^{n+1}(\mathrm{Epilin}(\mathbb{R}^{n+2}, \mathbb{R}^2))$$

- agree with proj. near the bdry
- have no sing.

(surj. linear maps)

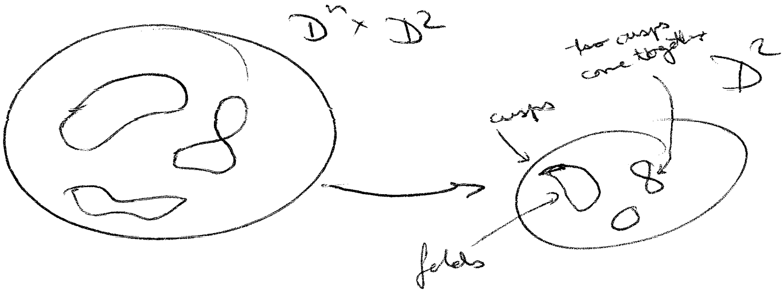
is rat. nullhomotopic (with $O(2)$-equiv., rel. to $\mathrm{Reg}_L(D^n \times D^2, D^2)$ (smooth maps ~~[illegible]~~ respecting proj. to L)

(L = line in $\mathbb{R}^2$) for all $L \subset \mathbb{R}^2$.

(RHS: $\simeq_{\mathbb{Q}} K(\mathbb{Q}, n-1)$.)

<u>Another reduction:</u> Introduce $\mathrm{Gen}(D^n \times D^2, D^2)$ space of "generic functions". Allow only sing of rank ≥ 1.

singularity picture



23.06

local models : $(x_1, \dots, x_{n+2}) \longmapsto (x_1, A(x_2, x_3, \dots))$

$$" \qquad \longmapsto (x_1, x_1 x_2 + x_2^3 + B(x_3, \dots))$$

$$" \qquad \longmapsto (x_1, x_2 C(x_1, x_2) + B(x_3, \dots))$$

A, B, C

are nondeg. quad. forms

Hypotheses 3 (implies Hypo 2)

The map
$$\Sigma^{n+2} \mathcal{G}en(D^n \times D^2, D^2) \xrightarrow{\text{Thom}^2 \text{ of sing. loc.}} K(\mathbb{Z}, 2n+2)$$
is rat. nullhomotopic, with $O(2)$-equiv., rel. to $\Sigma^{n+2} \mathcal{R}eg(D^n \times D^2, D^2)$ and $\mathcal{G}en_L(D^n \times D^2, D^2)$, $\forall L \subset \mathbb{R}^2$.

(roughly: the core of the statement of Hypo 2)

Explanation of maps:

Choose approximation $M \longrightarrow \mathcal{G}en(D^n \times D^2, D^2)$

Get adjoint $M \times D^n \times D^2 \xrightarrow{f} D^2$ (smooth)

with (mild) transversality conditions, the set of fiberwise singularities of f will be a submanifold C of $M \times D^n \times D^2$, codim $n+1$.

C = sing. locus.

(it/its normal bundle has a Thom class, we want to square it, ...)

get $\Sigma^{n+2} M \longrightarrow K(\mathbb{Z}, 2n+2)$.

Reason for Hypo 3 ⇒ Hypo 2.

Need to show that $Gen(D^n \times D^2, D^2)$ is not "too far" from being $\simeq *$,

Can use: $Gen(D^n \times D^2, D^2) \simeq \Omega^{n+2} Q(\mathbb{R}^{n+2}, \mathbb{R}^2)$,

where $Q(\mathbb{R}^{n+2}, \mathbb{R}^2)$ = space of polynomial functions $\mathbb{R}^{n+2} \longrightarrow \mathbb{R}^2$

which are generic at 0 in the sense previously defined.

(that is quite a hard thm.)

Idea of proof of Hypo 3:

Normal bundle of C (C = sing. locus in $M \times D^n \times D^2$)

is identified with the kernel of vertical diff. of f,

Hence cocycle in Hypo 3 can also be described

as $i_! e(\ker d_M f / C)$ (i: inclusion of C)

$D^n \times D^2 \times M \xrightarrow{f} D^2$

C sing. loc.

(rel. easy from that point onwards).

Sketch of proof of Hypo 3:

$C = \bigcup C_L \qquad u : C \longrightarrow \mathbb{R}P^1 = \mathbb{R}P(\mathbb{R}^2)$

$C_L = u^{-1}(L)$

$i_! e(\ker d_M f | C) = \underbrace{\bigcup_L \mathrm{Link}(C_L, C_{L^\perp})}_{\text{rationally}}$

By symmetry:

$$\bigcup_L \mathrm{Link}(C_L, C_{L^\perp}) = \bigcup_L \mathrm{Link}(C_{L^\perp}, C_L).$$

By construction of linking cocycles

$$\bigcup_L \mathrm{Link}(C_L, C_{L^\perp}) = -\bigcup_L \mathrm{Link}(C_{L^\perp}, C_L).$$

What we are trying to prove:

Splitting for second derivatives rationally like that for first.

This implies results about Pontryagin classes. (in BTOP(n) & BTOP).

Discussion Session

Q: What is known about DIFF outside the stable range?

$$\begin{tikzcd}
\widetilde{\mathrm{DIFF}}(*) \arrow[r, phantom, "\simeq"] \arrow[d, phantom, "\wr"] & \Omega(\mathrm{TOP}/\mathrm{O}) \\
\bigcup_i \mathrm{DIFF}^b(\mathbb{R}^i) & = \Omega(\mathrm{TOP}(n)/\mathrm{O}(n))
\end{tikzcd}$$

$E \longrightarrow B$

smooth bdle of exotic disks $\mathcal{E}$

fiber dim n

$T_{vr}E$ n-dim'l v. bdle over E

(these are "transgression classes".)

$$S^n = \mathrm{O}(n+1)/\mathrm{O}(n) \qquad \Omega^n S^n$$

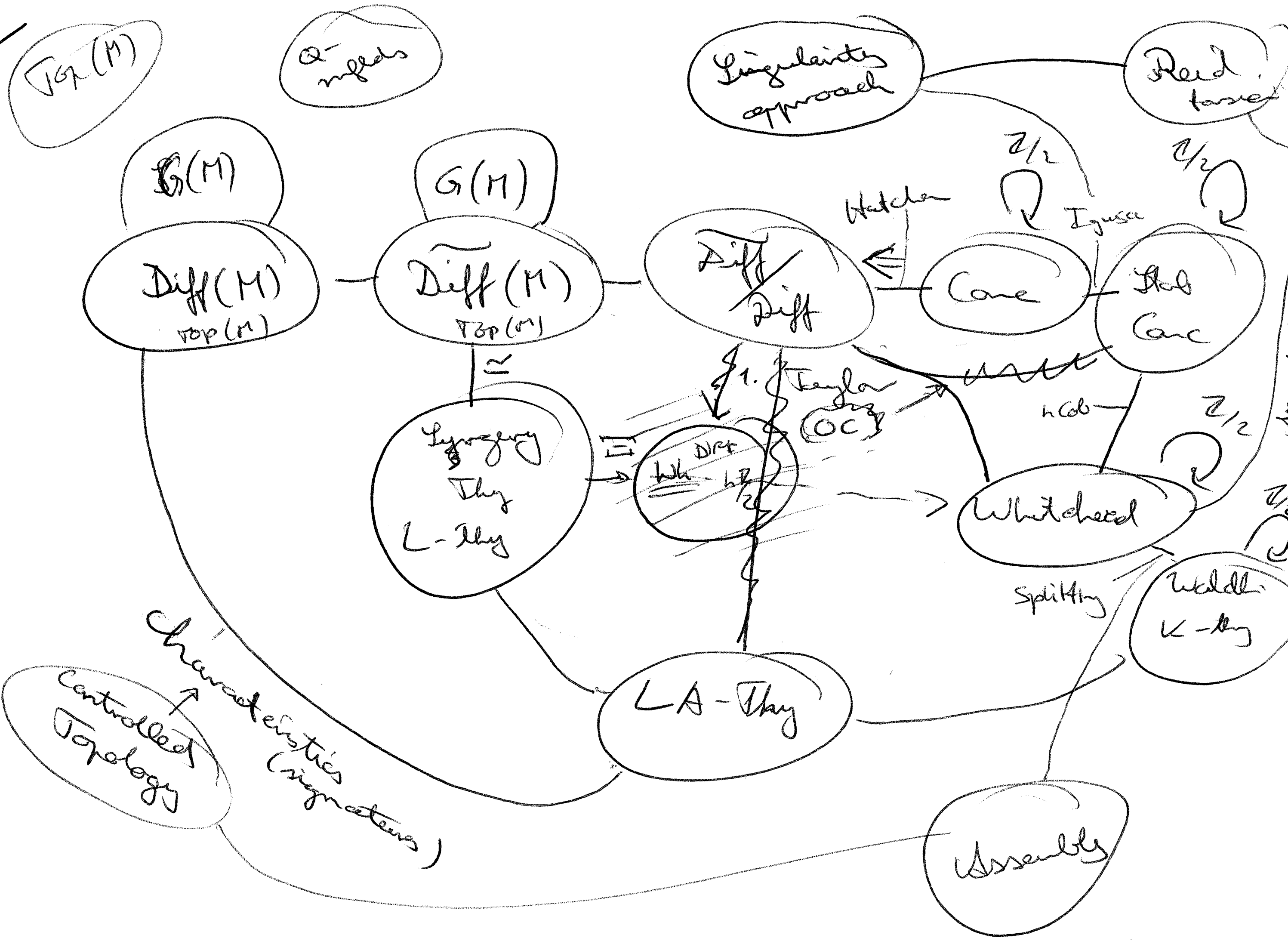
$$\begin{tikzcd}
\Omega^n S^n \arrow[d, "\text{about } n\text{-conn.}"] \\
\Omega^{n+1} S^{n+1}
\end{tikzcd}$$

$$S^{n+1} = \mathrm{O}(n+2)/\mathrm{O}(n+1)$$

$$\begin{tikzcd}
\Omega^n(\mathrm{TOP}(n+1)/\mathrm{TOP}(n)) \arrow[d] \arrow[r, leftrightarrow, "\text{roughly same conn.}"'] & h\mathrm{cob}^{\mathrm{DIFF}}(D^n) \arrow[d] \\
\Omega^{n+1}(\mathrm{TOP}(n+2)/\mathrm{TOP}(n+1)) & h\mathrm{cob}^{\mathrm{DIFF}}(D^{n+1})
\end{tikzcd}$$

$\Longrightarrow$ we could expect $\longrightarrow$ to be n-conn, we only know $n/3$, it is probably somewhere in between.

576



---

ad: Euler char.:

E → B bundle of cpct ~~smooth~~ top mflds

$B \to A^{\%}_{B}(E)$ fiberwise excessive Euler char.

dual to Euler class of vertical tangent bundle
(lives in ~~?~~-th cohomology of $(E, \partial E)$ in $\underline{\underline{A}}(*)$))

[Euler classes of ~~tangent~~ bundles vanish near the boundary

This Euler class of $T_{vert}E$ is also an obstruction to "splitting off a line bundle".

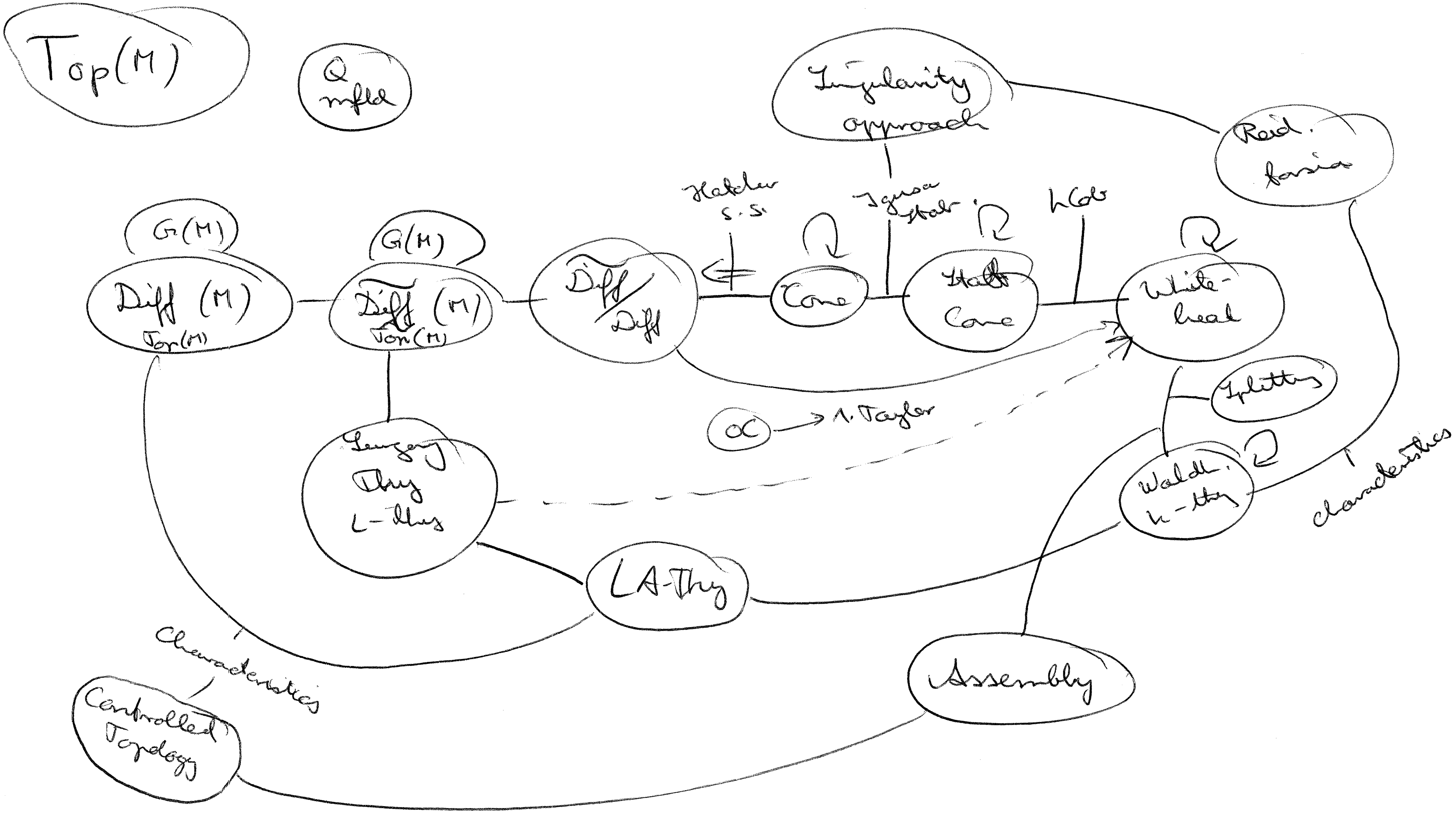
Related ~~to~~

$$\begin{tikzcd} & BTOP(n-1) \arrow[d] \\ E \arrow[ur, dashed] \arrow[r] & BTOP(n) \end{tikzcd}$$

rel. to $\frac{TOP(n)}{TOP(n-1)}$ ← unstable form of $\underline{\underline{A}}(*)$

has implications to Euler class of t

Euler class:
stabilized obstruction
to reducing the str. gr
~~from~~ TOP(n) to TOP(n-1)
(splitting of a line bdle)
smooth case: O(n) ⇝ O(n-
rel to sphere spectrum (vanishing th


Top(M)
Q mfld
Singularity approach
Reid. torsion
G(M)
Diff (M)
Top(M)
G(M)
$\widetilde{\text{Diff}}$ (M)
$\widetilde{\text{Top}}$(M)
$\widetilde{\text{Diff}}$/Diff
Hatcher s.s.
Cone
Igusa Hatcher
Hatt Cone
h-Cob
White-head
Splitting
Waldh. K-thy
characteristics
OC → A. Taylor
Surgery Thy L-Thy
LA-Thy
Assembly
Characteristics
Controlled Topology

3.3.06

(58)