Talbot 2005

Geometric Langlands

with host David Ben-Zvi

and organizers Chris Douglas
John Francis
Andre Henriques
Mike Hill

Scott : Langlands to geometric Langlands.

Langlands program:

$\boxed{Q}$: How to parametrize (autom.) rep'ns of a (reductive algebraic) group / over K (global) local or

unipotent radical trivial

May associate to G red. alg $\rightsquigarrow$ root datum

$(X^\bullet, \Phi^\bullet, X_\bullet, \Phi_\bullet)$ } determines G up to some iso$^{\underline{m}}$.

weight lattice, root system, coweight lattice, coroot system

natural bijection

Now look at $\hat{G}$, complex group with dual root data [switch roots, coroots]

Also ${}^L G$ Langlands dual gp $\boxed{\hat{G} \rtimes \mathrm{Gal}(\hat{K}/K)}$

product of mult. gps

if torus for G split, action trivial
(even if not, usually small ... factors through finite gp ...)

Examples:

G	$\hat{G}$
GL_n	GL_n
SL_n	PGL_m
SO_{2n+1}	Sp_{2n}
SO_{2n}	SO_{2n} or $Spin_{2n}$
G_n	G_n

simply conn gps $\longleftrightarrow$ gps w/ no center

(center and π_1 switch roles?)

$K =$ local: $\mathbb{R}, \mathbb{C}, \mathbb{F}_q((t))$
fin. ext. of $\mathbb{Q}_p$
global: fin. ext. of $\mathbb{Q}$ or $\mathbb{F}_p((t))$

Can associate to a rep'n a Langlands parameter

unramified absolute value

$\longleftrightarrow$ conjugacy class in ${}^L G$

$\{ c_v(\rho) \}_{v \in |K|}$

rep

analogue of "highest weight"

"place" (abs. value)

Langlands Functoriality Conjecture:

$$\begin{tikzcd} {}^LG \arrow[rr, "\pi"] \arrow[dr] & & {}^LG' \arrow[dl] \\ & \mathrm{Gal}(\bar{K}/K) & \end{tikzcd}$$

and suppose ρ is an auto$^{\underline{m}}$ rep'n of G.

Then $\exists$ ρ' auto$^{\underline{m}}$ rep. of G' s.t. $\{c_v(\rho')\} = \{\pi(c_v(\rho))\}$

Special Case: G° trivial, ${}^LG$ = Galois group, get "Langlands correspondence"

$\{$ rep'ns of $\mathrm{Gal}(\bar{K}/K)$ into $\widehat{G}'$ $\} \longrightarrow \{$ autom rep'ns of G' $\}$

$\boxed{Q}$ Is there an arrow going the other way?

In general no: Galois gp is "too small" [GL_n: Lots more on RHS than LHS]

Idea: refine/extend the Galois group! ... using motives.

Lafforge's work [geometry]

X smooth projective $\underbrace{\text{geometrically}}$ conn. alg. curve $/\mathbb{F}_q$

after $\otimes$ with $\bar{k}$, remains connected

F = field of fns X

↳ has absolute values at every closed pt ...

$\mathbb{A}_F$ = adèles of F

$= \prod'_{\text{points}}$ Laurent series at the point $= \prod'$ formal germs of mero. fns

only finitely many have poles

$\left\{ \begin{array}{l} \text{Isom classes of} \\ \text{cuspidal aut. irreps} \\ \text{of } GL_n(\mathbb{A}_F) \text{ with} \\ \text{fin. order central charac.} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{Iso. classes of irreps} \\ \text{of } \mathrm{Gal}(\bar{F}/F) \text{ on a } n\text{-dim'l } \overline{\mathbb{Q}}_\ell\text{-} \\ \text{v.s. a.e. unramified with fin. det} \end{array} \right\}$

Picture: for GL over function fields (?) only.

Lafforge moduli of Shtukas truncations compactifications

A-S trace formula

Geometry
moduli spaces

ℓ-adic coh.
Groth-L-Verdier trace formula
Pink-Fujiwara on Deligne conjecture

Spectral aut. rep'ns

Gal. rep'ns Algebraic

Weil II (weights)
Groth.'s cohom. interpretation of L-functions.
Laumon's determination of ε-constants
prod. formula

L-functions
ε-factors
Analytic

Converse Thm

Rankin-Selberg convolutions

(Jacquet, PS, Shalika)

Langlands $\rightsquigarrow$ geom. Langlands

"a moral arrow: it's not a functor" - Scott.

Three key insights:

1) Auto$^{\underline{m}}$ representations arise as functions on $G(F)\backslash G\mathbb{L}(\mathbb{A})/K$

K open compact in $G(\mathcal{O})$

and $G(F)\backslash G(\mathbb{A})/G(\mathcal{O})$ is in bijection with the set of iso$^{\underline{m}}$ classes of principal G-bundles on X.

$\Rightarrow$ should look for functions on a moduli sp. of bundles $Bun_G X$.

2) Grothendieck's Function - Sheaf dictionary

sheaf $\longrightarrow$ Function : alternating trace of $Frob_x$ on stalk at x, $\forall x$.

$\Rightarrow$ look for perverse $\overline{\mathbb{Q}}_\ell$-sheaves on $Bun_G X$.

Properties of this map:

(i) Induces a ring homo$^{\underline{m}}$ $\underbrace{K(X, \overline{\mathbb{Q}}_\ell)}_{\text{Groth. gp of } D^b_c(X,\overline{\mathbb{Q}}_\ell)} \xrightarrow{\eta} \text{Maps}(X(\mathbb{F}_q), \overline{\mathbb{Q}}_\ell)$

(RHS has ring str by $\otimes$ sheaves)

(ii) $K(X, \bar{\mathbb{Q}}_\ell) \hookrightarrow \prod_i \mathrm{Maps}\,(X(\mathbb{F}_{q^i}), \bar{\mathbb{Q}}_\ell)$

[image of map not well-understood]

(iii) For any $f : A \longrightarrow B$ morphism (stacks/~~shea~~ schemes), f^* commutes with η

(iv) $Rf_!$ commutes with η

3) $\underbrace{\text{Gal. rep'ns}}_{\text{Rep'ns of are fundamental gps}}$ can be realized by local systems.

monodromy gps of

$\underbrace{\text{fundamental gps}}_{\text{using étale topology}}$

Ref: SGA 1.

Now want a correspondence

$$\left\{ \begin{array}{c} \hat{G} \text{ local systems} \\ \text{on a curve } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{(Hecke eigen)sheaves} \\ \text{on } \mathrm{Bun}_G X \end{array} \right\}$$

Elizabeth Mann ①
Mon 28 Feb 2005

Hecke algebras

1. Motivation
2. Examples
3. p-adic gps, affine Hecke alg.
4. Bernstein isom/classical Satake

1. Motivation

Want to understand $\mathrm{Rep}(G)$

$$G = G(\mathbb{C}),\quad G(\mathbb{F}_q),\quad G(\mathbb{Q}_p)$$

For $G(\mathbb{C})$, all repns appear in $L^2(G)$

For others: $\mathbb{C}[G] = \mathrm{Ind}_{\{0\}}^{G}(\mathbb{1})$

$H \subset G$: look at $\mathrm{Ind}_H^G(\mathbb{1})$ which ones appear?

Frobenius reciprocity : $\langle V, \mathrm{Ind}_H^G(\mathbb{1})\rangle = \langle V|_H, \mathbb{1}\rangle$
$= \dim V^H$

V appears $\iff V^H \neq 0$.

(so start talking about <u>spherical</u> reps ($V^H \neq 0$))

What acts on V^H? Say G finite for simplicity.

$\mathbb{C}[{}_H\backslash^G/_H] \longleftrightarrow {}^H\mathbb{C}[G]^H$ acts. $h \in H$ $f \in \mathbb{C}[G]$

$$\left.\begin{aligned} fv &= h(fv) \\ &= f(hv) \end{aligned}\right\} \Rightarrow hf = f = fh$$

$\to \mathcal{H}(G, H)$

what's the multiplication? Convolution. again G finite.

$\mathbb{C}[G]$ = group ring.

$$f = \sum_{x \in G} c_x \delta_x \qquad \delta_x(y) = \delta_{x,y}$$

$$(f \cdot g) = \Big(\sum_{x \in G} a_x \delta_x\Big)\Big(\sum_{y \in G} b_y \delta_y\Big) = \sum_{x,y} a_x b_y \delta_{x,y}$$

$$= \sum \Big(\sum a_{zy^{-1}} b_y\Big) \delta_z$$

normalize Haar measure so $|B| = 1$.

More generally, $(f * g)(x) = \int_G f(xy^{-1})\, g(y)\, dy$. "perverse sheaves are almost fcns"

can make more general : perverse sheaves, pushforwards.

With good G, H (H needs to be "right size")

$$\left\{ \begin{array}{c} (\text{irr.}) \text{ rep'ns } V \text{ of } G \\ \text{with } V^H \neq 0 \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} (\text{irr}) \text{ reps of} \\ \mathcal{H}(G,H) \end{array} \right\}$$

$$V \longmapsto V^H$$

2. Baby example: $G = SL_2(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\}$

$$H = B = B(\mathbb{F}_q) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}$$

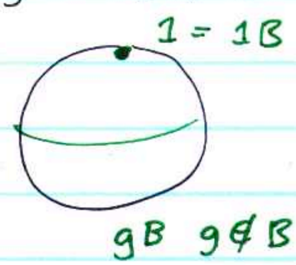
$$G/H = \left\{ \begin{pmatrix} a & * \\ c & * \end{pmatrix} \right\} \Big/ \begin{pmatrix} a \\ c \end{pmatrix} \sim \begin{pmatrix} ax \\ cx \end{pmatrix} = \mathbb{P}^1(\mathbb{F}_q)$$

$$|G| = (q+1)|B|$$

What are the H-invariant functions?

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} x & * \\ y & * \end{pmatrix} = \begin{pmatrix} ax + by & * \\ a^{-1}y & * \end{pmatrix}$$

orbits: $\left. \begin{array}{l} y = 0 \\ y \neq 0 \end{array} \right\}$ two orbits!



Two elts in Hecke alg: x_1

x_s

functions on G/H

$$x_1(gB) = \begin{cases} 1 & \text{if } g \in B \\ 0 & \text{else} \end{cases}$$

$$x_s(gB) = \begin{cases} 1 & \text{if } g \notin B \\ 0 & \text{else} \end{cases}.$$

Exercise: $x_1 * x_1 = ?$

$x_1 * x_s = ?$

$$(\chi_s * \chi_s)(gB) = ?$$

Case 1 : $g \in B$

$$(\chi_s * \chi_s)(\underset{x}{\underset{\parallel}{gB}}) = \int_G \chi_s(xy^{-1})\,\chi_s(y)\,dy$$

$$= \frac{1}{|B|} \# \{ y \in G \mid y \notin B,\ xy^{-1} \notin B \}$$

$\Updownarrow$ $x \in B$, so $y^{-1} \notin B$

$$= \frac{1}{|B|} \# \{ y \in G \mid y \notin B \}$$

$$= \frac{1}{|B|} (|G| - |B|)$$

$$= (q+1-1) = q.$$

Case 2 : $g \notin B$.

$$(\chi_s * \chi_s)(x) = \frac{1}{|B|} \# \left\{ y \in G \;\middle|\; \begin{matrix} y \notin B \\ y \notin xB \end{matrix} \right\}$$

$$= \frac{1}{|B|} (|G| - |B| - |B|) = q - 1.$$

$$\boxed{\chi_s^2 = (q-1)\,\chi_s + q\,\chi_1.}$$

This generalizes to $\mathcal{H}(G(\mathbb{F}_q), B(\mathbb{F}_q))$

use B-orbits on $G/B \longleftrightarrow$ Weyl gp elements

= Coxeter gp gen. by reflections

$s_1, \dots, s_n$

relations : $s_i^2 = 1$

$(s_i s_j)^m = 1$

$m = 2, 3, 4, 6$

according to Dynkin diagram

"constructible sheaf constant on that orbit"

$T_i = \chi_{s_i}$

so $\mathcal{H}(G(\mathbb{F}_q), B(\mathbb{F}_q))$:

generated by $T_1, \dots, T_n$ (algebra generators...)

$$(T_i)^2 = (q-1)\,T_i + q$$

setting

Exercise: $(T_i - q)(T_i + 1) = 0$. Exercise: Replace B by P.

Vectorspace basis for $\mathcal{H}$: $\{ T_w \mid w \in W \}$

define $T_w = T_{s_{i_1}} T_{s_{i_2}} \cdots T_{s_{i_k}}$

$w = s_{i_1} \cdots s_{i_k}$ reduced word

Relations: $T_w T_{w'} = T_{ww'}$

if $\ell(ww') = \ell(w) + \ell(w')$

$T_s^2 = (q-1)T_s + q$ [encoding complexity of convolution]

Exercise: write down mult. in $\mathcal{H}(SL_3, B)$, $\mathcal{H}(SO_5, B)$

Can do this for any Coxeter gp, so what about W_{aff}?

3. p-adic groups:

$G(\mathcal{K})$ $\quad \mathcal{K} = \mathbb{Q}_p \quad (\mathbb{F}_p((t)), \mathbb{C}((t)))$

$G(\mathcal{O})$ $\quad \mathcal{O} = \mathbb{Z}_p \quad (\mathbb{F}_p[[t]], \mathbb{C}[[t]])$

$ev: G(\mathcal{O}) \to G(\mathcal{O}/m)$ "evaluate at 0"?

Def^n: $I = ev^{-1}(B)$

Thm: (Iwahori-Matsumoto) $\mathcal{H}(G(\mathcal{K}), I) =$ Hecke alg. of W_{aff} //

in particular W_{aff} parametrizes double cosets $I \backslash G(\mathcal{K}) / I$?

$\langle s_0, s_1, \ldots, s_n \rangle$ acts on $\mathfrak{h}$

$s_i^2 = 1$

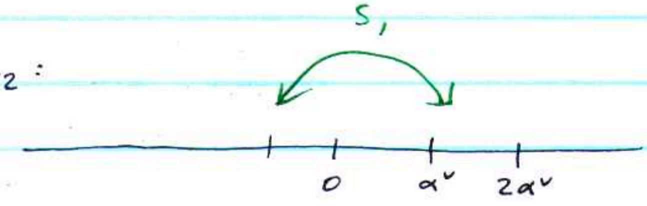
$(s_i s_j)^m = 1$

$m = 2, 3, 4, 6, \infty$

$s_0 = r_{\theta^\vee} e^{-\theta^\vee}$

θ: longest root of G

Picture for SL_2:



$s_1(x) = x$

$s_0(x) = (1 - x)$

where $\alpha^\vee = 1/2$.

$\mathbb{Q}[q^{\pm 1/2}]$ or $\mathbb{Z}[q^{\pm 1/2}]$

$\mathcal{H}_{aff}$: alg. / $\cancel{\mathbb{Z}[q]}$
gen. by T_i $\quad i = 0, \dots, n$

relations $T_i^2 = (q-1)T_i + q$

Basis $\{ T_w \mid w \in W_{aff} \}$

Exercise : $\mathcal{H}(SL_2(\mathbb{Q}_p), I)$

Exercise : (Assume $\pi_1 G = 0$) $\quad Y$ = coroot lattice $= \mathrm{Hom}(\mathbb{C}^\times, T) \subset \mathfrak{h}$.

$$W_{aff} \cong W \ltimes Y$$

$$(we^\lambda)(w'e^\mu) = ww' e^{(w')^{-1}\lambda + \mu}.$$

Can write Hecke alg. in terms of reflections and translations.

Geometric picture:

$$G(\mathbb{C}((t)))/I =: \mathcal{F}\ell \text{ affine flag variety}$$
$$\downarrow$$
$$\mathcal{L}G/\mathcal{L}^+G = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]) =: \text{affine Grassmannian } \mathcal{G}r$$

in case $\mathbb{C}$: what is q ?
integration → pushforward
counting pts → cohomology w/ compact support
take Euler charac. or keep degrees,
(q is keeping track of grading)
=
" q is Tate twist "

Question : What is
$$\mathcal{H}(G(\mathcal{K}), G(\mathcal{O})) \subset \mathcal{H}(G(\mathcal{K}), I) \ ?$$

$\mathcal{H}(G(\mathcal{K}), I)$: $\{ T_w \mid w \in W_{aff} \} = \{ T_{we^\lambda} \}$
$\mathcal{H}(G(\mathcal{K}), G(\mathcal{O})) = ?$

$\underline{\text{geometry}}$: $\mathcal{G}(\mathcal{O})$-orbits : $\lambda \in Y^+$ dominant

$\underline{\text{Exercise}}$: $\{ T_{e^\lambda} \mid \lambda \in Y \}$
- is not a subalg
- not at all commutative.

$\underline{\text{Observation/exercise}}$: If λ dominant, $\ell(e^\lambda) = \underbrace{\langle 2\rho, \lambda \rangle}_{\text{linear!}}$

$\underline{\text{Corollary}}$: $\ell(e^\lambda e^\mu) = \ell(e^\lambda) + \ell(e^\mu)$

$\underline{\text{Corollary}}$: $\underbrace{T_{e^\lambda} T_{e^\mu} = T_{e^{\lambda+\mu}}}_{\text{so there's a part that behaves nicely ... but don't have inverses}} = T_{e^\mu} \cdot T_{e^\lambda}$

$\underline{\text{Def}^{\text{n}}}$ (Bernstein) $\lambda \in Y$, $\lambda = \lambda_1 - \lambda_2$, $\lambda_1, \lambda_2 \in Y^+$

$$\Theta_\lambda := T_{e^{\lambda_1}} T_{e^{\lambda_2}}^{-1} q^{\langle \rho, \lambda_2 - \lambda_1 \rangle}$$

($\underline{\text{Exercise}}$: What is T_i^{-1} ? (allow $q^{\pm 1/2}$))

$\underline{\text{Exercise}}$: - indep. of λ_1, λ_2
- Θ_λ span $\underline{\text{commutative}}$ subalg. of $\mathcal{H}$
- $\underbrace{T_w \Theta_\lambda}_{\text{translation + reflection}}$ is v. sp. basis for $\mathcal{H}_{aff}$

CAUTION : multiplication complicated

$$\underbrace{T_s \Theta_\lambda - \Theta_{s(\lambda)} T_s}_{\text{in Weyl gp should be 0, but now have other stuff}} = (q-1) \frac{\Theta_\lambda - \Theta_{s(\lambda)}}{1 - \Theta_{-\alpha^\vee}}$$

$\alpha^\vee \leftrightarrow$ root corr. to s

(but for $q = 1$, get 0, reality check ✓)

$\underline{\text{Corollary/Exercise}}$: $T_s(\Theta_\lambda + \Theta_{s(\lambda)}) = (\Theta_\lambda + \Theta_{s(\lambda)}) T_s$

$\underline{\text{Cor}}$: $z_\lambda := \sum_{w \in W} \Theta_{w(\lambda)} \in Z(\mathcal{H}_{aff})$

<u>Thm</u>: (Bernstein) $Z(\mathcal{H}aff) = \text{span}\{z_\lambda\}$

<u>Thm</u>: (Bernstein? Lusztig?) 1) $Z(\mathcal{H}aff) \cong \mathcal{H}(G(\mathcal{K}), G(\mathcal{O}))$

2) $\exists$ basis c_λ of Z s.t.

$$c_\lambda c_\mu = \sum_\nu \underbrace{c_{\lambda\mu}^\nu}_{} c_\nu$$

structure const. A priori poly's in q, but actually constants! (only in Z)

$c_{\lambda\mu}^\nu$ is same as

$$V_\lambda \otimes V_\mu = \sum c_{\lambda\mu}^\nu V_\nu$$

for irred. high wt rep'ns of $\widehat{G}$, $\lambda \in Y$

$\updownarrow$

rep'ns of G.

<u>Re-interpretation</u>: (David BZ)

Satake-Langlands: $\subset \mathcal{H}(G(\mathbb{Q}_p), I)$

$$\underbrace{\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p))}_{\text{(commutative)}} = \operatorname{Rep}_{\mathbb{C}} G^\vee \underset{\text{high wt theory}}{=} \mathbb{C}[G^\vee]^{G^\vee \text{ conjugation}} = \mathbb{C}[T^\vee]^W$$

Now rep'n theory:

$$\underbrace{\left\{ \begin{array}{c} G(\mathbb{Q}_p)\text{-rep'ns } V \\ \text{gen. by } V^{G(\mathbb{Z}_p)} \neq 0 \end{array} \right\}}_{\text{= unramified rep'ns}} \longrightarrow \text{Rep'ns of } \mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$$

$$V \longmapsto V^{G(\mathbb{Z}_p)}$$

Irred. unramified $\longrightarrow V^{G(\mathbb{Z}_p)}$ irrep of $\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$

$\cong \mathbb{C}$ since $\mathcal{H}(-)$, comm.

now:

semisimple conj. classes $G^\vee_{ss}/G^\vee \cong T^\vee/W$

$\cup$

c characters of $\mathcal{H}(G(\mathbb{Q}_p), G(\mathbb{Z}_p))$

$\updownarrow$

irred. unramified rep'ns.

Can also do tamely ramified $\longleftrightarrow$ use <u>Iwahori Hecke algebra</u>

(not commutative, so need to classify rep'ns of $\mathcal{H}(G(\mathbb{Q}_p), I)$

(Kazhdan-Lusztig?)

Chris
Mon 28 Feb '05

The moduli stack of G-bundles

$X = \text{curve}/k$
$F = k(X)$

Classical Lang. $\left\{ \begin{matrix} \text{repns of } Gal(\bar{F}/F) \\ \text{into } \check{G} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Autom Reps} \\ \text{of } G \end{matrix} \right\}$

Dictionary:

$Gal(\bar{F}/F)$	$\pi_1 X$
Rep of $\uparrow$ into G	~~G-bundle on X~~ G-local system
Autom Rep of G	fncs/sheaves on $G(X)\backslash G(\mathbb{A})/G(\mathbb{O})$ $= Bun_G(X)$

Geometric Langlands:

$\left\{ \begin{matrix} \check{G}\text{-local systems} \\ \text{on } X \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{Hecke eigensheaves} \\ \text{on } Bun_G X \end{matrix} \right\}$

$Bun_G X = G(X)\backslash G(\mathbb{A})/G(\mathbb{O})$ stack-theoretic double quotient

$\parallel$ Theorem

$\mathcal{M}_{G,X} = \mathcal{L}_X G\backslash \mathcal{L}G/\mathcal{L}_+G$

Point: G-bundle over X is trivial away from a pt (if G semi-simple)
so suffices to study at a point.

principal G-bundle/X
$\parallel$
locally trivial G-bdle/X
$\hookrightarrow$ means: in some Groth. topology e.g. Zariski
étale
⋮

$\mathcal{M}_{G,X}$ = "space" of prin. G-bundles over X

need to discuss this!

<u>Moduli Spaces</u>

Z scheme

$\rightsquigarrow$ funct Scheme $\longrightarrow$ Sets

$$\mathcal{U} \longmapsto Z(\mathcal{U}) = \mathrm{Hom}(\mathcal{U}, Z)$$

<u>idea</u>: Having the functor is enough.

<u>Def^n</u>: functor is <u>representable</u> if $\exists$ scheme Z s.t. $F(\mathcal{U}) = \mathrm{Hom}(\mathcal{U}, Z) = Z(\mathcal{U})$

Moduli functor:

$$\text{Scheme} \longrightarrow \text{Sets}$$
$$\mathcal{U} \longmapsto \left\{ \begin{matrix} \text{prin } G\text{-bdle} \\ X \times \mathcal{U} \end{matrix} \right\} \Big/\!\!\Big/ \text{iso}^{\underline{m}}$$

is <u>NOT</u> representable.

<u>The fix</u>: enlarge to groupoids, ie. now have a <u>scheme</u> $\xrightarrow{\mathcal{M}_{G,X}}$ <u>Gpds</u>

$$\mathcal{U} \longmapsto \{\text{prin } G\text{-bdles}/_{\mathcal{U}\times X}\}$$
$$\mathcal{M}_{G,X} + \{\text{iso}^{\underline{m}}\}$$

This is a stack; in fact it's representable as a quotient stack.

<u>Stacks</u>

$$\Gamma = \left\{ \begin{matrix} \text{open subsets} \\ \text{complex mfds} \end{matrix} \right\}$$

Fibered gpoid = presheaf (up to natural iso$^{\underline{m}}$) of groupoids

$$\left[\ \mathcal{U} \xrightarrow{i_{VU}} V \xrightarrow{i_{WV}} W \quad \text{composition of restrictions...}\ \right]$$

i_{WU}

Prestack = fibered gpd s.t. Hom sets are sheaves

(our site will be Scheme$/_k$ with étale top.)

Stack = prestack where the objects glue.
Algebraic stacks = additional technical conditions ...

<u>Quotient Stack</u>

$$Z \text{ scheme} \circlearrowleft G \text{ group} \quad \rightsquigarrow \quad Sch/_K \xrightarrow{[Z/G]} Gpd$$

$$\mathcal{U} \longmapsto \left\{ \begin{array}{c} Z' \\ \downarrow \\ \mathcal{U} \end{array} \xrightarrow{\text{eqvt}} Z \right\}$$

prin. G-bdle

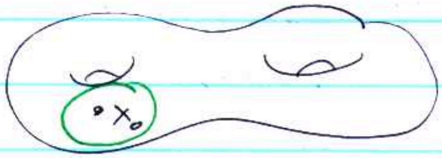
Note this makes sense when G acts freely ...

$$\begin{array}{ccccc} Z & & Z' & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ Z/G & & \mathcal{U} & \longrightarrow & Z/G \end{array}$$

// Now: represent $\mathcal{M}_{G,X}$ as a quotient stack:
Topological analogue:

$\mathcal{M}^{top}_{G,X}$ represented by $Map(X, BG)$

<u>Fact</u>: $\pi_0 \mathcal{M}^{top}_{G,X} = \pi_0 Map(X, BG) = \pi_0 \left({}_{L_X G}\backslash LG / L_+ G \right)$



<u>Pf</u>: Key fact: $X \setminus x_0 \simeq$ 1-complex
$\therefore$ prin G-bundles over $X \setminus x_0$ are trivial.

G conn, semisimple ...?
hard thm in alg. category

$\mathcal{L}G = Hom(D \setminus x_0, G)$

$\mathcal{L}_+ G = Hom(D, G)$

$\mathcal{L}_X G = Hom(X \setminus x_0, G)$

Thm: $\mathcal{M}_{g,X} = \left[{}_{\mathcal{L}_X G}\backslash^{\mathcal{L}G}/_{\mathcal{L}_+ G} \right]$

(in alg. geom. category)

FACT: $\mathcal{L}G/\mathcal{L}_+G$ is an ind-scheme

$\mathcal{L}G = \cancel{\mathrm{Hom}(\mathbb{C}^\times, G)}$ ← $\mathbb{C}^\times$ too big!

$\mathcal{L}G = G(\mathbb{C}((t)))$

$\mathcal{K} = \mathbb{C}((t))$

$\mathcal{O} = \mathbb{C}[[t]]$

$\mathcal{L}_+G = G(\mathbb{C}[[t]])$

$\mathcal{L}_X G = \mathrm{Hom}(X \setminus x_0, G)$

Example: $G = GL_n$

$\mathcal{L}G/\mathcal{L}_+G = \{ \mathcal{O}\text{-lattice in } \mathcal{K}^n \}$ ($\mathcal{O}$-submodule $\cong$ to $\mathcal{O}^n$)

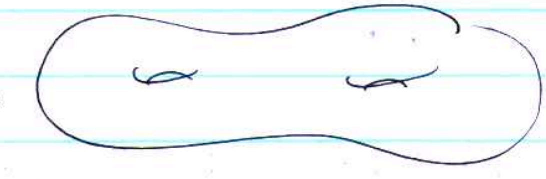
since $G(\mathcal{O}) \to G(\mathcal{K}) \to gr(\mathcal{O}^n, \mathcal{K}^n)$

$gr^{aff}_{(m)} = \{ \text{lattices } W \text{ s.t. } t^{-m}\mathcal{O}^n \subset W \subset t^m\mathcal{O}^n \}$.

$\Rightarrow$ gr^{aff} is an ind-scheme

The other description of G-bundles:

X / arbitrary field



Given a vector bundle, can find basis of meromorphic sections.

Let V = v. bdle, trivialize V away from $x_1, \ldots, x_n$

↑ v. bdles are always locally Zariski trivial.

Trivialize V locally near every point – "formal" neighbord at every x.

$\Rightarrow$ at every point $x \in X$, get $g_x \in G(\mathcal{K}_x)$ ← Laurent series at that point

g_x is transition mx between the two local trivializations.

at all but $x_1, \ldots, x_n$, this lies in $G(\mathcal{O}_x)$.

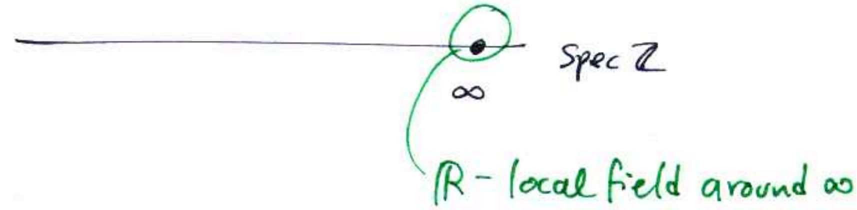
$$\nabla \Rightarrow \{g_x\}_{x\in X} \in G(\mathbb{A})$$

(can go backwards by gluing them together)

Then $\mathcal{M}_{G,X} = G(F_X) \backslash G(\mathbb{A}) / G(\mathcal{O}_{\mathbb{A}})$

Elliptic Curves:

$$SL_2(\mathbb{Q}) \backslash SL_2(\mathbb{A}) / SL_2(\prod \mathbb{Z}_p) = \underbrace{SL_2\mathbb{Z} \backslash \underbrace{SL_2\mathbb{R} / SO_2}_{\text{upper half-plane}}}_{\text{moduli of elliptic curves}}$$



$SO_2 \to L_+ G$

$SL_2\mathbb{Z} \to L_X G$

André Henriques

Perverse Sheaves

Tricky issue: when have a complex

$$V \longrightarrow V \longrightarrow V \longrightarrow V \longrightarrow V \longrightarrow V$$
$$0$$

V^1 means: differentials increase degree

V_1 means: differentials decrease deg

ACK!

X – variety

$D(X)$ – derived category of constructible sheaves on X

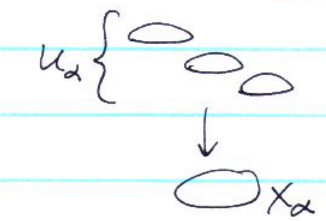
– complexes of sheaves of [fill in blank] ← s.t. get abelian category

such that cohom. sheaves

$H^i(-)$ constructible, vanish for all but finitely many i

F is <u>constructible</u> if $X = \coprod X_\alpha$ ← strata, such that $F|_{X_\alpha}$ is locally constant.

Locally constant: $\exists$ $\mathcal{U}_\alpha \downarrow X_\alpha$ cover of X_α in étale topology s.t. $F|_{\mathcal{U}_\alpha}$ is constant



$F(V) = A^{\pi_0(V)}$

$A \in$ [fill in blank]

↑

same A for every V!

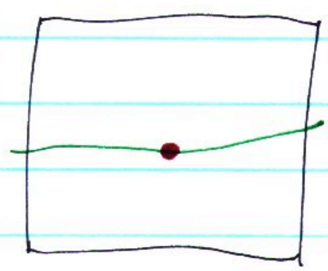
(e.g. $A = \mathbb{C}$)

constructible $\neq$ coherent!
two orthogonal worlds

Two perspectives on $D(X)$ / constructible sheaves

(1) Topologist's

$\mathcal{F}$ sheaf



point $x \in X \rightsquigarrow A_x$

path $x \longrightarrow y \rightsquigarrow A_x \longrightarrow A_y$

path cannot go into a stratum of smaller dimension.

If stays in stratum, should be an iso$^{\underline{m}}$.

$$\mathcal{F}(\mathcal{U}) = \left\{ \begin{array}{l} \text{give an element } a_x \text{ in each } A_x \text{ for } x \in \mathcal{U} \\ \text{such that } \begin{array}{ccc} A_x & \longrightarrow & A_y \\ x \bullet & \rightsquigarrow & \bullet\, y \\ a_x & \longmapsto & a_y \end{array} \end{array} \right\}$$

back to other perspective

Constructible sheaves $\hookrightarrow D(X)$ by taking $\mathcal{F}$ to $\left(\underbrace{\to 0}_{} \to \mathcal{F} \to \underbrace{0 \to \cdots}_{} \right)$ 0's everywhere else

② Constructible Cosheaves : paths that fall into strata but don't go out.

otherwise the same. Again get a local system.

$$\mathcal{C} \quad (\text{cosheaves}) : \begin{pmatrix} \mathcal{U} \\ \downarrow \\ X \end{pmatrix} \longrightarrow \begin{pmatrix} \text{abelian} \\ \text{group} \end{pmatrix}$$

$$\begin{pmatrix} \mathcal{U} \\ \downarrow \\ V \\ \downarrow \\ X \end{pmatrix} \longrightarrow \left(\mathcal{C}(\mathcal{U}) \longrightarrow \mathcal{C}(V) \right)$$

$$\mathcal{C}(\mathcal{U}) = \left\{ \begin{array}{l} \text{a finite collection} \\ \text{of points of } \mathcal{U} \text{ and} \\ \text{for each one an} \\ \text{element of } A_x \end{array} \right\}$$

mod out by a relation :

$$\begin{array}{ccc} A_x & \xrightarrow{\gamma} & A_y \\ \Downarrow & & \\ a_x & \longmapsto & \gamma_*(a_x) \\ \bullet_x & \overset{\gamma}{\rightsquigarrow} & \bullet_y \end{array}$$

identify $\boxed{a_x \approx \gamma(a_x).}$

How do these sit in $D(X)$?

start with
complexes of sheaves
whose cohomology is constructible ... ?

sheaf $\longmapsto$ complex of cosheaves

$\mathcal{F} \longmapsto \mathcal{C}$

cosheaf $\longmapsto$ complex of sheaves

$\mathcal{C} \longmapsto \mathcal{F}$

take an injective resolution of $\mathcal{F}$ — if nice enough, functorial in $\mathcal{U}$?

$\boxed{\mathcal{C}(\mathcal{U}) := \text{compactly supported cochains in } \mathcal{U} \text{ with values in } \mathcal{F}} = C^*(X, X \setminus \mathcal{U}; \mathcal{F})$

$\boxed{\mathcal{F}(\mathcal{U}) := \text{noncompactly supported chains on } \mathcal{U} \text{ with values in } \mathcal{C}} = C_*(X, X \setminus \mathcal{U}; \mathcal{C})$

" What sheaves want is to have their cohomology taken ...
what cosheaves want is to have their homology taken " — André

<u>Theorem</u> [Verdier duality]

$$\mathcal{F} \longleftrightarrow \mathcal{C}$$

(What other people might mean by Verdier duality: do the same, but then take dual of everything to get <u>sheaves</u>, not cosheaves.)

<u>Operations</u> :

$f^*_{\bullet} \quad f^{\bullet}_{*} \quad f_! \quad f^!$

<u>sheaves</u> | <u>cosheaves</u>

f^* easy: $f : X \longrightarrow Y$

$$(f^*A)_x = A_{f(x)}$$

$\mathcal{U}$ open $\quad f_*\mathcal{F}(\mathcal{U}) = \mathcal{F}(f^{-1}(\mathcal{U}))$

C closed $\quad f_!\,\mathcal{F}(C) = \mathcal{F}_{cpt}(f^{-1}(C))$

$\Big\}$ but need derived functors Rf_*

adjoint $Rf_! \longrightarrow Rf^!$ (def^n of $f^!$)

for cosheaves: backwards. $f^!$ easy, pullback costalks, etc.
so then f^* harder.

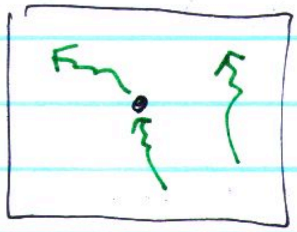
Def^n: Perverse sheaves
$P \in D(X)$ is perverse if $\dim_{\mathbb{C}}\{x \in X \mid H^{-i}((i_x)^* P) \neq 0\} \leq i$
$\dim_{\mathbb{C}}\{x \in X \mid H_{-i}((i_x)^! P) \neq 0\} \leq i$

Topologist's perspective on perverse sheaves:

Examples:

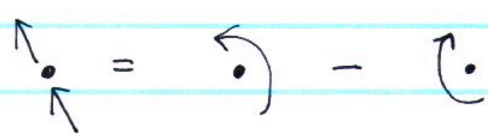
1) 1 stratum: X smooth. $\longrightarrow$ local system.

2) 2 strata: $\mathbb{C}$ codim 1.



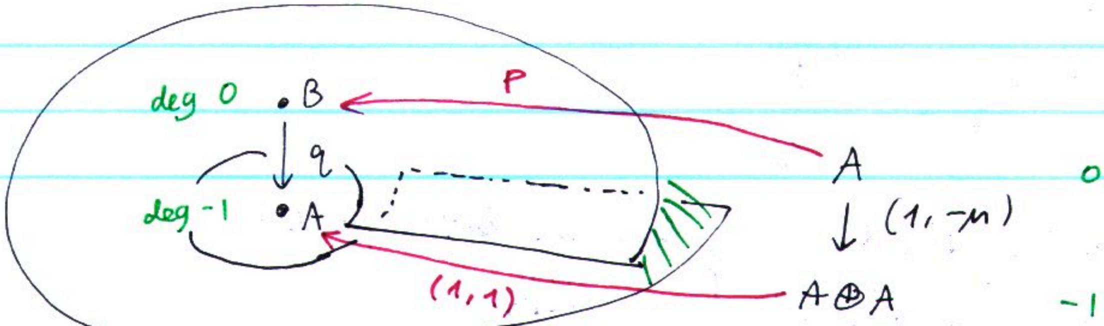
if rel. $\dim_{\mathbb{C}} > 1$, no maps: just a direct sum.

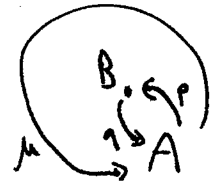
"Now I'm in perverse land" - André



(Reminder: Goresky-MacPherson have a magical Morse theory interpretation of these pictures...)

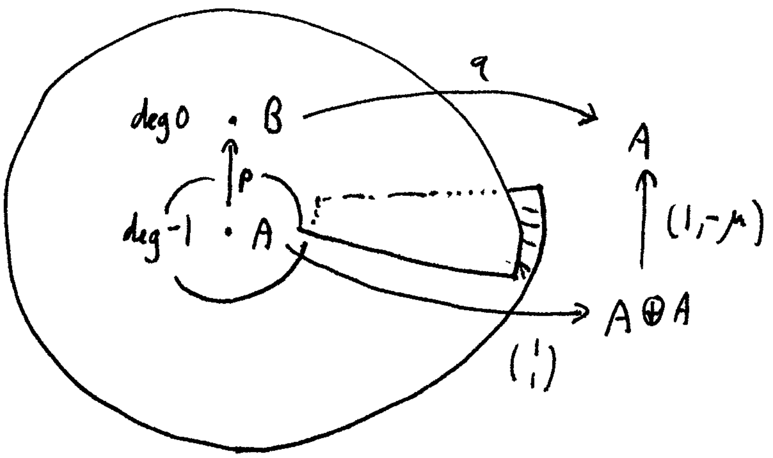
A constructible cosheaf for this picture:



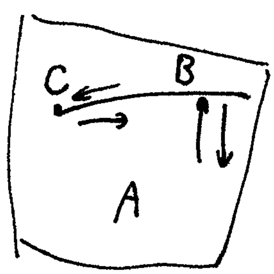
μ is defined by 

We have the complex $0 \to B \to A \to 0$ living on the small stratum.
A lives on the big stratum, but to define the arrow in, we use the resolution
$0 \to A \to A \oplus A \to 0$.

Here is the Verdier dual picture, with sheaves:



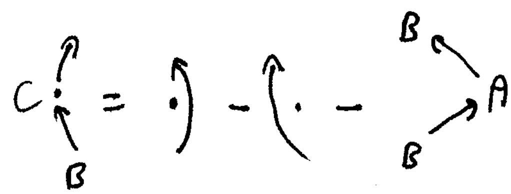
3) 3 strata



relations: $A \to B \to C = 0$

$C \to B \to A = 0$

AB picture same as above (local $\Rightarrow$ C is irrelevant)



$A \rightleftarrows B \rightleftarrows C$

this + this $= 1 - \mu_B$

<u>Cotangent bundle</u>

smooth $X = \coprod X_\alpha$. Sheaves don't in general live on the strata. They live on conormals.

$$N^*X_\alpha := \{(x,\xi) \in T^*X \mid x \in X_\alpha,\ \xi \text{ kills } T_x X_\alpha\}$$

$$N^* := \bigcup_\alpha N^*X_\alpha, \quad \overset{\circ}{N}{}^* := N^* \setminus \bigcup_{\alpha,\beta} (N^*X_\alpha \cap N^*X_\beta) \text{ smooth locus,}$$

Microlocal stalks live in $\overset{\circ}{N}{}^*$, and form a local system.
= local in both position and momentum.

$\rightsquigarrow$ Morse theoretic construction of cohomology of the "Milnor fiber". (appears in SGA as nearby cycles.)

Joel Kamnitzer ①
Tues 1 March 2005

Geometric Satake Correspondence

Classical Satake: G = split reductive gp over $\mathbb{F}_q$ (eg GL_n)

$\mathcal{K} = \mathbb{F}_q((t))$, $\mathcal{O} = \mathbb{F}_q[[t]]$

contains torus isom to $(\mathbb{F}_q^{\times})^n$ some n

Elizabeth mentioned $\boxed{\underbrace{\mathcal{H}(G(\mathcal{K}), G(\mathcal{O}))}_{\text{convolution of fns}} \cong \underbrace{\mathrm{Rep}\,\check{G}}_{\text{f.d. }\mathbb{C}\text{-rep'ns}}}$

GOAL: Geometrify and categorify the above
provide categories whose Grothendieck gp is above ; in fact show categories equivalent (with tensors)

For RHS: clearly want rep'n category $\underline{\mathrm{Rep}}\,\check{G}$.

X variety over $\mathbb{F}_q$, $\mathcal{F}^\bullet$ complex of sheaves on X.

$$\chi_{\mathcal{F}^\bullet} : X(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_\ell = \mathbb{C} \quad \text{cohomology of stalks}$$
$$x \longmapsto \sum_i (-1)^i \, tr(Fr_x \mid \mathcal{H}^i_x \mathcal{F}^\bullet)$$

Frobenius

[geometrically: local system, and go around little loop at each pt, take monodromy]

Philosophy: $P(X)$ = perverse sheaves

$$\text{Grothendieck } K(P(X)) \longrightarrow \mathrm{Fun}(X(\mathbb{F}_q))$$
$$\mathcal{F} \overset{\text{by above procedure}}{\longmapsto} \chi_{\mathcal{F}}$$

actually a ring map by taking derived tensor product from $P(X)$

Can think of $\mathcal{H}(G(\mathcal{K}), G(\mathcal{O}))$ three different ways:

$$\mathcal{H}(-) = \mathrm{Fun}_{G(\mathcal{O})\times G(\mathcal{O})}(G(\mathcal{K}))$$
$$= \mathrm{Fun}_{G(\mathcal{O})}(G(\mathcal{K})/G(\mathcal{O})) \longleftarrow \text{This is the good option}$$
$$= \mathrm{Fun}\left({}_{G(\mathcal{O})}\backslash G(\mathcal{K})/G(\mathcal{O})\right)$$

There exists an ind-scheme over $\mathbb{F}_q$, Gr^{aff} s.t. $Gr(\mathbb{F}_q) = G(\mathcal{K})/G(\mathcal{O})$

So: the category to replace the LHS:

$$\boxed{K\left(\begin{matrix} \text{perverse sheaves on} \\ G(\mathcal{K})/G(\mathcal{O}) \\ \underline{\text{pure weight } 0} \end{matrix} \right) = \mathcal{H}(G(\mathcal{K}), G(\mathcal{O})).}$$

$G(\mathcal{O})$-equivariant

technical cond
re: Frobenius,
to make ring map an isom ?

Now the $\mathbb{C}$ world — things are easier!

G = reductive over $\mathbb{C}$, $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$

TC $\quad Gr^{aff} = G(\mathcal{K})/G(\mathcal{O})$ ind-scheme

X smooth curve over $\mathbb{C}$, $x \in X$

$$Gr^{aff} \cong \left\{ \begin{matrix} \text{principal } G\text{-bundles on } X \\ \text{along with a trivial. away from } x \end{matrix} \right\}$$

Properties: $G(\mathcal{O}) \circlearrowright Gr^{aff}$.

For each $\mu \in Hom(\mathbb{C}^\times, T) = \Lambda$, there is a point $t^\mu \in Gr$

By doing $\mathcal{O}$-row and column operations, can get any element $\in G(\mathcal{K})$ to the form t^λ for λ dominant $\in \Lambda_+$.

$$Gr = \bigsqcup_{\lambda \in \Lambda_+} G(\mathcal{O}).t^\lambda$$

Aside: $Q \backslash G/P = W_Q \backslash W / W_P$

$G(\mathcal{O}) \backslash G(\mathcal{K})/G(\mathcal{O}) = W \backslash W \ltimes \Lambda / W = \Lambda / W.$

Prop: $\overline{Gr^\lambda} = \bigcup_{\mu \leq \lambda} Gr^\mu$

finite-dim'l projective varieties

Gr^λ a v. bdle over a flag variety.

$$\dim Gr^\lambda = 2\langle \lambda, \rho \rangle$$

Def: A G(O)-equivariant perverse sheaf on Gr is a perverse sheaf which is constructible w.r.t. this stratification (by Gr^λ's).

[probably need to say, since Gr an ind-scheme, ie: it's a perverse sheaf supported on some finite piece, ie. $\overline{Gr^\lambda}$.]

Denote the category $\mathcal{P} := \mathcal{P}_{G(O)}(Gr)$

X complex projective variety
$\cup$
X° smooth locus

There is a ~~unique~~ smallest perverse sheaf on X which is the constant sheaf when restricted to X°, called IC_X = the intersection cohomology ~~of~~ sheaf (actually a complex of sheaves)
[Deligne-Goresky-MacPherson]

Now: Cohomology [actually derived functor of global sections functor]

$$\mathbb{H}^*(IC_X) = IH^*(X)$$

(FACT: Verdier dual of IC_X is IC_X)

Example of G(O)-equt perverse sheaf: $IC_{\overline{Gr^\lambda}} =: IC_\lambda$.

Theorem [Lusztig, Ginzburg, Beilinson-Drinfeld, Mirković-Vilonen]
There exists a structure of tensor category on $\mathcal{P}$ such that $\exists$ equivalence of categories

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\cong"] \arrow[dr, "\mathbb{H}^*"'] & \operatorname{Rep} \check{G} \arrow[d, "\text{forget}"] \\ & \text{Vect} \end{tikzcd}$$

POINT: Don't want (derived functor of) naïve tensor product, because want ours to correspond to convolution product of functions, not ptwise mult.

Under this equivalence, the simple objects in the categories correspond:

$$IC_\lambda \longmapsto V_\lambda, \quad \lambda \in \Lambda_+$$

so $IH^*(\overline{Gr^\lambda}) \cong V_\lambda$

Example: $G = GL_n$

$$\lambda = \omega_k = (\underbrace{1, \dots 1}_{k}, 0, \dots 0)$$

$$\overline{Gr_{\omega_k}} = Gr_{\omega_k} \cong Gr(k,n) = GL(n,\mathbb{C}) /$$

$$g(t)\cdot t^{\omega_k} \longmapsto [g(0)]$$

Intersection cohomology

$$IH(\overline{Gr}_{\omega_k}) = IH(Gr(k,n)) = H(Gr(k,n)) \leftarrow$$ has a basis indexed by k-element subsets of $\{1, \dots, n\}$

$$V_{\omega_k} = \Lambda^k \mathbb{C}^n \leftarrow$$

so $IH(\overline{Gr}_{\omega_k}) = V_{\omega_k}$ ✓

Proof of Thm: Tannakian duality.

"Theorem": Let $\mathcal{C}$ be a rigid $\otimes$ category along with an exact $\underbrace{\text{}}_{\mathbb{C}\text{-linear}}$ functor $F: \mathcal{C} \to Vect$

$\circledast : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$

$\alpha : (A \circledast B) \circledast C \to A \circledast (B \circledast C)$

$\sigma : A \circledast B \to B \circledast A$

$1 \in \mathcal{C}$

$\star : \mathcal{C} \to \mathcal{C}$

Then there exists an affine group scheme H over $\mathbb{C}$ whose rep^n category (over $\mathbb{C}$, f-dim'l) is $\mathcal{C}$.

$$H = \mathrm{Aut}(F, \circledast) = \left\{ \{ h_A : F(A) \to F(A) \}_{A \in \mathcal{C}} : \begin{array}{l} h_{F(A \circledast B)} \\ = h_{F(A)} \otimes h_{F(B)} \end{array} \right\}$$

So then the point is that to prove the theorem, need to construct all the maps $\circledast$, α, σ, etc.

Tues 1 March 2005.

Joel, part II.

<u>Exercise</u>: Find the group whose rep category is diff. graded v. spaces $\overset{\shortparallel}{\text{complex of vector spaces}}$

$\mathcal{P}_{G(O)} Gr$: want to give this the str of a Tannakian category

Produce a $\otimes$-product (some sort of convolution)

Work for a moment with $\mathcal{P}_{G(O)\times G(O)}(G(K))$:

$$\begin{aligned} &\ \ \psi \\ &\mathcal{F}, \mathcal{G} \longmapsto \mathcal{F} \boxtimes \mathcal{G} \in \mathcal{P}(G(K) \times G(K)) \end{aligned}$$

use multiplic. $m : G(K) \times G(K) \longrightarrow G(K)$

$$\mathcal{F} \boxtimes \mathcal{G} \longmapsto \underbrace{R_{m_*}}_{\text{derived functor for } m} (\mathcal{F} \boxtimes \mathcal{G}) =: \mathcal{F} \circledast \mathcal{G}$$

Why is $R_{m_*}(\mathcal{F} \boxtimes \mathcal{G})$ perverse?

<u>Theorem</u>: $\mathcal{F} \circledast \mathcal{G}$ is a perverse sheaf.

Given $\mathcal{F}, \mathcal{G}$: $\mathcal{F} \boxtimes \mathcal{G} \in \mathcal{P}_{G(O)\times G(O)}(Gr \times Gr)$

$$\downarrow$$

$$p^*(\mathcal{F} \boxtimes \mathcal{G}) \in \mathcal{P}_{G(O)\times G(O)\times G(O)}(G(K) \times Gr)$$

$$\downarrow$$

$$\mathcal{F} \tilde{\boxtimes} \mathcal{G} \in \mathcal{P}_{G(O)}(G(K) \times_{G(O)} Gr)$$

$$\downarrow$$

$$R_{m_*}(\mathcal{F} \tilde{\boxtimes} \mathcal{G}) \in$$

$$m : G(K) \times_{G(O)} Gr \longrightarrow Gr \quad [\text{this is stratified semi-small}]$$

$$[g, L] \longmapsto g \cdot L$$

$\hookrightarrow$ so $\mathcal{F} \otimes \mathcal{G}$ is perverse.

Another def^{n} of this tensor product : allows to prove properties in a simpler way.

<u>Beilinson-Drinfeld Grassmannian:</u> Gr_X $\quad$ X = smooth curve over $\mathbb{C}$

There exists an ind-scheme over $\cancel{X}$, whose $\mathbb{C}$-points are

$$\left\{ (x, F, \nu) : F \text{ is a principal } G\text{-bundle on } X,\ x \in X,\ \nu \text{ is triv. of } F \text{ over } X \setminus x \right\}$$

$$\begin{array}{c} Gr_X \\ \downarrow \\ X \end{array}$$ $\longleftrightarrow$ fiber = affine Grassmannian assoc. to pt x

$$Gr_X^{(2)} = \left\{ (x, y, F, \nu) : x, y \in X,\ \nu \text{ a triv. of } F \text{ over } X \setminus \{x, y\} \right\}$$

$$\begin{array}{ccc} Gr_X & & Gr_X^{(2)} \\ \downarrow & & \downarrow \pi \\ X & \xrightarrow[\text{diag}]{} & X^2 \end{array}$$

fiber over diagonal — affine Grassmannian.

fibers away from diagonal — two copies of the affine Grassmannian

<u>Claim</u>: $\left\{ \text{Two bundles } F_1, F_2 \text{ along w/ triv's of } F_1 \text{ away from } x \text{ and } F_2 \text{ away from } y \right\}$

$$\Updownarrow$$

$\left\{ \text{One bundle } F \text{ and a triv. away from } \{x, y\} \right\}$

<u>Pf</u>: Locally, glue triv's. (work)

so $\pi^{-1}(x, y) = Gr \times Gr$ if $x \neq y$

$\pi^{-1}(x, x) = Gr$.

Joel part II

Want to define $\circledast$ again:

$X = \mathbb{A}^1$, $\begin{array}{c} Gr_X \\ \downarrow \\ X \end{array}$ is trivial.

$\begin{array}{ccc} G(\mathcal{O}_x) & \in & G_X(\mathcal{O}) \\ \downarrow & & \downarrow \\ x & \in & X \end{array}$ bundle of groups over X acts: $\circlearrowright \left(\begin{array}{c} Gr_X \\ \downarrow \\ X \end{array} \right)$

and $\circlearrowright P_{G_X(\mathcal{O})}(Gr_X) = \underset{=: \mathcal{P}}{P_{G(\mathcal{O})}(Gr)}$

Given $A, B \in \mathcal{P}$:

$$Gr_X^{(2)}|_U = Gr_X \times Gr_X|_U = Gr \times Gr \times U \xhookrightarrow[j]{} Gr_X^{(2)}$$

$A \boxtimes B \in \mathcal{P}(Gr_X \times Gr_X|_U)$

Now pull out intermediate extension functor

$$j_{!*}(A \boxtimes B) \in \mathcal{P}(Gr_X^{(2)})$$

Now use $i: \begin{array}{ccc} Gr_X & \xhookrightarrow{i} & Gr_X^{(2)} \\ \downarrow & & \downarrow \\ X & \xrightarrow[\text{diag}]{} & X \times X \end{array}$ and pullback $i^*(j_{!*}(A \boxtimes B)) \in \mathcal{P}(Gr_X)$.

Thm: This agrees with our previous definition.

"merging of bundles" is somehow the multiplication.

Using the second def$^{\underline{n}}$, commutativity is obvious:

use $\begin{array}{ccc} Gr_X^{(2)} & \longrightarrow & Gr_X^{(2)} \\ (x, y) & \longmapsto & (y, x). \end{array}$ (recall for Tannakian category, $A \circledast B \cong B \circledast A$,) Also need $(A \circledast B) \circledast C \cong A \circledast (B \circledast C)$

Suppose we believe Tannakian category stuff.
Why should we get Langlands dual $G^\vee$?

Weight Functors

$$V_\lambda \cong \bigoplus_\mu V_\lambda(\mu) \qquad \mu \in \Lambda = \mathrm{Hom}(T^\vee, \mathbb{C}^\times) = \mathrm{Hom}(\mathbb{C}^\times, T)$$

$\|$

$$IH(\overline{Gr_\lambda}) = \bigoplus_\mu ?$$

$G(\mathcal{O}) \circlearrowright Gr$
$N(K) \circlearrowright Gr$ } using these, can take any el't to $\begin{bmatrix} t^{\mu_1} & & \\ & \ddots & \\ & & t^{\mu_n} \end{bmatrix}$

but can't reorder μ_i's,

$$Gr = \bigcup_{\mu \in \Lambda} \overbrace{\underbrace{N(K) \cdot t^\mu}_{\text{infinite-dim'l and codimension'l.}}}^{S^\mu}$$

$$\mathbb{H}^k_c(S^\mu, A) \qquad A \in P_{G(\mathcal{O})}(Gr)$$

Theorem (M-V) ① For all $H_c^k(S^\mu, A) = 0$ unless $k = 2\langle \mu, \rho \rangle$
② $\mathbb{H}^*(A) = \bigoplus_\mu \underbrace{\mathbb{H}^k_c(S^\mu, A)}$

so dual torus naturally acts, by weight μ, on this piece.

so $T^\vee \circlearrowright \mathbb{H}(A)$ forall $A \in P$.

E.g: $IH(\overline{Gr_\lambda})$ is spanned by the components of $\underbrace{Gr^\lambda \cap S^\mu}_{\text{MV-cycles.}}$

for $\lambda = \omega_k$, these are Schubert varieties.

Tues 1 March 2005

Dave BenZvi.

- Why $\mathcal{D}$-modules?
- what is geometric Langlands?
- $GL(1)$

1. $\mathcal{D}$-modules.

What we <u>really</u> want: functions.

smooth, $\mathcal{L}^2$, distributions $\left(\dfrac{G(\mathbb{A})}{G(\mathbb{F})} \right)$ ... to get same richness in alg. geom, need more.

$$\text{eg. want: } e^{tx} : \mathbb{R} \longrightarrow U(1)$$

$$\text{derivative } \partial_x e^{tx} = t e^{tx}$$

$$\text{so } \underbrace{(\partial_x - t) e^{tx} = 0}_{}$$

e^{tx} satisfies an <u>algebraic</u> D.E. A-ha!

Ex:

$$M = \left. \underset{\mathbb{C}\langle x, \partial_x \rangle}{\overset{\mathcal{D}}{\shortparallel}} \text{ differential operators} \middle/ \mathcal{D}(\partial - t) \right.$$

then e^{tx} is a solution of M, ie. satisfies relation in M

so anytime have a notion of differential

$$M \longrightarrow \text{Fun} \quad \left(\text{image of } 1 \text{ is a solution + the diff'l eq'ns} \right)$$

have $\mathcal{D}^q \longrightarrow \mathcal{D}^p \longrightarrow M \longrightarrow 0$ coherent $\mathcal{D}$-module

Another pt. of view: $\mathcal{O} \longrightarrow M$ map of $\mathcal{D}$-modules.

M = sections of V v. bdle with <u>flat</u> connections.

$\downarrow$

relations in v. fields satisfied, can extend to diff'l op's

$\mathcal{O} \to M \Leftrightarrow$ flat sections (image of 1)

$$H^0_{dR}(M) = \operatorname{Ker}(M \xrightarrow{\nabla} M \otimes \Omega^1)$$

In terms of perverse sheaf,

$$M \underset{\substack{dR\\ \text{functor}}}{\longmapsto} \{ M \to M \otimes \Omega^1 \to \ \dots \ \to M \otimes \Omega^n \}$$

complex whose cohomology is constructible if M good

$f \in \mathrm{Fun}$ [where we know how to differentiate]

$\mathcal{D} f \subset \mathrm{Fun}$. $M_f := \mathcal{D} f$. If f holonomic, f satisfies lots of DE's.

(For ~~any~~ arbitrary f, could be $\mathcal{D}f \cong \mathcal{D} \dots$)
WANT them to be as small as possible

If M holonomic, then $dR(M) \in D_{constr}(X)$

$$\left\{ M \text{ regular holonomic} \right\} \xrightarrow[\simeq]{\text{perverse}} \left\{ \begin{matrix} \text{perverse} \\ \text{sheaves} \end{matrix} \right\}$$

Riemann-Hilbert

Example: $\partial - t$, $\mathcal{D}/\partial - t$ NOT regular, essential singularity at ∞.
The perverse sheaf you get is constant sheaf.

$\mathcal{D}$ has a geometric filtration $\mathcal{D}_i$ = diff'l ops of order $\le i$.

Can also form $\underbrace{\mathrm{Sym}\, \underbrace{T_X}_{\text{tangent bdle}}}_{\text{commutative!}} = \underbrace{gr\, \mathcal{D}}_{\text{associated graded of } \mathcal{D}}$

$\quad = \mathbb{C}[T^*X]$

Now: $\mathrm{Sym}\, T_X$-module = (coherent) sheaf on T^*X.

Given M $\mathcal{D}$-module, $\Longrightarrow$ $gr(M)$ a $gr(\mathcal{D}) = \mathrm{Sym}\, T_X$ - module
$$\| $$
sheaf T^*X

get a characteristic $M \subset T^*X$ (support)

Holonomic $\Leftrightarrow$ dim characteristic variety = dim X
($\Rightarrow$ Lagrangian)

Example: M flat vector bundle
$\mathrm{char}\, M = X \subset T^*X$

Another perspective: $\mathfrak{g} \longrightarrow \mathrm{Vect}$
$U\mathfrak{g} \longrightarrow \mathcal{D}$
so close to rep'n theory.

Tannakian categories

Classical Satake: $\mathrm{Fun}\left({}_{G(O)}\backslash G(K)/G(O)\right) \simeq \mathrm{Rep}\, \check{G}$

Joel: $\underset{G(O)}{\mathcal{P}}\left(G(K)/G(O)\right) \simeq \underbrace{\mathrm{Rep}\ G^\vee}$

if you have category of rep'ns, can recover the group !!
in some sense this is the right def^n of the Langlands dual gp.

Example $\left\{\begin{matrix}\text{graded}\\ \text{vector spaces}\end{matrix}\right\} \xrightarrow[\text{forgetful}]{} \mathrm{Vect}$

$G = \mathrm{Aut}(\text{fiber functor})$
$G = \mathbb{C}^\times$

Example: $\left\{ \begin{matrix} \text{local systems} \\ \text{on a space } X \end{matrix} \right\} \xrightarrow[\text{fiber at } x]{} \underline{\mathrm{Vect}}$

fix $x \in X$

and now the group is $\pi_1(X)$.

What is geometric Langlands?

$\mathrm{Fun}\left({}_{G(F)} \backslash G(\mathbb{A}) \right) \circlearrowleft G(\mathbb{A})$

$\prod G(\mathbb{Z}_p)$ (in number field case)

As Elizabeth explained, can reduce to study of $G(\mathcal{O}_{\mathbb{A}})$-invariants and action of Hecke algebra on it. "everywhere unramified"

$\mathrm{Fun}\left({}_{G(F)} \backslash G(\mathbb{A}) \right)^{G(\mathcal{O}_{\mathbb{A}})} \circlearrowleft$ (some Hecke algebra)

$\|$

$\underbrace{\mathrm{Fun}\left({}_{G(F)} \backslash G(\mathbb{A}) / G(\mathcal{O}_{\mathbb{A}}) \right)}_{\mathrm{Bun}_G(X)}$

For each $x \in X$, $\mathrm{Fun}(\mathrm{Bun}_G(X)) \circlearrowleft \underbrace{\mathcal{H}(G(K_x), G(\mathcal{O}_x))}_{\text{commutative algebra}}$ $= \mathrm{Rep}\, G^\vee$

(spherical Hecke algebra)

$\underbrace{\text{$f$ eigenfunctions of $\mathcal{H}_{sph}$ at each $x \in X$}}_{\text{simultaneously diagonalize!}}$

Which eigenvalues appear?

Langlands: the eigenvalues that appear $\Longleftrightarrow$

$$[\, \mathrm{Gal}(F(X)) \longrightarrow G^\vee \,] \quad G^\vee\text{-local system on } X$$

$$\mathrm{Frob}_x \longrightarrow \text{semisimple conjugacy class in } G^\vee$$

$$\mathrm{Rep}\, G^\vee = \mathbb{C}[G^\vee]^{G^\vee}$$

Geometric Langlands: [next time]

Weds 2 March '05

Dave BenZvi : $GL(1)$

L rank 1 local system on a curve $X/\mathbb{C}$
$\iff$ line bundle with flat connection

How to describe L? $\to$ give monodromy, ie: $\pi_1(X) \longrightarrow \mathbb{C}^\times$.

$\searrow H_1(X;\mathbb{Z}) \nearrow$ factors through abelianization.

$\underbrace{\cong H^1(X;\mathbb{Z})}$ fundamental gp of $\mathrm{Jac}(X)$.

$X \rightsquigarrow \mathrm{Jac}\, X$ abelian variety $\simeq (S^1)^{2g}$, g = genus
= line bundles of deg 0 on X
$= H^1(X, \mathcal{O}) / H^1(X;\mathbb{Z}) = H^1(X, \mathbb{R}/\mathbb{Z})$

structure sheaf

Given $L \iff K_L$ rank 1 local system on $\mathrm{Jac}(X)$

geometric Langlands (case $GL(1)$)

or: local system on X $\longleftrightarrow$ perverse sheaf / $\mathcal{D}$-module on $\mathrm{Jac}(X)$

But: have annoying deg 0 condition - really want something on $\mathrm{Pic}\, X$.

$G = GL(1)$

$$\mathrm{Bun}_{GL(1)}(X) = \mathcal{M}_{GL(1),X} = \mathrm{Pic}\, X \simeq \mathrm{Jac}(X) \times \mathbb{Z}$$

what's the identification?

$x \in X \quad \mathcal{L} \in \mathrm{Pic}^n$
$\Rightarrow \mathcal{L}(x) \in \mathrm{Pic}^{n+1}$, but this depends on choice of x

Build on $\mathrm{Pic}(X)$:

L on $X \Rightarrow K_L$ on $\mathrm{Pic}(X)$ as follows:

(normalization: $K_L\big|_{\{\mathcal{O}\} \in \mathrm{Pic}^0} = \mathbb{C}$)

Problem: naïve suggestion $K_L|_{\mathcal{L}(x)} = K_L|_{\mathcal{L}}$ depends on x.

Better: Abel-Jacobi: $X \hookrightarrow \mathrm{Pic}^1$

$x \longmapsto \mathcal{O}(x)$

take a point to line bdle w/ that pt as divisor

now $K_L|_{X \subset \mathrm{Pic}^1} := L$.

another way: $K_L|_{\mathcal{O}(x)} = \underbrace{K_L|_{\mathcal{O}}}_{\overset{\shortparallel}{\mathbb{C}}} \otimes L|_x$ Hecke eigensheaf condition

$\mathbb{C}$ by normalization

Now repeat same rule:

$$K_L|_{\mathcal{O}(x+y)} = K_L|_{\mathcal{O}(x)} \otimes L|_y$$

$\mathcal{O}(x+y) = \mathcal{O}(x)(y)$

so $K_L|_{\mathcal{L}(x)} = \underbrace{L|_x}_{\text{"e'value"}} \otimes \underbrace{K_L|_{\mathcal{L}}}_{\text{"eigenvector"}}$ "eigenvalue eq'n" $\forall x \in X$

Hecke operation = "modify at x"

Modification: $\mu: X \times \mathrm{Pic} \longrightarrow \mathrm{Pic}$

$x, \mathcal{L} \longmapsto \mathcal{L}(x)$

so want $\boxed{\mu^* K_L = L \boxtimes K_L}$ Hecke eigensheaf condition

– overdetermines K_L.

Don't yet know it exists!

But it does, and is unique.

Deligne: K_L exists.

$$X \hookrightarrow Pic^1 X$$
$$Sym^2 X \longrightarrow Pic^2 X$$
$$x \cdot y \qquad \mathcal{O}(x+y)$$
$$\vdots$$
$$Sym^n X \longrightarrow Pic^n X$$

To define on $Sym^n X$, $K_L|_{x_1+\dots+x_n} = L|_{x_1} \otimes L|_{x_2} \otimes \dots \otimes L|_{x_n}$

But for higher n, ie. $n \geq g$, $Sym^n X \longrightarrow Pic^n$ surjective

but too big!

• $n \geq 2g-2$, $Sym^n X \longrightarrow Pic$

fiber bundle, fibers $\mathbb{P}^{n-g}$ complete "linear ~~systems~~ series" =

Deligne: "projective space is simply connected"!

$L \longmapsto Sym^n L$ local system on $Sym^n X$

since fibers simply-conn, local system trivial on fibers, descends to Pic.

$\Rightarrow Sym^n L$ descends to Pic^n

this is why we want geometric Langlands: for function fields, would have to prove that function is constant along fibers

Now have for high enough n, take negatives for lower...

$$\mu: X \times Pic \longrightarrow Pic$$
$$\cap$$
$$\tilde{\mu}: Pic \times Pic \xrightarrow{mult} Pic$$

$$\mu^* K_L = L \boxtimes K_L, \quad \tilde{\mu}^* K_L \cong K_L \boxtimes K_L$$

Another point of view: draw the graph instead.

$$\begin{tikzcd} & \mathcal{H}ecke = \{(x, \mathcal{L}, \mathcal{L}') : \mathcal{L}' \text{ modified by } x \text{ from } \mathcal{L}\} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \times Pic & & Pic \end{tikzcd}$$

So our condition is $\pi_{1*}(\pi_2^* K_L) = L \boxtimes K_L$

<u>Relate to adèles in $Gl(1)$ case:</u>

$$GL_1(\mathbb{C}(x)) \backslash GL_1(\mathbb{A}) / GL_1(\mathcal{O}_{\mathbb{A}})$$

$$GL_1(\mathbb{A})/GL_1(\mathcal{O}_{\mathbb{A}}) \simeq \prod_{x \in X}' GL_1(K_x)/GL_1(\mathcal{O}_x)$$

$$K_x \simeq \mathbb{C}((t)) \supset \mathcal{O}_x = \mathbb{C}[[t]]$$

so $GL_1(K_x)/GL_1(\mathcal{O}_x) = \mathbb{C}((t))^* / \mathbb{C}[[t]]^*$

$= \{\text{non-zero Laurent}\} / \{\text{Taylor w/ nonzero constant term}\}$

$= \mathbb{Z}$

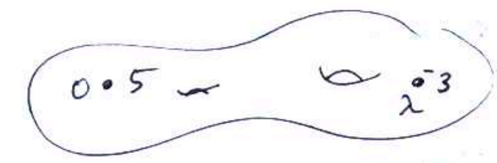
$$GL_1(\mathbb{A})/GL_1(\mathcal{O}_{\mathbb{A}}) = \prod_{x \in X}' \mathbb{Z} = \underline{\underline{\text{divisors}}}$$

so $GL_1(\mathbb{C}(x)) \backslash GL_1(\mathbb{A}) / GL_1(\mathcal{O}_{\mathbb{A}}) = \mathbb{C}(X)^* \backslash \text{divisors} = \underline{\underline{Pic}}$.

$$G(\mathcal{O}_x) \backslash G(K_x) / G(\mathcal{O}_x) = \mathbb{Z} \text{ acts on } Pic.$$

$$\mathcal{L} \longmapsto \mathcal{L}(x).$$

Why are the integers a group?



-3 approaches 5

$\lambda \to 0$

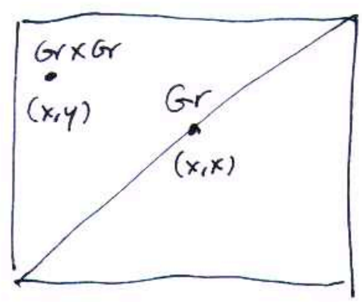
$$5 \in \mathbb{C}((t))^* / \mathbb{C}[[t]]^* \qquad t^5$$

$$-3 = (t-\lambda)^{-3} \in \mathbb{C}((t-\lambda))^* / \mathbb{C}[[t-\lambda]]^*$$

$$\lim_{\lambda \to 0} \frac{t^5}{(t-\lambda)^3} = t^2$$

Joel: Last time: $G(K_x)/G(O_x) = Gr =$ G-bundles on X with trivialization $X \setminus \{x\}$

X^2



$= Gr^{(2)}$

$=$ G-bundles + triv. away from $x, y \in X$

so as x, y get close, two copies of Gr somehow "glue", this is the multiplication.

What IS geometric Langlands?

$Bun_G(X)$ $\qquad$ L: $G^\vee$ local system on X

* Geometric Langlands conj: For every L $G^\vee$-local system, can assign K_L (perverse sheaf / $\mathcal{D}$-module) on $Bun_G(X)$ which is a Hecke eigensheaf with eigenvalue L.

Hecke condition:

$x \in X, \quad V \in \operatorname{Rep} G^\vee$

$$\begin{array}{c}\text{irred}\\\text{repns of } G^\vee\end{array} \longleftrightarrow G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})$$

$$\mathbb{Z} \longleftrightarrow \mathcal{K}^*/\mathcal{O}^*$$

$$\underbrace{H_{V,x}}_{} \circ K_L \xrightarrow[\cong]{} (L_x)_V \otimes K_L$$

some operator $Sh(Bun_G) \xrightarrow[H_V]{} Sh(Bun_G \times X)$ (here: derived category)

AJ
Weds 2 March '05.

Beilinson - Bernstein Localization

- method for building D-modules
- loosely connected to associated bdle constr. from diff'l geom.
- ref: Frenkel, BenZvi.

D_S - modules : S a scheme/$\mathbb{R}$ $\quad \mathcal{O}_S$ functions on S

Θ_S vect. fields on S

$\mathcal{D}_S$ = sheaf of diff. op's on S

= sheaf on algs. gen. by Θ_S, $\mathcal{O}_S$ w/ $[X, f] = X(f)$

$\mathcal{D}_S = U(\Theta_S)$

A $\mathcal{D}_S$ - module is a quasi-coherent $\mathcal{O}_S$ - module M
w/(left) action of $\mathcal{D}_S$: $\mathcal{D}_S \otimes_{\mathcal{O}_S} M \to M$.

Recall our mantra: $\mathcal{D}$-modules $\Leftrightarrow$ sheaves w/flat connection

Construct: $M \in$ R-module $\}$ problems: 1) trivial
$\mathcal{O}_S \otimes_R M$ $\quad$ 2) Fiber isn't M.

If $S = \operatorname{spec} A$, $\mathcal{O}_S \otimes_A M$ (M A-mod)

Obvious way to get $\mathcal{D}_S$-modules: $\mathcal{D}_S \otimes_R M$

problems with this: 1) trivial

2) HUGE $\quad 0^{th}$ approx. to B-B.

~~###~~ Localization (abelian):

$$S \quad \mathcal{O}_S\text{-module (quasi-coherent sheaves)} \underset{\Delta}{\overset{\text{global sections } \Gamma}{\rightleftarrows}} \text{Modules for } \Gamma(\mathcal{O}_S)$$

$$\mathcal{O}_S \otimes_{\Gamma(\mathcal{O}_S)} M \longleftrightarrow M$$

$$\operatorname{Hom}(\Delta(M), \mathcal{F}) = \operatorname{Hom}(M, \Gamma(\mathcal{F})).$$

Now 1^{st} approx in non-commutative setting:

1) M $\mathfrak{g}$-module

2) Z a space w/ transitive $\mathfrak{g}$-action (infinitesmal)

$$a: \mathfrak{g} \otimes \mathcal{O}_Z \twoheadrightarrow \Theta_Z \qquad \text{homo}^{\underline{m}} \text{ of Lie alg's}$$

surjective = transitive
isomorphism = simply transitive

$$\Rightarrow \mathcal{U}(\mathfrak{g}) \otimes \mathcal{O}_Z \twoheadrightarrow \mathcal{D}_Z$$

$$\widetilde{\Delta}(M) = \mathcal{D}_Z \otimes_{\mathcal{U}(\mathfrak{g})} M \qquad (\text{analogous to } P \times_G V \text{ assoc. bdle})$$

if a simple transitive: fiber $\simeq M$
if a transitive: fiber $\simeq M/\ker(a)$

Aside: Γ: $\mathcal{D}_Z$-modules $\rightarrow$ $\mathfrak{g}$-modules

(w/ good adjectives, can make this an iso$^{\underline{m}}$)
$\mathfrak{g}$ - semisimple
Z - flag variety of G

"The coolest theorem in representation theory" - Dave BZ
(Beilinson-Bernstein)

Inspiration: the associated bundle construction

$$\begin{array}{ll} Z & \text{a } K\text{-bundle} \qquad V \text{ a } K\text{-module} \\ \downarrow & \\ S & \end{array}$$

then $Z \times_K V$ a v. bundle on S w/ fiber V and str. gp K

Def^{n}: 1) A $(\mathfrak{g}, K)$-module is a $\mathfrak{g}$-module M w/ compatible K-action ($\mathrm{Lie}(K) \subset \mathfrak{g}$)

2) a $(\mathfrak{g}, K)$-structure on S is a principal K-bundle $\begin{array}{c} Z \\ \downarrow \pi \\ S \end{array}$ w/ a compatible transitive $\mathfrak{g}$-action on Z

B-B localization: $\Delta_Z : (\mathfrak{g}, K)\text{-module} \longrightarrow \mathcal{D}_S\text{-module}$

$$\Delta_Z(M) = \left(\pi_* (\mathcal{D}_Z \otimes_{\mathcal{U}(\mathfrak{g})} M) \right)^K$$

NOTE: This is some intermediary between the two extremes discussed so far: K = trivial, 1st approx.
K = everything, then just assoc. bdle construction

Example: 1) $Z = G$ $\quad$ $M = V$ a G-module.
$S = G/K$

$\Delta(M)$ = Vector bundle on G/K w/ fiber V and flat connection.

"algebraic version of Borel-Weil"

2) X = curve / k $\quad (k = \mathbb{C})$

$D = \operatorname{Spec} \mathbb{C}[[t]]$ $\quad D^\times = \operatorname{spec} \mathbb{C}((t))$

$\mathfrak{g} = \operatorname{Vect}(D^\times)$ $\quad K = \operatorname{Aut}(D^\times)$ = formal power series, non-zero leading coeff

$= \mathbb{C}((t))\partial_t$

$$\begin{array}{c} \operatorname{Aut}_X = \{ (x, z_x) \mid z_x \text{ a generator of } \mathcal{O}_{X,x} \cong \mathbb{C}[[z_x]] \} \\ \Big\downarrow \pi \\ X \end{array}$$

If V is a conformal vertex algebra, V is a $(\mathfrak{g}, K)$-module

$\Delta(V)$ is a sheaf of vertex algebras: $\partial_z + L_{-1}$

Another generalization: fix line bundle $\mathcal{L}$ on Z

$\mathcal{D}_\mathcal{L} = \mathcal{L} \otimes \mathcal{D}_Z \otimes \mathcal{L}^{-1}$ diff op's acting on $\mathcal{L}$

$\mathcal{D}_Z$ = diff. op's acting on $\mathcal{O}_Z$

$\mathcal{D}_{\mathcal{L}}$-modules = "twisted" $\mathcal{D}_Z$-modules

B-B works here as well: $\Delta_{\mathcal{L},Z}(M) = \left(\pi_*\left(\mathcal{D}_{\mathcal{L}} \otimes_{\mathcal{U}(\mathfrak{g})} M\right)\right)^K$

<u>Example</u>: $\mathfrak{g}$ Lie alg.

$$0 \to \mathbb{C}1 \to \hat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0$$

tensor w/ $\mathcal{O}_S$

$$0 \to \mathcal{O}_S \to \hat{\mathfrak{g}} \otimes \mathcal{O}_S \to \mathfrak{g}((t)) \otimes \mathcal{O}_S \to 0 \quad \text{splits over } a.$$

$\exists$ maps: $a : \mathfrak{g}((t)) \otimes \mathcal{O}_S \to \Theta_S$

$\hat{a} : \hat{\mathfrak{g}} \otimes \mathcal{O}_S \to \Theta_S$

We get: $0 \to \mathcal{O}_S \to \overset{\tau}{\overset{\parallel}{\hat{\mathfrak{g}} \otimes \mathcal{O}_S / \ker(a)}} \to \Theta_S \to 0$

(Analogue to $0 \to \mathcal{O}_S \to \mathcal{D}_S^{\leq 1} \to \Theta_S \to 0$)

$$\begin{array}{ccc} & \mathcal{D}_\tau = \mathcal{U}(\tau) & \\ {}^{i}\nearrow & & \nwarrow^{j} \\ \mathcal{O}_S & & \tau \\ \cup & & \cup \\ f & & x \end{array} \qquad [j(x), i(f)] = i(a(x) \cdot f)$$

Weds 2 March '05 (1)

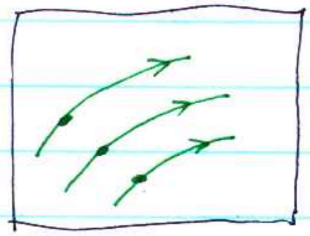
Alexei, Classical Hitchin systems

Hamiltonian dynamical systems:
- Phase space
- Poisson bracket
- Hamiltonian.

$\{ , \} : Fun_p \otimes Fun_p \to Fun_p$ satisfying Jacobi and Leibniz rule.

Example: $M = T^*\mathbb{R}^n = \mathbb{R}^{2n}$ (p_i, q_i)

$$\{p_i, q_j\} = \delta_{ij}$$
$$\{p_i, p_j\} = \{q_i, q_j\} = 0$$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Want to understand: $\boxed{\frac{d}{dt} F(x(t)) = \{H, F\}.}$

Example: $T^*\mathbb{R}^n$, $H = \sum p_i^2$

$$\frac{dq_i}{dt} = \{q_i, \textstyle\sum p_j^2\} = p_i$$

$$\frac{dp_i}{dt} = \{p_i, \textstyle\sum p_j^2\} = 0.$$

Integrable system: $I \in Fun_p$ s.t. $\{I, H\} = 0 \Rightarrow I$ integral of motion

$$\frac{dI}{dt} = 0.$$

Def^n: H dynamical system is (completely) integrable if $\exists I_i$ $i = 1, .., n$, which are functionally indep. $\{I_i, I_j\} = 0$, $\{I_i, H\} = 0$.

$$n = \frac{\dim P}{2}.$$

Example above: the p_i are integrals.

Geometry:

$$P \xrightarrow[\vec{I}]{} \mathbb{C}^n$$

$$I(p,q) = (I_1(p,q), \dots, I_n(p,q))$$

Th^m: Generic fiber of I is torus and evolution is linear motion on the torus. These tori are Lagrangian tori.

Remark: In many examples, these tori are Jacobians of curves.
(Primians ?)

Hitchin integrable system

Σ, genus $= g$. G complex semisimple Lie group

We will study Bun_G and build a Poisson str on it and an integrable system on it.

$\mathcal{A} = \{ A \in \Omega^{0,1}(\Sigma; \mathrm{Lie}(G)) \}$ $\quad A = A_{\bar{z}}(z, \bar{z}) d\bar{z}$

$\mathcal{G} \circlearrowright \mathcal{A}$ $\quad T^*_{\mathcal{A}} = \{ \Phi \in \Omega^{1,0}(\Sigma, \mathrm{Lie}(G) \}$

$\phi = X(z, \bar{z}) dz$

$\mathcal{G} = \mathrm{Maps}(\Sigma, G)$

Define $\langle \phi, X \rangle = \int_\Sigma \mathrm{Tr}(\phi_z X_{\bar{z}}) dz\, d\bar{z}$

$\{ \delta\phi_z, \delta A_{\bar z} \} = \int_\Sigma \mathrm{Tr}(\delta\phi_z \wedge \delta A_{\bar{z}})\, dz\, d\bar{z}$

$A_0 - A_1 = A'$: recall $g \cdot A' = g^{-1} A g$

Moment map: $\mu = \bar{\partial}_A \phi = \partial\phi + A \wedge \phi + \phi \wedge A : T^*\mathcal{A} \longrightarrow \mathfrak{g}^*$

$$P = \mu^{-1}(0) / \mathrm{Maps}(\Sigma, G)$$

Thm: (Narasimhan-Seshadri) $P = T^* Bun_G^{\,0}$ $\leftarrow$ open piece of $T^* Bun_G$

Alg-geom: $T^* Bun_G X$

$\swarrow$ gauge, twisted by P

$T_P Bun_G X = H^1(X, \mathfrak{g}_P)$ $\leftarrow$ infinitesmal

$T^*_P Bun_G X = H^0(X, \mathfrak{g}_P^* \otimes \Omega^1)$ Higgs fields.

$T^* Bun_G = \text{Higgs bundles}_G = (P, \phi)$

$\uparrow$ Higgs fields

Thm 2: $\dim P = 2 \dim Bun_G = 2(g-1) \dim G$.

Alex ②

canonical bdle

For $P \in \mathbb{C}[Lie(G)]^G$, given Higgs field $\phi \in H^0(X, Lie(G) \otimes \omega)$, $X = \Sigma$

$$P(\phi) := \sum \underbrace{P_i(\phi)}_{=\,H_{P,i}} \omega^i \in \bigoplus_{i=1}^{\deg P} H^0(\Sigma, \omega^i)$$

Thm: $\{H_{P,j}, H_{P,i}\} = 0$ and # integrals = dim Bun_G.

in Gl_n, $T^*Bun_n X = \{ \underset{\text{rk } n \text{ bundle}}{\mathcal{E}}, \ \phi$ holo^{m} mx of 1-forms on $\mathcal{E}$ $\ \operatorname{End} \mathcal{E} \otimes \Omega \}$

$$\downarrow$$

commuting Ham^{n}'s. $\{ tr\,\phi, tr\,\phi^2, \dots$

$\uparrow$

v. space of right dim^{n}! $\longrightarrow H^0(X, \Omega^1) \oplus H^0(X, \Omega^{\otimes 2}) \oplus \dots \oplus H^0(X, \Omega^{\otimes n})$

Now where do these Jacobians come from? $r = \text{rk bdle}$.

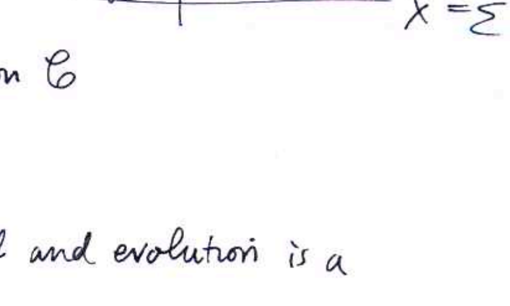
$$P = \underbrace{\det(\phi - E)}_{\text{char. poly.}} = (-1)^r E^r + tr(\phi)(-1)^{r-1} E^{r-1} + \dots$$

$P: K \longrightarrow K^{\otimes r} \supset X = \Sigma$ zero section

$P^{-1}(X) = \mathcal{C}$ generically r-fold cover of X

$\uparrow$

spectral curve



F = v. b. on Σ, $\longleftrightarrow$ line bundle L on $\mathcal{C}$

$$\pi_*(L) = F.$$

Proposition: Spectral curve C is an integral and evolution is a linear flow.

Weds 2 March '05.

David BenZvi – Part III

Sheaves $(\mathrm{Bun}_G(X))$ "harmonic analysis" for Hecke.

$\circlearrowleft$ Hecke operators

For each point $x \in X$, have an action of the category $(\underline{\mathrm{Rep}}\, G^\vee, \otimes)$ on derived cat. of sheaves $D(\mathrm{Bun}_G X)$

("everything gets categorified")

Recall: this morning: GL_1, $G = GL_1$, $G^\vee = GL_1$, "$\mathrm{Rep}\, G^\vee$" $\cong \mathbb{Z}$

irred rep'n of $GL_1 \leftarrow n$

$z \mapsto z^n \circlearrowright \mathbb{C}$.

what's the operator on sheaves on $\mathrm{Bun}_G(X)$?

$$\begin{array}{ccc} \mathrm{Pic} & \longrightarrow & \mathrm{Pic} \\ \mathcal{L} & \longmapsto & \mathcal{L}(nx) \end{array}$$

translating sheaves from component to component

and now want eigensheaves.

get an action of $\mathrm{Rep}\, G^\vee$ for each $x \in X$

$x \in X$

$V \in \mathrm{Rep}\, G^\vee$

V_x sheaf on $\mathcal{H}ecke_x$ (depends on x and V)

$$\begin{tikzcd} & \mathcal{H}ecke_x \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Bun}_G X & & \mathrm{Bun}_G X \end{tikzcd}$$

Another persp:

$$\begin{tikzcd} & \mathcal{H}ecke \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}\, G \times X \arrow[rr, dashed] & & \mathrm{Bun}_G X \end{tikzcd}$$

$\Rightarrow$ operator $Sh(\mathrm{Bun}_G X) \to Sh(\mathrm{Bun}_G X)$

ψ

$$H_{V,x}(\mathcal{F}) \cong \pi_{1*}(\pi_2^*(\mathcal{F}) \otimes V_x)$$

"kernel operator"

$\mathcal{F}$ is $\mathcal{D}$-module

Analogue: $\hat{f}(t) = \int e^{itx} f(x)\, dx$

$$\begin{tikzcd} & \mathbb{R} \times \mathbb{R} \arrow[dl] \arrow[dr] & \\ \mathbb{R} & & \mathbb{R} \end{tikzcd}$$

$\mathbb{R} \times \mathbb{R} \leftarrow e^{itx}$

Hecke eigensheaf condition: $H_{V,x}(\mathcal{F}) = (\mathcal{P})_V \otimes \mathcal{F}$

part of the claim: output is a $\mathcal{D}$-module – and .

$\mathcal{P} = G^\vee$-local system on X

$(\mathcal{P})_V$ assoc. v. bdle $\mathcal{P} \times_{G^\vee} V$ $\longrightarrow$ take fibers at x.

Operators $\{H_{V,x}\}$ defines action of $\mathrm{Rep}\, G^\vee$ on $D(\mathrm{Bun}_G X)$

$$H_{V_1,x} \circ H_{V_2,x} \xrightarrow{\simeq} H_{V_1 \otimes V_2, x}$$

$\mathcal{F} \in \mathcal{D}\text{-mod}(\mathrm{Bun}_G X)$ eigenfn w/ eigenvalue given by $\mathcal{P}$ a $G^\vee$-local system
(and Langlands will say that given $\mathcal{P}$ $\exists$ essentially unique $\mathcal{F}$)

Geometric Langlands :

For every $\mathcal{P}$, $\exists$ essentially unique Hecke eigensheaf on Bun_G with e'value $\mathcal{P}$.

note the interest is in the monodromy in this local system.

$$\mathcal{H}ecke_x = \{ \mathcal{E}_1, \mathcal{E}_2, \eta : \mathcal{E}_1|_{X \setminus x} \xrightarrow{\sim} \mathcal{E}_2|_{X \setminus x} \}$$

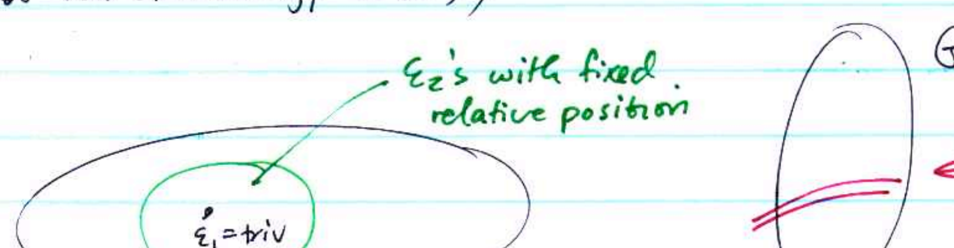
$$\begin{array}{ccc} & \mathcal{H}ecke_x & \\ \swarrow \pi_1 & & \searrow \pi_2 \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \\ \ni \mathcal{E}_1 & & \ni \mathcal{E}_2 \end{array}$$

"relative positions of bundles"
"modifications"

Example : $\mathcal{E}_1$ = trivial. Then $\eta : \left\{ \mathrm{triv}|_{X \setminus x} \longrightarrow \mathcal{E}_2|_{X \setminus x} \right\}$

$$= Gr_x$$

If $\mathcal{E}_1$ not trivial, get something iso$^{\underline{m}}$ to Gr_x, just twisted by $\mathcal{E}_1$.
So $\mathcal{H}ecke_x$ is bundle of fiber Gr_x over Bun_G with π_1
(and has structure gp $G(\mathcal{O})$)



Bun_G

$$F(\mathcal{E}_1) = \int F(\mathcal{E}_2)$$

analogue of harmonic analysis: taking

Satake.

V_x $\mathcal{D}\text{-mod}_{G(O)}(\mathcal{G}r) = \operatorname{Rep} G^\vee$]

Each fiber Gr comes with a stratification: recall Joel's talk

$$Gr_{GL_n} \supset Gr(k,n) \quad \text{orbit}_{\omega_k} \longleftrightarrow \Lambda^k \mathbb{C}^n \in \operatorname{Rep} GL_n$$

$\longleftrightarrow$ take a k-dim subspace fiber of trivial bundle at 0

Given a rank n v. bdle V on X

and $W \subset V|_x$ k-dim subspace

$\Rightarrow [V]_{W/x}$ rk n vector bundle whose sections = sections of $V,|_x \subset W$.

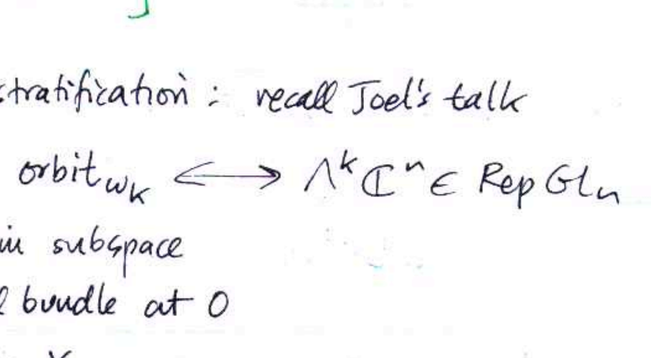
"elementary modification at x"

Picture: $\underbrace{Gr(k,n)}_{\text{appropriate subset of } Gr^{aff}} = Gr(k, V|_x) \xrightarrow{\pi_1}$

- and we're using the constant sheaf on $Gr(k,n)$

$\downarrow \pi_2$

$\bullet$ $V \in Bun_G$



just a small piece of the bigger diagram

$$H_{\Lambda^k \mathbb{C}^n, x} \cdot \mathcal{F}\Big|_V = \pi_{2*}(\pi_1^* \mathcal{F})$$

$\mathcal{H}ecke_x$

$Bun_G \swarrow \quad \searrow Bun_G$

Example: GL_1

(leading to GL_n: quantization of Hitchin system (Beilinson-Drinfeld))

we're trying to find $\mathcal{D}$-modules on Pic.

One strategy: $\mathcal{D}$ / cut it down (as in AJ's talk).

$\rightsquigarrow$ want: line bundles w/ flat connection

Classical Route

First: symbols of DO's ... $\Longleftrightarrow$ functions on T^* Jac.

$$T^*\mathrm{Jac} \cong \mathrm{Jac} \times H^0(X,\Omega) \quad !$$

$$\downarrow$$

$$H^0(X,\Omega)$$

$$\parallel$$

$$T^*\mathrm{Jac}_0$$

(fibers are tori !! completely int. system.

fibers all Jacobians. (Hitchin int. sys.) for GL_1

Now look at functions $\mathbb{C}[T^*\mathrm{Jac}]$ — not so many; all fns coming from $H^0(X,\Omega)$, i.e. components of the int. system.

$$\parallel$$

$$\mathbb{C}[H^0(X,\Omega)].$$

Quantize. go back to DO's, not just the symbols.

Are there honest DO's whose symbols are these ?

$$A = \Gamma_{\mathrm{Jac}}(\mathcal{D}) = \mathrm{Sym}\, \underbrace{T\mathrm{Jac}|_0} = \mathbb{C}[T^*\mathrm{Jac}_0]$$

abelian and projective; 1st-order DO's determined at one pt

so everything extends.

- A is a commutative algebra (because we're on torus)

POINT: Have an easy construction of $\mathcal{D}$-modules.

Take M an A-module. $\longmapsto$ $\underbrace{\mathcal{D}_{\mathrm{Jac}} \otimes_A M}_{\text{left } \mathcal{D}\text{-module}}$.

functor: A-mod $\to$ $\mathcal{D}$-mod.

Overkill! simple A-modules $\leftrightarrow$ pts of $\mathrm{Spec}\, A = H^0(X,\Omega)$.

Suppose $\eta \in H^0(X,\Omega)$. Then the assoc. $\mathcal{D}$-module K_η is

$$K_\eta = \mathcal{D} \otimes_A \underbrace{\mathbb{C}_\eta}_{\text{skyscraper at } \eta} = \mathcal{D}/(\partial - \overset{\in \mathbb{C}}{\eta(\partial)})$$

$$\partial \subset T(\mathrm{Jac})_0$$

Exercises: • K_η is a line bundle w/ flat connection

(b/c have a 1^{st}-order D.E.)

• K_η is the Hecke $\mathcal{D}$-module corresponding to ...

(fill in blank)

[There's a rank 1 local system hiding in the picture

$H^0(X, \Omega)$ = connections on the trivial line bundle]

$= d \pm \eta$ "GL_1 opers"

• In fact we've done it in families – given M A-modules produce corresponding $\mathcal{D}$, not just for η.

John Francis ①
Fourier-Mukai.

Outline

(a) Intro/dictionary, with calculus. "abelian harmonic analysis"
(b) Preliminaries on abelian varieties
(c) Derived categories of coherent sheaves
(d) Cite ref's.

Dictionary

abelian harmonic analysis	abelian alg. geom.
abel. topological gp	abelian varieties
— dual	— dual
(smooth) functions	(coherent) sheaves
L^2 functions	derived cat. of coherent sheaves
Diff'l operators	Functors between derived categories
	^ Derived
- kernel	- kernel sheaf
- exponential	- Poincaré sheaf
Pontrjagin duality	T-duality (Fourier-Mukai transform gives equivalence of derived categories)

string theorists would say: T-duality on branes.

Fourier transform

Isometry $L^2\mathbb{R} \cong L^2\mathbb{R}^*$, more generally $L^2 V \cong L^2 V^\vee$ (dual v.sp.)

Fourier transform: $\int_V e^{2\pi i \langle x,y\rangle} f(x)$

$\exp^{2\pi i \langle , \rangle}$ ← integral kernel

Think of this as
$$\begin{tikzcd} & V \times V^\vee \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ V & & V^\vee \end{tikzcd}$$

so rewrite as

" $\pi_{2*}(\pi_1^* f \cdot e^{2\pi i \langle , \rangle})$ "

Fourier-Mukai transform : A, $A^\vee$ abelian varieties ($A^\vee$ dual of A)

$$\begin{tikzcd} & A \times A^\vee \arrow[dl] \arrow[dr] & \\ A & & A^\vee \end{tikzcd}$$

$A \times A^\vee$ — Poincaré sheaf $\mathcal{P}$ (some analogue of $\exp^{2\pi i\langle , \rangle}$)

$$\mathcal{F} \longmapsto \pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{P})$$

$$\mathcal{D}_c^b(A) \xrightarrow{\sim} \mathcal{D}_c^b(A^\vee)$$ equivalence of bounded derived categories of coherent sheaves.

- exact
- exchanges products and convolutions, just like usual Fourier
- preserves triangulation str ?

Def : An abelian variety is a complete group scheme$/k$ (very strong condition)
a.k.a. a complex torus $\mathbb{C}^n/\Gamma$ Γ = lattice
w/ a pos-def hermitian form H on $\mathbb{C}^n$ s.t. $\mathrm{Im}(H)$ takes $\mathbb{Z}$-values on Γ

Rigidity: $X \times Y \to Z$ all abelian varieties
suppose $X \times \{pt\} \to Z$ Then $\exists$ factorization

$$\begin{tikzcd} X \times \{pt\} \arrow[r] \arrow[dr] & pt \\ & Y \arrow[u, dashed] \end{tikzcd}$$

"Theorem of Cube": Given bundle over $X \times Y \times Z$ abelian, it's determined by restrictions to X, Y, Z.

some culture... rigidity.

The dual $\widehat{A} := \mathrm{Pic}^0$, so topologically trivial bundles.

$$\begin{array}{c} P \\ \downarrow \\ A \times \widehat{A} \end{array}$$

Poincaré line bundle, the universal line bundle s.t. $P|_{A \times \{L\}} = L$.

Example $A = \mathbb{C}^n/\Gamma$, $\widehat{A} = \mathrm{Hom}(\Gamma, U(1))$ also a torus

Denote $\mathcal{P}$ = sheaf of sections of P.

Derived category of coherent sheaves

$\mathcal{A}$ abelian category : $\begin{cases} 0 \text{ object} \\ \text{Hom sets - abelian gps} \\ \oplus \text{ exists} \\ \#4 \text{ axiom : have homology exist and is unique} \end{cases}$

$\mathrm{Kom}(\mathcal{A})$ category of chain complexes w/ homotopy equivalences
(consider two elements in Hom set same if induce same map on homology) ? or chain homotopy ?

$D(\mathcal{A})$ - localization of $\mathrm{Kom}(\mathcal{A})$ w.r.t. quasi-isom.
(induce the identity on homology)

universal prop:

$$\begin{tikzcd} \mathrm{Kom}(\mathcal{A}) \arrow[r] \arrow[d] & D(\mathcal{A}) \arrow[dl, "!\varphi"] \\ \mathcal{B} & \end{tikzcd}$$

$\mathrm{Kom}(\mathcal{A}) \xrightarrow{f} \mathcal{B}$ if $f \in \mathrm{Kom}(\mathcal{A})$ induces an isom on homology, then $\exists f^{-1}$ in $\mathcal{B}$ then $\exists$ map (unique) $D(\mathcal{A}) \xrightarrow[\varphi]{} \mathcal{B}$.

Example : Derived category of $\mathcal{O}_X$-modules on $X \longrightarrow D(X)$.
in here : have full subcategories

$D_{qc}(X)$ - complexes of sheaves whose coh. sh. are quasi-coherent.

$D_c(X)$ - same, but coherent.

Also : $D^b_{qc}(X)$ - bounded — " —
(below and above)

Recall : $\{R\text{-mod}\} \longleftrightarrow \{\text{quasi-coh. sheaves on Spec } R\}$

$\{\text{fin. gen. } R\text{-mod}\} \longleftrightarrow \{\text{coherent — " —}\}$

Fourier - Mukai : X, Y varieties

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & Y \end{tikzcd}$$

$\mathcal{P}$ sheaf $/ X \times Y$

$\mathcal{F}$ a sheaf $\in D^b_c(X)$.

Define : $\Phi_{\mathcal{P}, X\to Y}(\mathcal{F}) = \pi_{2*}(\pi_1^* \mathcal{F} \otimes \mathcal{P})$

(subtleties about L, R, but ignore those)

("kernel sheaf" ~~of~~ $\mathcal{P}$)

NOTE : If $\mathcal{P}$ has finite Tor-dim, preserves subcategories (bounded, q-coh, coh, etc)

Theorem : For $\mathcal{P}$ the Poincaré sheaf on $A \times A^\vee$,

$* = b, +, -$ $$\Phi_{\mathcal{P}, A\to \check{A}} : D^*_{qc}(A) \longrightarrow D^*_{qc}(A^\vee)$$

is an equivalence of categories.

(if compose with $\Phi_{\mathcal{P}, \check{A}\to A}$, get "identity" up to minus sign and a shift in degree)

Thm (Orlov) If Φ is a fully faithful functor, $D^b_c(X) \longrightarrow D^b_c(Y)$ (X, Y proj, smooth, connected) then $\exists\, \mathcal{P} \in D^b_c(X\times Y)$ s.t. (! up to iso$\underline{\text{m}}$)

$$\Phi = \Phi_{\mathcal{P}, X\to Y}.$$

(Dave BZ: In fact should be able to drop some of these adjectives ... ?)
Also: think of this as analogue : any linear map between v. spaces given by a matrix ...

Thm : If ω_X or ω_X^{-1} is ample and if $D^b_c(X) \cong D^b_c(Y)$ then $X \cong Y$.

"if you have either pos or neg curvature, then you are completely determined by your (triangulated) derived category."

$\rightsquigarrow$ so the only place where you get a Fourier-Mukai transform is when it's "flat", e.g. tori [A, $\check{A}$] or Calabi-Yaus.

This time: Explain how $\left\{\begin{array}{l}\text{vertex alg's}\\ \text{B-B}\\ \text{Hitchin}\\ \text{Fourier-Mukai}\end{array}\right\}$ has to do w/ Langlands. David BZ. ①

Quantization of Hitchin's Hamiltonians

$$A = \Gamma(\mathcal{D}_{\mathrm{Jac}X}) \simeq \mathbb{C}[H^0(X,\Omega)] = \mathbb{C}[\text{Connections on trivial bundle}]$$

Given $d+\eta$ a connection, $d+\eta \longmapsto \mathcal{D}$-module on $\mathrm{Jac}(X)$

$$\mathcal{D}\otimes_A \{\text{eval. at } \eta\in H^0(X,\Omega)\} = \mathcal{D}/(\partial-\eta(\partial))$$

Actually, stronger: have functor

$$\begin{array}{ccc} \text{Coh. sheaves on } \{\mathcal{C}\mathrm{onn}_{triv}\} & = & A\text{-module } M \\ \downarrow & & \downarrow \\ \mathcal{D}_{\mathrm{Jac}}\text{-mod} & & \mathcal{D}\otimes_A M \end{array}$$

will be a Fourier transform (later today)

---

want to do this in non-abelian case: Beilinson-Drinfeld.

- Build $\mathcal{D}$-modules on Bun_G (recall: that's our goal in Langlands)

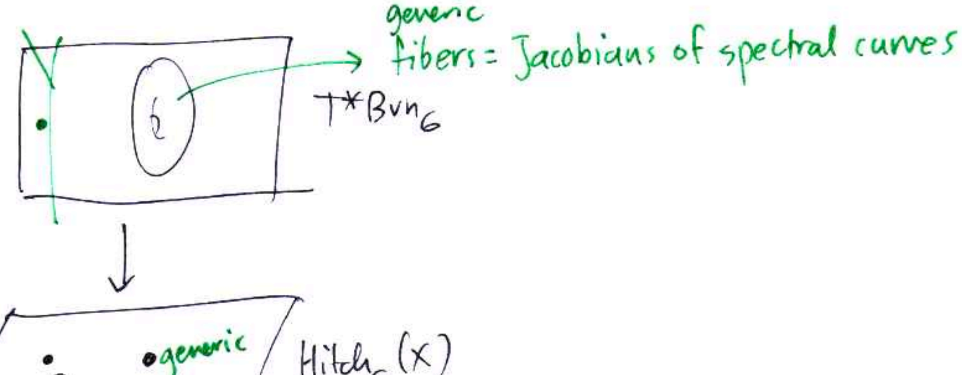
Symbols (assoc graded $\mathrm{gr}\mathcal{D}$) = functions on $T^*\mathrm{Bun}_G$

$\Gamma(\mathcal{O}_{T^*\mathrm{Bun}_G})$ and ask: are they symbols of global DO's?

$$\begin{array}{ccc} \underline{\text{Alex}}:\ T^*\mathrm{Bun}_G & = & \{\mathcal{E}\in \mathrm{Bun}_G,\ \phi\in \mathrm{End}\,\mathcal{E}\otimes\Omega\} \\ \downarrow & & \downarrow \\ \mathrm{Hitch}_G(X) := \bigoplus H^0(X,\Omega^{\otimes i}) & & \mathrm{tr}(\text{powers of } \phi) \end{array}$$

Hitchin's int. system

picture:



$\mathrm{Hitchin}^{-1}(0) = \text{Lagrangian} = \mathcal{E}, \eta \quad \eta$ nilpotent

nilpotent cone

$\cup$

Bun_G is an irred. component

IDEA: Bun_G is "approximated" by these Jacobian tori; have a chance to use this to build $\mathcal{D}$-modules on Bun_G.

Note. $\mathbb{C}[T^* Bun_G] = \underbrace{\mathbb{C}[\mathrm{Hitch}_G(X)]}^{= A_{classical}}$ (all functions come from base)

T^*Bun_G is stacky, but not so bad (abelian)

Coherent sheaf on T^*Bun_G are now easy to write down; functions are on base:

pick η $\quad \mathcal{O}_{T^*Bun_G} \underset{A_{classical}}{\otimes} \mathbb{C}_\eta$ $\quad$ "functions on the fiber" — just evaluate all Hitchin functions (also determines the fiber)

$= \mathcal{O}_{\mathrm{Hitch}^{-1}(\eta)}$

$\mathcal{D}$-module: non-commutative version of coherent sheaf on T^*Bun_G.

$\rightsquigarrow$ want to quantize the $\mathcal{O}_{\mathrm{Hitch}^{-1}(\eta)}$'s.

This is what Beilinson-Drinfeld do.

Quantization.

- $A' := \Gamma(\mathcal{D}_{Bun_G}) = \mathbb{C}$, so NO GOOD. Not enough! (for G = semisimple)
- More clever: $A = \Gamma(\underbrace{\mathcal{D}_{Bun_G}(\mathcal{L})}_{\text{what AJ called } \mathcal{D}_\lambda})$

$\mathcal{L}$ line bundle on Bun_G

"sheaf of twisted diff op's"

NOTE: $gr(\mathcal{D}(\mathcal{L})) = \mathcal{O}_{T^*Bun_G}$ $\quad$ symbols are the same! You get to assoc graded, don't see difference. [same symbols!]

<u>Thm</u>: $\Gamma(\mathcal{D}_{Bun_G}(\mathcal{L})) = \begin{cases} \mathbb{C} & \text{most of time} \\ A \text{ comm. alg. when } \mathcal{L} = \underbrace{\omega^{1/2}_{Bun G}}_{\text{"spin structure"}} \end{cases}$

(Beilinson-Drinfeld)

ω = top exterior power ...

For $GL(1)$, recall $A = \Gamma(\mathcal{D}_{Jac}) = \mathbb{C}[Conn_{triv}]$

For G = semisimple, want $A = \Gamma(\mathcal{D}_{Bun_G}(\mathcal{L})) = \mathbb{C}[Op_{G^\vee}]$

where $Op_{G^\vee}$ = <u>affine space</u> modelled on $Hitch_G X = \bigoplus H^0(X, \Omega^{\otimes i})$

"affine space of right size"

Note: only makes sense to talk about <u>filtration</u> (by order) on affine.

<u>Associated graded</u> will be $gr A = A_{cl} = \mathbb{C}[\bigoplus H^0(X, \Omega^{\otimes i})]$

Now: the black box $Op_{G^\vee}$:

$\rho \in Op_{G^\vee} \implies M'_\rho = \mathcal{D}_{Bun_G}(\mathcal{L}) \otimes_A \mathbb{C}_\rho$, a $\mathcal{D}(\mathcal{L})$-module

$\rightsquigarrow$

$M_\rho = M'_\rho \otimes \mathcal{L}^{-1}$ to get rid of twist

$\mathcal{D}(\mathcal{L})\text{-modules} \underset{\otimes \mathcal{L}}{\overset{\otimes \mathcal{L}^{-1}}{\rightleftarrows}} \mathcal{D}\text{-modules}$

(Think of $\mathcal{L}$ as crutch - we use it to construct the modules, but at the end of the day, they're the same.)

Elizabeth: this is the "same" as a <u>ρ-shift</u>. $\rightsquigarrow$ geometric analogue is $\omega^{1/2}$

$1/2$-forms have an inner product, so want them for unitary rep'ns....

$Op_{G^\vee} = G^\vee\text{-opers} \subset \underbrace{\mathcal{C}onn_{G^\vee} X}$ so a special kind of Connection

$\rho \in$ $G^\vee$- local systems on X

$\swarrow$ M_ρ on $Bun_G X$

$\downarrow$

$Bun_{G^\vee} X$

$G^\vee$-oper is a half-dim'l fiber of $Conn_{G^\vee} X \to Bun_{G^\vee} X$ (there's a special $G^\vee$-bundle of which one takes fiber ...)

(*) **Theorem:** M_ρ is a (irred) Hecke eigensheaf with e'val ρ.
(Beilinson-Drinfeld)

(so have geom. Langlands for 1/2-dim'l piece)

GL_n-oper:

in local coordinates: $\left(\partial^n - q_1 \partial^{n-1} \dots - q_n\right) f = 0$

$$\Longleftrightarrow \text{ linear algebra! } \left(\partial - \begin{pmatrix} q_1 & \cdots & & q_n \\ 1 & & & \\ & 1 & & 0 \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix}\right)\begin{pmatrix} f_1 \\ \vdots \\ \vdots \\ f^{(n-1)} \end{pmatrix} = 0$$

a connection on a trivial bundle!

"rational cyclic form"?

For general G: Kostant, Drinfeld-Sokolov

look at $\partial - \begin{pmatrix} 1 & & \star \\ & 1 & \\ 0 & & 1 \ddots \end{pmatrix} \Big/ N = \begin{pmatrix} 1 & \star \\ 0 & \ddots \; 1 \end{pmatrix}$ - gauge eq'ce.

1's on sub-diagonal

can translate to gen'l G: sub-diagonal has meaning: negative simple roots.

Def: GL_n oper on X curve is rank n vector bundle $\mathcal{E}$ and a flag $0 \subset \mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots \subset \mathcal{E}_n = \mathcal{E}$ of sub-bundles and ∇ a connection: $\mathcal{E} \longrightarrow \mathcal{E} \otimes \Omega^1$

Dave BZ ③

such that • $\nabla : \mathcal{E}_i \longrightarrow \mathcal{E}_{i+1} \otimes \Omega$ "0's below subdiagonal"

• $\nabla^{gr} : \mathcal{E}_i / \mathcal{E}_{i-1} \xrightarrow[\simeq]{} \mathcal{E}_{i+1}/\mathcal{E}_i \otimes \Omega$ "1's on subdiag"

Jacob: think of this as variation of Hodge structure
e.g. a nontrivial family of line bundles on Σ gives an oper ?

Example: SL_2 oper : same, and additionally $\det \mathcal{E} \cong \mathcal{O}$

Exercise: $\mathcal{E} = \begin{pmatrix} \mathcal{E}_1 \\ \cup \\ \mathcal{E}_2/\mathcal{E}_1 \end{pmatrix}$, get $\mathcal{E} = \begin{pmatrix} \Omega^{1/2} \\ \hline \Omega^{-1/2} \end{pmatrix}$ ← line bundle sub-bundle
← quotient of $\mathcal{E}$ wrt top.

comes up in Teichmuller theory: same as
projective structures on a Riemann surface

Where do these $\mathcal{D}$-modules come from? (idea of Beilinson-Drinfeld's big thm)

• $\hat{\mathfrak{g}}$ affine K-M Lie algebra

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathfrak{g}(\mathbb{C}((t))) \to 0.$$

B-B localization : $\mathcal{O} = \mathbb{C}[[t]]$

$$(\hat{\mathfrak{g}}, G(\mathcal{O}_x))\text{- modules} \leftarrow \text{purely } \underline{\text{local}} \text{ gadget}$$
$$\downarrow$$
$$\mathcal{D}_{Bun_G X} \quad (+ \text{ possible twist})$$

Recall: $x \in X$

$$\begin{array}{c} \widehat{Bun_G(X,x)} = \{ G\text{-bundles + trivialization near } x \} \\ \Big\downarrow G(\mathcal{O})\text{-prin. bdle} \\ Bun_G(X) \end{array}$$

$\hat{\mathfrak{g}}$

(think of $\hat{\mathfrak{g}}$ as change-of-coordinates and gluing near x — acts transitively)

Idea: M $(\hat{\mathfrak{g}}, G(\mathcal{O}_x))$-module.

$\longrightarrow \mathcal{D}_{\widehat{Bun_G}} \otimes_{U\hat{\mathfrak{g}}} M$ $G(\mathcal{O})$-equivariant,
so descends to $\Delta(M)$ on $Bun_G(X)$.

Example: $(\mathfrak{g}, K)$ in general $V_{\mathfrak{g},K}$ vacuum.

$$\Delta_Z(V_{\mathfrak{g},K}) = \mathcal{D}_Z$$

so in particular $(\hat{\mathfrak{g}}, G(O))$ $V_{\hat{\mathfrak{g}},G(O)}$ $\Delta_{Bun_G}(V_{\hat{\mathfrak{g}},G(O)}) = \mathcal{D}_{Bun_G}(\mathcal{L})$

twist assoc. to central extension

so can get $\mathcal{D}_{Bun_G}$ out of this construction as well

Thm (Feigin-Frenkel)

$$\underbrace{End_{\hat{\mathfrak{g}}}(V_{\hat{\mathfrak{g}},\mathfrak{g}(O)})}_{\text{this is a Hecke algebra: re: Elizabeth's talk.}} = Fun[Op_{G^\vee}(D)] = \mathbb{C}[Op_{G^\vee}(D)]$$

purely rep-theoretic result: now apply B-B localization.

Proof uses vertex algebras.

Mark Behrens (1)

Eichler-Shimura $G = GL_n$, $n=2$.

Basic purpose: take circle in Scott's talk, go around it for $GL(2)$, for number fields.

Recall:

Geometric Langlands (replace functions by sheaves) $\left\{\begin{matrix} GL_n \\ \text{local systems} \\ \text{on } X\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{Hecke} \\ \text{e'sheaves} \\ \text{on } Bun_G X\end{matrix}\right\}$

$\uparrow$

Function field Langlands $\{\pi_1(X) \to GL_n\} \leftrightarrow \left\{\text{Hecke e'fns on } \underbrace{G(k)\backslash G(\mathbb{A})/G(\mathcal{O})}_{\text{"Bun"}}\right\}$, $k = k(X)$

$\downarrow$

Number field Langlands

($\mathbb{Q}$) (replace curve with the mysterious curve $\mathrm{Spec}(\mathbb{Z})$)

$\left\{Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n\right\} \leftrightarrow \left\{\text{Hecke e'fns on } G(\mathbb{Q})\backslash G(\mathbb{A})/K\right\}$, K gp

generate $G(\mathcal{O})$-fixed vectors in $G(\mathbb{A})$-rep'ns ...

$x \in X$ $\leftrightarrow$ "place" p prime or ∞; $X \leftrightarrow \mathrm{Spec}\,\mathbb{Z}$

Start with thinking about $G(\mathbb{Q})\backslash G(\mathbb{A})/K$: is it "Bun" of anything?

function fields	Number fields
$G(k)\backslash G(\mathbb{A})/G(\mathcal{O})$	$G(\mathbb{Q})\backslash G(\mathbb{A})/K$ ← open compact part in non-Archimedean places ...
$\parallel$	$\parallel$
Bun_G	a space $Sh(G,K)$ "Shimura variety" (not always, but ok when $G=GL_n$, $n=2$)
$\parallel$	$\parallel$
Space of v. bdles $/X$	Space of abelian varities

$$\mathbb{A} = \prod_{v \in |\mathbb{Q}|}{}' \mathbb{Q}_v$$

$\downarrow$ Some analogue of space of v. bdles ... ?

(Case $n=1$: global class field theory)

Now $n=2$:

$G^+(\mathbb{A})$: $\det > 0$

$G^+(\mathbb{R}) = GL_2^+(\mathbb{R})$ ⟸ identify $\mathbb{R}^2$ w/ $\mathbb{C}$

$$G(\mathbb{Q})\backslash G(\mathbb{A})/K$$

$$= G^+(\mathbb{Q})\backslash G^+(\mathbb{A}) \Big/ \left(\prod_p G(\mathbb{Z}_p) \times GL_1(\mathbb{C})\right)$$

Recall: can pull a trick about trivializing ~~at~~ away from one pt

$$G(K)\backslash G(\mathbb{A})/G(\mathbb{O}) = \mathcal{L}_X G\backslash \mathcal{L}G/\mathcal{L}_+G$$

In this setting, a "strong approximation" allows to do something similar

$$G^+(\mathbb{Q})\backslash G^+(\mathbb{A})/\left(\prod_p G(\mathbb{Z}_p)\times GL_1(\mathbb{C})\right) = \Gamma\backslash GL_2^+(\mathbb{R})/GL_1(\mathbb{C})$$

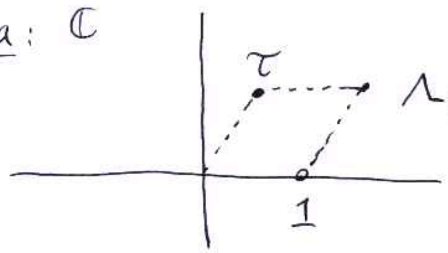
$$\left(\Gamma = SL_2(\mathbb{Z})\right)$$

$$= \Gamma\backslash \text{upper-half-plane} = \mathcal{H}$$

= space of elliptic curves !!

$= Y(1)$

Idea: $\mathbb{C}$



$\mathbb{C}/\Lambda_\tau = E_\tau$.

We'll take (as replacement of $\mathrm{Fun}(Y(1))$) $\left\{\begin{array}{l}\text{weight 2}\\ \text{modular forms}\end{array}\right\}$

$\Gamma_0(N) \subset \Gamma(1)$ $\qquad \Gamma_0(N)\backslash\mathcal{H} = \left\{\begin{array}{l}\text{space of elliptic}\\ \text{curves w/}\\ \text{"}\Gamma_0(N)\text{ structure"}\end{array}\right\}$

(E, H)

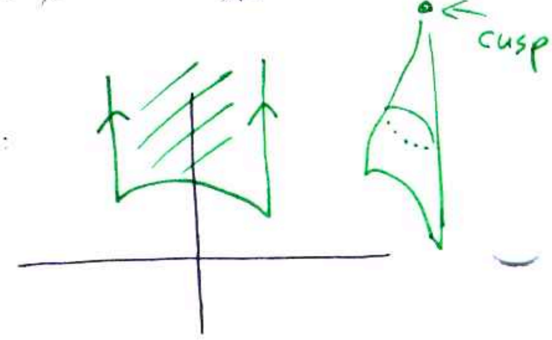
$H \leq E,\ H \cong \mathbb{Z}/N$.

NOTE: $Y_0(N)$ has "cusps"

$\rightsquigarrow$ need to compactify: get $\underbrace{X_0(N)}$

complex curve
– can have large
genus.

example:



Mark ②

Lying over $Y(1)$ is a line bundle :

$$\begin{tikzcd} \omega \arrow[d] & \omega_E = T^*_\varepsilon E \\ Y(1) \ni E & \end{tikzcd}$$

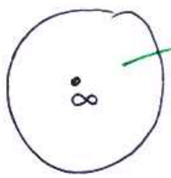
weight k modular forms $M_k(\Gamma) =$ sections $\Gamma(X(\Gamma), \omega^{\otimes k})$. $\quad \Gamma \subset \Gamma(1) \overset{\shortparallel}{} SL(2,\mathbb{Z})$

If view them on upper-half plane, can see them as certain kinds of fns.

"Cusp forms" $S_k(\Gamma) \subset M_k(\Gamma)$

$$\overset{\shortparallel}{\{ f \mid (\text{holom and})\ \underline{\text{vanishes}} \text{ at cusps} \}}$$

Near cusp at infinity,



— coordinate q near ∞

$$f = f(q) = \sum a_n q^n \quad \text{"}q\text{-expansion"}$$

(Didn't have to work over $\mathbb{C}$! Could have been over $\mathbb{Z}$, even.)

f cusp $\Rightarrow a_0 = 0$

f normalized cusp $\Rightarrow a_1 = 1$.

Now the Hecke algebras that act on them: $p \nmid N$.

$$\begin{tikzcd}[column sep=small] & X_0(pN) \arrow[dl, "s"'] \arrow[dr, "t"] & & V \cong \mathbb{Z}/p \\ X_0(N) & E, H\times V \arrow[dl, mapsto] \arrow[dr, mapsto] & X_0(p) & \\ (E,H) & & (E/V, \bar{H}) & \end{tikzcd}$$

Now pullback and pushforward, as before:

$$\begin{array}{ccccc} M_k(\Gamma_0(N)) & \xrightarrow{s^*} & M_k(\Gamma_0(pN)) & \xrightarrow{t_!} & M_k(\Gamma_0(N)) \\ \Psi & & & & \Psi \\ f & \longmapsto & & & T_p f \end{array}$$

Recall: $\mathcal{H}(GL_2(\mathbb{Q}_p), GL_2(\mathbb{Z}_p)) = \mathbb{C}[T^1, (T^2)^{\pm}] = \mathbb{C}[z_1, z_2]^W$

$\circlearrowright$

$M_k(\Gamma_0(N))$

(here $p \nmid N$)

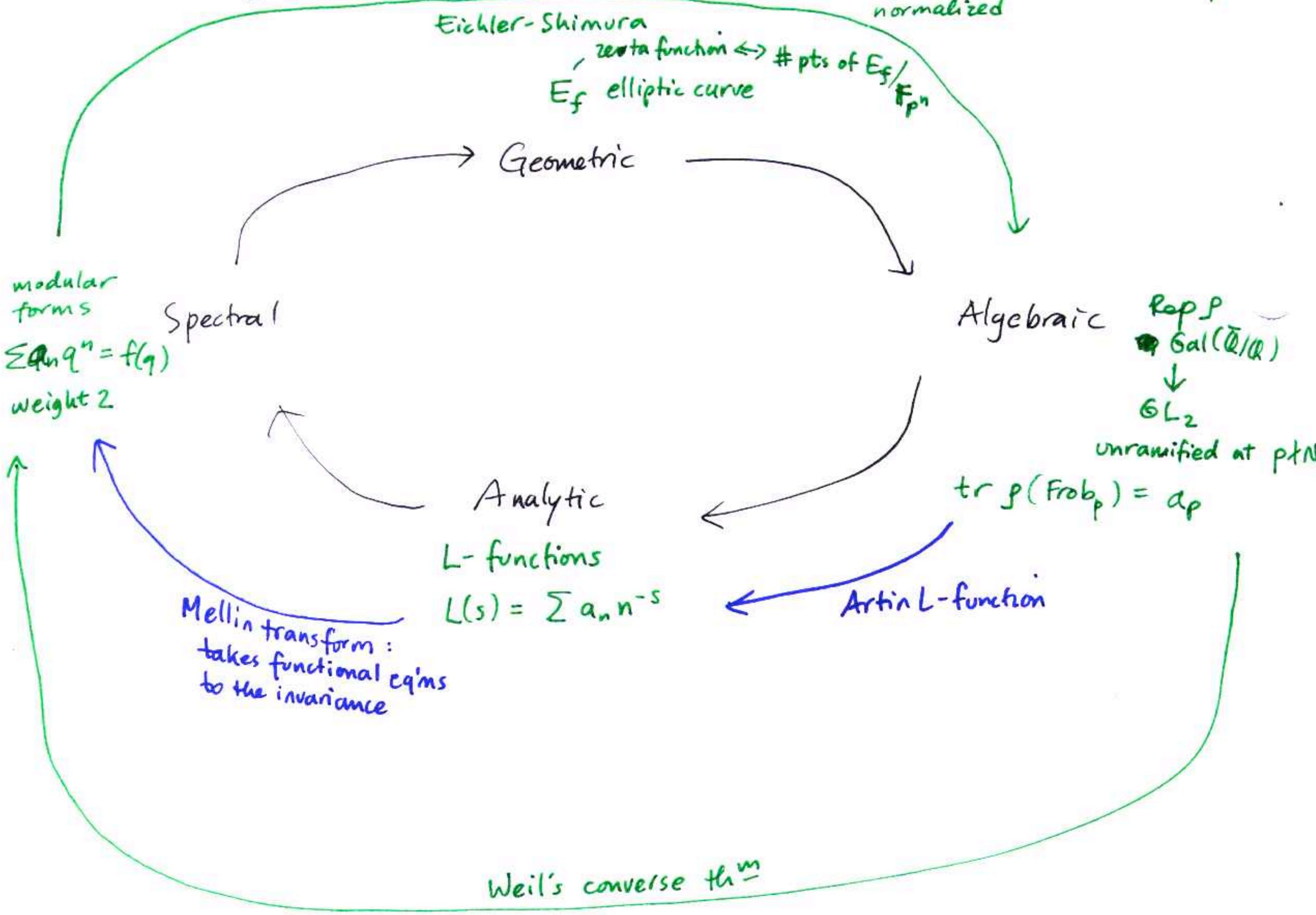
$T^1 \longleftrightarrow T_p$

$T^2 \longleftrightarrow R_p$ where $R_p f = p^{-k} f$

NOTE: $f \in S_k(\Gamma_0(N))$, $f(q) = \sum a_n q^n$

Assume f normalized Hecke e'sheaf, $T_p f = a_p f$

Now the diagram:



Mark (3)

The top half of the circle: constructing E_f.

$$J_0(N) = \mathrm{Jac}(X_0(N))$$

modify by $-\infty$? actually all cusps

Kodaira-Spencer: $\omega^{\otimes 2}_{(-\infty)} \cong \Omega^1_{X_0(N)}$

$$\underbrace{S_2(\Gamma_0(N))}_{\exists \text{ basis of Hecke e'fns.}} = \Gamma(\Omega^1_{X_0(N)})$$

$$= T_e^* J_0(N) \longrightarrow \mathbb{Q}(f)$$

kernel

Ψ f

1-dim'l

can realize this by a map $J_0(N) \longrightarrow E_f$

can view $\mathcal{H} \subset \mathrm{End}^\circ(J_0(N))$.

From $E_f \longrightarrow$ Rep ρ: Take $H^1(E_{f/\bar{\mathbb{Q}}}; \mathbb{Q}_\ell) = \mathbb{Q}_\ell^2$ $\circlearrowleft$ Get Galois gp action.

David B-Z ①
"Last talk ever!"

- Fourier-Mukai point of view on geom. Langlands ⟸
- What is a vertex algebra?

Dave's answer to what's the point of geom. Langlands is.
(Another answer: representation theory, Beilinson-Bernstein)
Also: take $X = \mathbb{P}^1$; geometry goes away and rep'n theory remains.

Various enhancements of Fourier-Mukai.

dual abel. varieties

$$\begin{array}{ccc} & A \times A^\vee \leftarrow \mathcal{P} \text{ Poincaré} & \\ \swarrow & & \searrow \\ A & & A^\vee = \mathrm{Pic}^0 A \ni [\mathcal{L}] \text{ line bundle on } A \end{array}$$

$$\underbrace{\mathbb{F}}_{\text{Fourier-Mukai}}\big(\underbrace{\mathcal{O}_{[\mathcal{L}]}}_{\text{skyscraper at } [\mathcal{L}]}\big) = \mathcal{L}$$

Normal Fourier transform: "any" "function" is an integral of exponentials
Fourier-Mukai: any sheaf $\in D(A)$ is a (direct) integral of line bundles

Enhanced F-M: any $\mathcal{D}$-module on A is a (direct) integral of line bundles w/ flat connection. (A = abelian variety)

i.e.

Thm: (Lamonde, Rothstein)

$$\boxed{D(\underbrace{\mathcal{D}_A\text{-mod}}_{}) \xleftrightarrow{\sim} D(\mathcal{O}_{A^\natural}\text{-module})}$$

take whatever flavor you want ... for your favorite analogue of harmonic analysis

$A^\natural$ = moduli of line bundles on A w/ flat connection

Mantra: "Geometric Langlands is harmonic analysis on Bun_G"

$$\begin{array}{l} A^\natural \leftarrow \text{affine, modelled on } T^*A^\vee \\ \downarrow \text{forget connection} \\ A^\vee \end{array}$$

M $\mathcal{D}$-module $\rightsquigarrow$ $F(M)$ is $\mathcal{O}_{A^\vee}$-module + action of functions on $A^\natural$

Another claim: $\mathcal{D}_{A^1}$-module $\xleftrightarrow{\simeq}$ $\mathcal{D}_{A^1}$-module (A^1 own dual)

$$\mathcal{D}_{A^1} = \mathbb{C}\langle x, \partial_x \rangle$$

* This is the usual Fourier transform!!

($\exists$ version of this that contains all of these as special case: Cartier duality)

Back to geometric Langlands:

$$A = \mathrm{Jac}\, X = A^\vee$$

$$\mathrm{Pic}^0(\mathrm{Jac}) \simeq \mathrm{Pic}^0 X = \mathrm{Jac}$$

$$\begin{array}{c} \uparrow \\ X \end{array}$$

Apply the above to this case:

$$D(\mathcal{D}_{\mathrm{Jac}}) \xleftrightarrow{\simeq} D(\mathcal{O}_{A^\natural}\text{-mod})$$

there are issues about how to topologize this space...

$$A^\natural = \mathrm{Conn}_{GL(1)} X \text{ ! (did this before)}$$

$$K_\rho \longleftarrow\!\!\shortmid \rho \quad \text{connection}_{GL(1)}$$

$\mathcal{D}$-module

think of it as skyscraper sheaf

Claim: This is geometric Langlands.

* But really, stronger! They span. All $\mathcal{D}$-modules can be written as a linear comb'n of Hecke eigenfns.

"Fourier decomposition of $\mathcal{D}$-modules"

Jacob: if want all of Pic, make it stacky, use gerbes.

Now: A completely symmetric formulation of geom. Langlands.

"A two-parameter family of theorems"....

$$\begin{tikzcd}
D_{coh}(T^*\mathrm{Jac}) \arrow[r, leftrightarrow, "\simeq", "\text{"classical"}"'] \arrow[d, squiggly, "\text{quantize}"] & D_{coh}(T^*\mathrm{Jac}) \arrow[d, squiggly, "\text{affinize}"] \\
D(\mathcal{D}_{\mathrm{Jac}}) =: D_{coh}(\underbrace{T^*_\hbar \mathrm{Jac}}_{\text{"non-comm. space"}}) \arrow[r, leftrightarrow, "\text{"Langlands"}"] & D_{coh}(\mathrm{Jac}^\natural)
\end{tikzcd}$$

usual F-M, just cross w/ vector space.

(so geometric Langlands has a "classical limit" !!)

Can go further and quantize $\mathrm{Jac}^\natural$:

$$\begin{tikzcd}
D(\mathcal{D}(\Theta^{1/k})) \arrow[r, leftrightarrow, "\text{"quantum Langlands"}"] \arrow[u, squiggly, "\text{quantize}"] & D(\mathcal{D}(\Theta^k)) \arrow[u, squiggly, "\text{quantize}"']
\end{tikzcd}$$

Θ is a nice line bundle on Jac: twist D.O.'s by Θ

$\Theta = \det \pi_* \mathcal{P} \Leftarrow$ Poincaré line bundle.

These are all theorems: we're in $GL(1)$.

Non-abelian version !? ... but how do we get started?

Use the abelian varieties that Alex gave us.

But first:

Conjecture: $D(\mathcal{D}_{\mathrm{Bun}_G}) \xleftrightarrow{\simeq} D(\mathcal{O}_{\mathrm{Conn}_{G^\vee}})$

$$\begin{array}{c} G^\vee\text{-bundle} \\ \text{w/ connection} \\ \text{on } X \end{array} = \mathrm{Conn}_{G^\vee} X \xrightarrow{\text{forget connection}} \mathrm{Bun}_{G^\vee} X$$

difference btw conn = Higgs field $\in H^0(X, \mathrm{End}\, \mathcal{E} \otimes \Omega^1)$

so $\begin{array}{c}\mathcal{C}onn_{G^\vee}X \\ \downarrow \\ Bun_{G^\vee}X\end{array}$ is an affine bundle modelled on $T^*Bun_{G^\vee}X = Higgs_{G^\vee}$

"twisted cotangent bundle"

What geom. Langlands said:

$$\rho \;\; G^\vee\text{-bdle w/conn} \longmapsto K_\rho \;\; \mathcal{D}\text{-module on } Bun_G .$$

Conjecture:

$$\underbrace{\mathcal{O}_{[\rho]}}_{\text{skyscraper}} \longmapsto K_\rho$$

and more: again, these form a basis.

Moreover $\exists$ ptwise action of $Rep\, G^\vee$ on both sides, equivalence equivariant.

("Convolution gets exchanged with tensor")

NOTE: We already wrote pieces of these: e.g.

- $A = \mathbb{C}[H^0(X,\Omega)] \longleftrightarrow \Gamma(\mathcal{D}_{Jac})$

$$\begin{array}{l} A\text{-mod} \\ = Coh(\underbrace{\mathcal{C}onn|_{GL_1,\, triv}}_{\text{fiber over triv. bdle}}) \end{array} \longmapsto \mathcal{D}_{Jac}\text{-mod}$$

This piece is the ~~[illegible]~~ harmonic analysis just for $\mathcal{D}$.

- Similarly $A = \mathbb{C}[Op_{G^\vee}] \longrightarrow \Gamma(\mathcal{D}_{Bun_G})$

Dave BZ ③

$$\begin{array}{ccc} T^*_\hbar \mathrm{Bun}_G & & \mathcal{C}onn_{G^\vee} \\ \downarrow & & \downarrow \\ \mathrm{Bun}_G & & \mathrm{Bun}_{G^\vee} \\ \rightsquigarrow \hbar\to 0 & & \rightsquigarrow \\ T^*\mathrm{Bun}_G & & T^*\mathrm{Bun}_{G^\vee} \\ & \searrow \quad \swarrow & \\ & \mathrm{Hitch} = \bigoplus H^0(X, \Omega^{\otimes i}) & \end{array}$$

← coherent sheaves on this is $\mathcal{D}$-modules.

← this is a stack: actually some fibers "empty" since don't admit connections but not really

$\mathcal{A}$ family of abelian varieties

$\mathcal{A}^\vee$ dual family of abel. varieties

invt polynomials of G, $G^\vee$ same (identify $\mathfrak{g}$, $\mathfrak{g}^*$)

(Hausel-Thaddeus ....)

Arinkin: $D_{coh}(T^*\overset{\circ}{\mathrm{Bun}}_G) \xleftrightarrow{\cong} D_{coh}(T^*\overset{\circ}{\mathrm{Bun}}_{G^\vee})$

Fourier-Mukai, fiber with fiber.

$D(\mathcal{D}_\hbar\text{-mod}) \longleftrightarrow D(\mathcal{O}_{\mathcal{C}onn_{G^\vee}}\text{-mod})$

"asymptotic" version where $\hbar$ formal variable ... can see germ

Has cool physics interpretation:

Fourier-Mukai is T-duality

$\rightsquigarrow$

get S-duality, geom. Langlands is some shadow ... (!)

David BZ ①
Vertex Algebras

What *is* a Vertex Algebra?

Out of this we'll get a conjectural candidate for: "Poincaré sheaf"

↳ the integral kernel for the F-M transform ....

$$\mathcal{D} \circlearrowright \mathcal{P} \circlearrowleft \mathcal{O}$$

$$\begin{array}{c} \mathcal{P} \\ \downarrow \\ Bun_G \times Conn_{G^\vee} \\ \swarrow \qquad \searrow \\ Bun_G \qquad\qquad Conn_{G^\vee} \end{array}$$

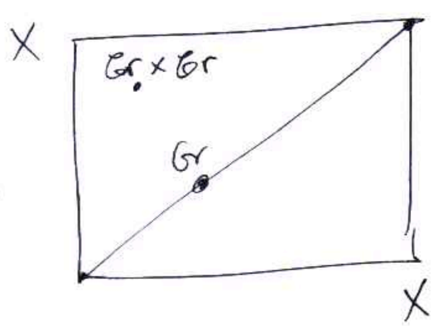
makes (conjecturally) geom. Langlands actually a functor.

... all from just thinking about vertex algebras "correctly". !!

Recall Joel's talk: fix G.

$$Gr^{(2)} = \{ \mathcal{P}\ G\text{-bundle},\ x_1, x_2 \in X,\ \text{triv. of } \mathcal{P}|_{X \setminus \{x_1, x_2\}} \}$$
$$\downarrow$$
$$X \times X$$

$$Gr_x \hookrightarrow Gr^{(1)} = \{ \mathcal{P}\ G\text{-bundle},\ x \in X,\ \text{triv. of } \mathcal{P}|_{X \setminus \{x\}} \}$$
$$Gr_x = G(K)/G(O), \qquad Gr^{(1)} \to X$$



$$Gr^{(2)}\Big|_{x_1 \neq x_2} \xrightarrow{\cong} Gr^{(1)}_{x_1} \times Gr^{(2)}_{x_2}$$

$$Gr^{(2)}\Big|_{x_1 = x_2} \xrightarrow{\cong} Gr_{x_1}$$

Keep going! $Gr^{(n)} = \{ P\ G\text{-bundle},\ x_1, \dots, x_n,\ \text{triv of } \mathcal{P}|_{X \setminus \{x_1 \dots x_n\}} \}$
$$\downarrow$$
$$X^n$$

Then $Gr^{(n)}_{x_1 \dots x_n}$ depends only on the subset $\{x_1, \dots, x_n\} \subset X$

doesn't care about order or multiplicity.

$$Gr_{\{x_i\} \amalg \{y_j\}} \xrightarrow{\cong} Gr_{\{x_i\}} \times Gr_{\{y_j\}}$$

"the local data don't interact"

- Factorization space

$$\mathcal{G} \qquad x_1, \dots, x_n \in X \longmapsto \mathcal{G}_{x_1, \dots, x_n} \quad \text{space}$$

with the following structure

- $\mathcal{G}_{x_1, \dots, x_n}$ ~~depends only on~~ independent of multiplicities

$$\mathcal{G}_{xxy} \xrightarrow{\simeq} \mathcal{G}_{xy}$$

- $$\mathcal{G}_{\{x_i\} \amalg \{y_j\}} \xrightarrow{\simeq} \mathcal{G}_{\{x_i\}} \times \mathcal{G}_{\{y_j\}}$$

but want them to form some nice bundle also.

$$\begin{tikzcd} \Delta^* Gr^{(n)} \cong Gr^{(m)} \arrow[d] & Gr^{(n)} \arrow[d] \\ X^m \arrow[r, "\Delta"] & X^n \end{tikzcd}$$

$x \longmapsto (x, x)$ "partial diagonals" (can be permutations also)

(so they fit together nicely)

actually contractible

Formalize this: $\mathrm{Ran}(X)$ = space of finite subsets of X

$$\text{"}\exp X \setminus pt\text{"} = \lim_{\text{all } \Delta\text{'s}} X^n.$$

not algebraic in any sense, (no functions, even locally) YUCK!

so then Factorization Space is: a space $\mathcal{G}$

$$\left.\begin{matrix} \mathcal{G} \\ \downarrow \\ \mathrm{Ran}(X). \end{matrix}\right\} \text{ will be a "flat family"}$$

- $\mathrm{Ran}(X)$ is a semigroup under $\amalg$.
- $\mathcal{G}$ is "multiplicative under $\amalg$", ie $\mathcal{G}_{\dots} = \mathcal{G} \times \mathcal{G}$

Dave BZ ②
Vertex alg's.

<u>Def</u>: A <u>factorization algebra</u> $\mathcal{V}$

$\mathcal{V}^{(n)}$ quasi-coherent sheaf, satisfying all same properties
$\downarrow$ except multiplication becomes $\otimes$
X^n

$$\mathcal{V}_{\{x_i\} \amalg \{y_j\}} = \mathcal{V}_{\{x_i\}} \otimes \mathcal{V}_{\{y_j\}}$$

[Another way to say: $\mathcal{V} \downarrow \mathrm{Ran}(X)$ a quasi-coh. sheaf ... but $\mathrm{Ran}(X)$ not algebraic, so really just means $\uparrow$ above defn.]

<u>Example</u>: $\mathcal{V}^{(n)} = \mathcal{O}_{X^n}$ [corresponds to <u>vacuum</u> vertex algebra]
$\mathcal{O} :=$

<u>Def</u>: unital fact. algebra: also have inclusion $\mathcal{O} \hookrightarrow \mathcal{V}$, compatible w/all str's.
"unit"

Theorem: (Beilinson-Drinfeld) Unital factorization algebras (here X is a <u>curve</u>?)
on disc D = Spec $\mathbb{C}[[z]]$ disc of radius 0
this version is called a <u>chiral</u> alg's

$\updownarrow$

Vertex algebras.

(*) A "def'n" of vertex algebra with no formulas!
⇝ and can have <u>non-linear analogues</u>! e.g. affine Grass.

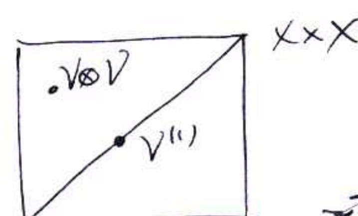
here even have analogue of unit - triv. bdle, trivialized trivially

<u>Claim</u>: the "∂" in Reimundo's defn also just comes out of the geometry.

* We're going to <u>build</u> vertex algebras ... one of these (the "chiral Hecke algebra") will give us that $\mathcal{P}$.

<u>Idea of above Thm</u>: $\mathcal{V} = \mathcal{V}^{(1)}$ q-c sheaf on X

$\mathcal{V}^{(2)}$



$j: X \times X \setminus \Delta \hookrightarrow X \times X$

polar parts $\to 0$ of $\mathcal{V}^{(2)}$ along Δ

(need some flatness along diagonal)

$$\mathcal{O} \to \mathcal{V}^{(2)} \hookrightarrow \mathcal{V} \otimes \mathcal{V}, \text{ poles along } \Delta$$

polar parts $= \Delta_+ \Delta^* \mathcal{V}^{(2)}$. (standard sheaf theory ...)

$= \underbrace{\Delta_+}_{\text{"}\mathcal{D}\text{-module push forward"}} \mathcal{V}$ because $\mathcal{V}^{(2)}$ on Δ is $\mathcal{V}$

$= \mathcal{V} \boxtimes \mathcal{O}(\infty\Delta) / \mathcal{V} \boxtimes \mathcal{O}$ (Note: $\mathcal{V}$ is actually a $\mathcal{D}$-module ...)

= delta-fns on Δ valued in $\mathcal{V}$

Giving $\mathcal{V}^{(2)} \Longleftrightarrow$ giving the map $\underbrace{\mathcal{V} \boxtimes \mathcal{V}(\infty\Delta) \to \Delta_+ \mathcal{V}}$

in local coörd's, this is the Y in Reimundo's talk.

Local coörd's: $X = D \quad \mathcal{V} = D \times V$
$A, B \in V \ni |0\rangle =$ unit in $\mathcal{V}^{(1)}|_0$

$\mu : A \boxtimes B \longrightarrow Y(A,z)B$ mod regular things

$\mathcal{V} \boxtimes \mathcal{V}(\infty\Delta) \supset A \boxtimes B$

$Y(A,z)\cdot B \in V((z))$, $z = x-y$, transverse to diagonal (x, y coörds on $D_x \times D_y$)

Converse: Given a vertex algebra, write this down, take kernel $= \mathcal{V}^{(2)}$.
Claim: everything works for higher $\mathcal{V}^{(n)}$!

Geometric Langlands: using geom. ~~Langlands~~ Satake

From factorization space, build factorization algebra, has magical properties: Cohomology of it gives the "$\mathcal{P}$". Can show get Hecke e'sheaves, etc.

---> Just need to compute and show image is non-zero ... but this isn't done!

How do you linearize the factorization space $\mathcal{G}r$?

Recall: $\underbrace{\mathcal{D}_{G(0)}(Gr)}_{G(0)\text{-eqvt } \mathcal{D}\text{-modules on } Gr} \simeq \mathrm{Rep}\, G^\vee$

particular ones we like: $\boxed{\text{trivial rep} = \mathbb{C}} \in \mathrm{Rep}\, G^\vee \overset{\text{Satake}}{\longmapsto} M_{\mathbb{C}}$ $\mathcal{D}$-module on Gr.

Claim: $\Gamma_{Gr}(M_{\mathbb{C}}) = V_{\hat{\mathfrak{g}}, G(0)}$

$\| \quad \mathrm{Ind}_{\mathfrak{g}(0)}^{\hat{\mathfrak{g}}} \mathbb{C}$

Another way: δ-fns at point

$= U\hat{\mathfrak{g}} \otimes_{U\mathfrak{g}(0)} \mathbb{C}$

$\sim \mathrm{Sym}(\underbrace{\mathfrak{g}(K)/\mathfrak{g}(0)}_{\text{tgt space of } Gr})$

$\left(= V_k(\mathfrak{g}) \text{ in Reimundo's talk} \right)$

is a vertex algebra.

[We're using that $\mathbb{C}$ is a ring object in $\mathrm{Rep}\, G^\vee$?]

The other rep we like: $\boxed{\text{regular rep'n}} = \bigoplus_{\text{ireps of } G^\vee} V \otimes V^* = \mathbb{C}[G^\vee]$

$\downarrow$ Satake

M_{reg} $\mathcal{D}$-module on Gr

direct sum of stuff supported on pieces — using perverse sheaves,

$= \bigoplus IC_\lambda \otimes V_\lambda^*$

$\cup$

$M_{\mathbb{C}}$.

What we like: (about regular rep's)

- $\mathbb{C}[G^\vee]$ ring object in $G^\vee$-rep's.
- $\mathbb{C}[G^\vee]$ is a $G^\vee$-rep in $\mathrm{Rep}\, G^\vee$ (left and right action)

$\rightsquigarrow$ so M_{reg} carries an action of $G^\vee$.

"is the most canonical object that exists in the world"

$\Rightarrow \mathbb{H} := \Gamma(M_{reg})$

"... if you were stranded on a desert island and thought pure thoughts about geom. Langlands ... you'd come up with this object ... well, if you're Beilinson" — David.

- $G^\vee$-representation $\approx$ "$U\hat{\mathfrak{g}}$"
- $\hat{\mathfrak{g}}$-rep $\mathbb{H} \supset V_{\hat{\mathfrak{g}}, G(0)}$
- $\mathbb{H}$ factorization algebra: because used a ring object

combines the infinitesmal action of $\hat{\mathfrak{g}}$ and the global str of the Satake.

used in the gluing data

Case: $GL(1)$. Gr looks like $\mathbb{Z}$: $\vdots$

$\mathbb{H}_{GL(1)} =$ free fermions/bosons ψ, ψ^* "lattice vertex algebra assoc. to $\mathbb{Z}$"
α

carries action of $GL_1^\vee = GL_1$, given by grading.

"This is a machine that eats a $G^\vee$-local system and spits out $\mathcal{D}$-modules"

ρ $G^\vee$ local system on X $\Rightarrow$ take associated v. bdle $\mathbb{H}_\rho$,

factorization algebra on X

($G^\vee$ acts by automorphisms)

$\left(\mathbb{H}_\rho \supset V_{\hat{\mathfrak{g}}, G(0)}\right) \circlearrowleft \hat{\mathfrak{g}}$.

Apply B-B localization: take cohomology of $\mathbb{H}_\rho$ along X (slightly lying: need conformal blocks)

$\Rightarrow \mathcal{D}$-module on Bun_G. or: $H_{dR}(\mathrm{Ran}\, X, (\mathbb{H})_\rho)$.

The promised Poincaré sheaf:

$\rho_{G^\vee}$ local system $\Longrightarrow$ v.a. ${}_{G(O)}(H)_{\rho_{G^\vee}}$ on X

$\mathcal{E}_G$ G bundle $\Longrightarrow$ ${}_{\mathcal{E}_G}(H)_{\rho_{G^\vee}}$

then fiber of Poincaré sheaf at that point is $H_{dR}(\mathrm{Ran}\,X, {}_{\mathcal{E}_G}(H)_{\rho_{G^\vee}})$.