COMBINATORIAL INTERPRETATION OF HECKE ALGEBRA TRACES

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Outline

(1) The Hecke algebra and its traces
(2) The chromatic symmetric and quasisymmetric functions
(3) Zig-zag networks
(4) Formula for evaluating induced trivial characters
(5) Conjectured formula for evaluating power sum traces

The Hecke algebra $H_n(q)$

Generators over $\mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$: $T_{s_1}, \ldots, T_{s_{n-1}}$.

Relations:

$$\begin{aligned} T_{s_i}^2 &= (q-1)T_{s_i} + qT_e && \text{for } i = 1, \ldots, n-1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j} && \text{for } |i-j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i} && \text{for } |i-j| \geq 2. \end{aligned}$$

Natural basis: $\{T_w \,|\, w \in S_n\}$,

$$T_w = T_{s_{i_1}} \cdots T_{s_{i_\ell}}, \qquad (w = s_{i_1} \cdots s_{i_\ell} \text{ reduced}).$$

Kazhdan-Lusztig basis: $\{C'_w \,|\, w \in S_n\}$,

$$C'_w = q^{\frac{\ell(w)}{2}} \sum_{v \leq w} P_{v,w}(q) T_v.$$

Call $\theta_q : H_n(q) \to \mathbb{C}[q^{\frac{1}{2}}, q^{\frac{-1}{2}}]$ a *trace* if $\theta_q(gh) = \theta_q(hg)$ $\forall g, h$.

Examples of Hecke algebra traces

Irreducible characters: $\{\chi_q^\lambda \mid \lambda \vdash n\}$.

Induced trivial, sign characters: $\{\eta_q^\lambda \mid \lambda \vdash n\}$, $\{\epsilon_q^\lambda \mid \lambda \vdash n\}$.

Monomial, power sum traces: $\{\phi_q^\lambda \mid \lambda \vdash n\}$, $\{\psi_q^\lambda \mid \lambda \vdash n\}$.

$$\eta_q^\lambda = \sum_\mu K_{\mu,\lambda} \chi_q^\mu, \quad \epsilon_q^\lambda = \sum_\mu K_{\mu^\top,\lambda} \chi_q^\mu, \quad \phi_q^\lambda = \sum_\mu K_{\lambda,\mu}^{-1} \chi_q^\mu,$$

$$\psi_q^\lambda = \sum_\mu L_{\lambda,\mu} \phi_q^\mu.$$

$$h_\lambda = \sum_\mu K_{\mu,\lambda} s_\mu, \quad e_\lambda = \sum_\mu K_{\mu^\top,\lambda} s_\mu, \quad m_\lambda = \sum_\mu K_{\lambda,\mu}^{-1} s_\mu,$$

$$p_\lambda = \sum_\mu L_{\lambda,\mu} m_\mu.$$

Formulas for trace evaluations

θ_q	$\theta_q(T_w)$ in $\mathbb{N}[q]$?	interpretation of $\theta_q(T_w)$ as $\sum_k (-1)^{\lvert S_k \rvert} \lvert R_k \rvert q^k$?	$\theta_q(q^{\frac{\ell(w)}{2}} C'_w)$ in $\mathbb{N}[q]$?	interpretation of $\theta_q(q^{\frac{\ell(w)}{2}} C'_w)$ as $\sum_k \lvert R_k \rvert q^k$ for w avoiding 3412, 4231?
η_q^λ	no	open	H '92	CHSS '13
ϵ_q^λ	no	open	H '92	CHSS '12
χ_q^λ	no	open	H '92	CHSS '13
ψ_q^λ	no	open	(conj. H '92)	conj. CHSS '13
ϕ_q^λ	no	open	conj. H '92	open

Connection to chromatic symmetric functions

Stanley ('95) associated to each poset P a *chromatic symmetric function* X_P.

P an n-element unit interval order $\implies \exists w \in \mathfrak{S}_n$ avoiding 312 s.t.

$$X_P = \sum_{\lambda \vdash n} \epsilon^{\lambda}(C'_w(1)) m_\lambda = \sum_{\lambda \vdash n} \eta^{\lambda}(C'_w(1)) f_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(C'_w(1)) s_\lambda$$

$$= \sum_{\lambda \vdash n} \frac{\psi^{\lambda}(C'_w(1))}{(-1)^{n-\ell(\lambda)}} \frac{p_\lambda}{z_\lambda} = \sum_{\lambda \vdash n} \phi^{\lambda}(C'_w(1)) e_\lambda.$$

Conj: (SS '93) For P a unit interval order, $X_P \in \mathrm{span}_{\mathbb{N}}\{e_\lambda \,|\, \lambda \vdash n\}$. (c.f. H '92.)

Connection to chromatic quasisymmetric functions

Shareshian, Wachs ('12) introduced a quasisymmetric q-analog $X_{P,q}$ of X_P.

Thm: (CHSS '13) For P an appropriately labeled unit interval order, $\exists w \in \mathfrak{S}_n$ avoiding 312 s.t.

$$X_{P,q} = \sum_{\lambda \vdash n} \epsilon_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w) m_\lambda = \sum_{\lambda \vdash n} \eta_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w) f_\lambda = \sum_{\lambda \vdash n} \chi_q^{\lambda^\top}(q^{\frac{\ell(w)}{2}} C'_w) s_\lambda$$

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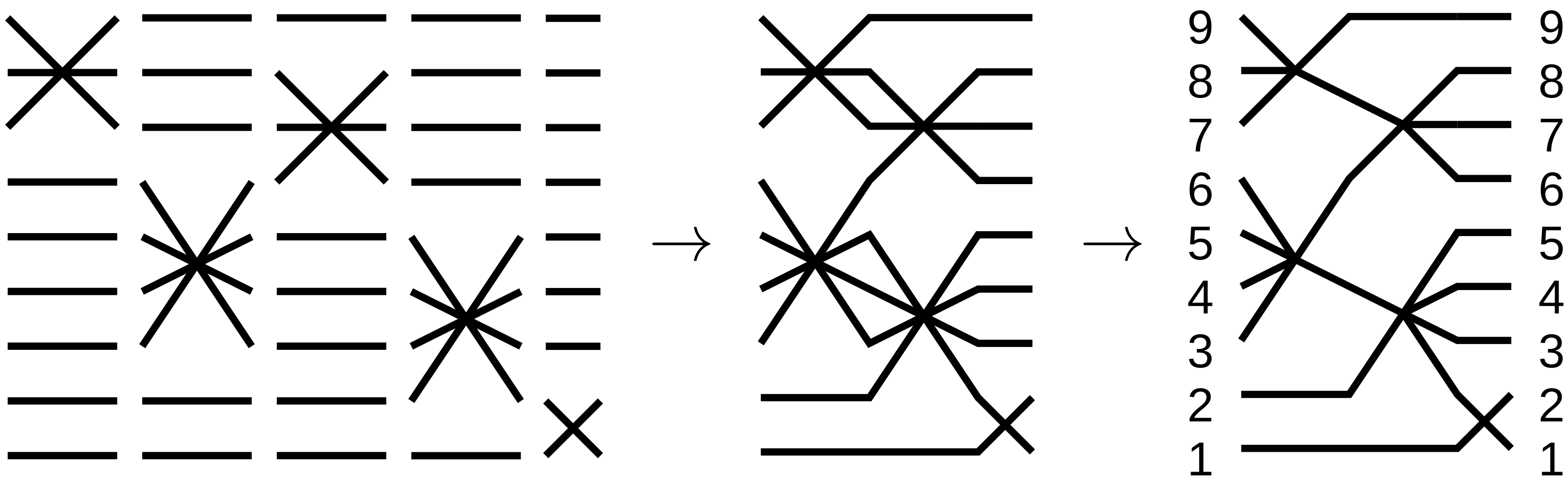
Conj: (SW '12) For P an appropriately labeled unit interval order, $X_{P,q} \in \mathrm{span}_{\mathbb{N}[q]}\{e_\lambda \,|\, \lambda \vdash n\}$. (c.f. H '92.)

Zig-zag networks

Thm: (S '08) For $w \in \mathfrak{S}_n$ avoiding $3412, 4231$, the element $C'_w \in H_n(q)$ can be encoded by a *zig-zag network of order* n.

Ex: $C'_{258431976}$ is encoded by the zig-zag network (c.f. BW '01)

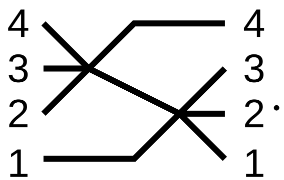


Call vertices on left *sources*, vertices on right *sinks*.

Special case: $w \in \mathfrak{S}_n$ avoids 312.

Path families and F-tableaux

Call a sequence of paths from sources $1, \ldots, n$ to sinks $w_1, \ldots, w_n$ a *path family of type* w.

Example: $F =$ 

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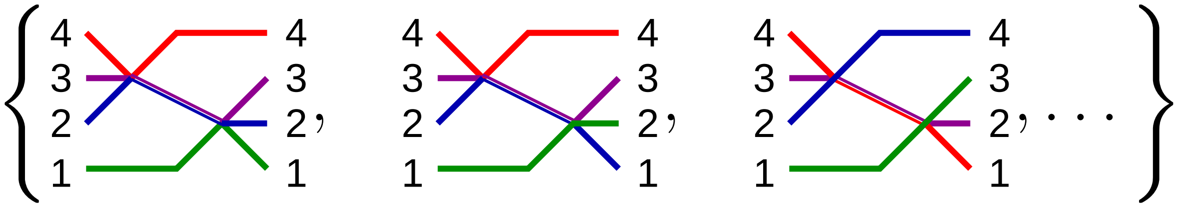
path families: 

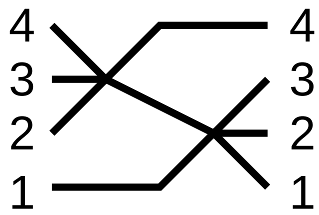
$$\text{types} : \left\{ \begin{pmatrix} 1234 \\ 1234 \end{pmatrix}, \quad \begin{pmatrix} 1234 \\ 2134 \end{pmatrix}, \quad \begin{pmatrix} 1234 \\ 3421 \end{pmatrix}, \ldots \right\}.$$

There is at most one path from source i to sink j. Call it (i, j).

Define an *F-tableau* of shape λ to be a placement of a path family $((1, w_1), \ldots, (n, w_n))$ into a Young diagram of shape λ.

Row-closed and left row-strict tableaux

Call an F-tableau *row-closed* if in each row, the sets of source and sink indices are equal. Call it *left row-strict* if source indices increase to the right.

Ex: Two such tableaux of shape 31 for $F =$ 

are

3,3		
1,2	2,4	4,1

,

4,4		
1,3	2,1	3,2

.

Let U_k denote row k of tableau U. Let $\circ$ denote concatenation. For each tableau U above, $U_1 \circ U_2$ is equal to

1,2	2,4	4,1	3,3

,

1,3	2,1	3,2	4,4

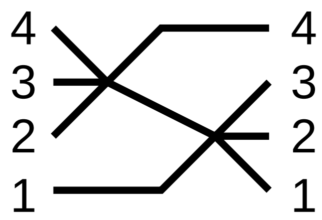
.

Right inversions in F-tableaux

Call intersecting paths (i, w_i) and (j, w_j) a *right inversion* in an F-tableau if $w_i > w_j$ and (i, w_i) appears earlier than (j, w_j).

Let $\textsc{rinv}(U)$ denote the number of right inversions in U.

The previous tableaux satisfy $\textsc{rinv}(U_1 \circ U_2) = 3, 2$, respectively:

$F =$ 

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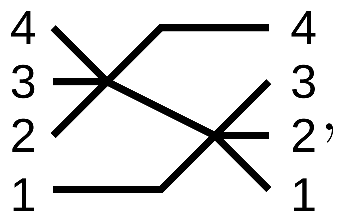
$F =$ 

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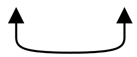
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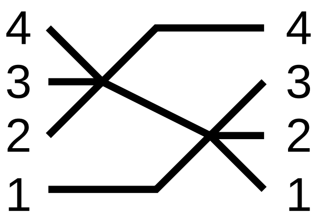
$F =$ 

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1,3	2,1	3,2	4,4

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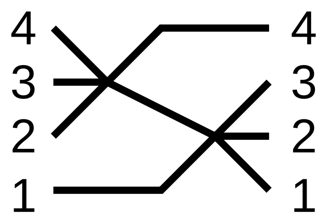

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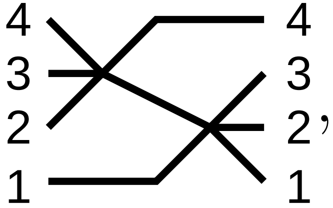
$F =$ 

1,2	2,4	4,1	3,3

,

1,3	2,1	3,2	4,4

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Induced trivial characters

Thm: (CHSS '13) For w avoiding 3412, 4231, we have

$$\eta_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w) = \sum_U q^{\mathrm{RINV}(U_1 \circ \cdots \circ U_r)},$$

where the sum is over all row-closed, left row-strict F-tableaux of shape $\lambda = (\lambda_1, \ldots, \lambda_r)$, and F corresponds to w.

Ex: For previous network F, we have $w = 3421$ and

$$\eta_q^{4}(q^{\frac{\ell(w)}{2}} C'_w) = 1 + 3q + 5q^2 + 5q^3 + 3q^4 + q^5,$$

$$\eta_q^{31}(q^{\frac{\ell(w)}{2}} C'_w) = 1 + 3q + 6q^2 + 6q^3 + 3q^4 + q^5,$$

$$\eta_q^{22}(q^{\frac{\ell(w)}{2}} C'_w) = 1 + 3q + 6q^2 + 6q^3 + 3q^4 + q^5,$$

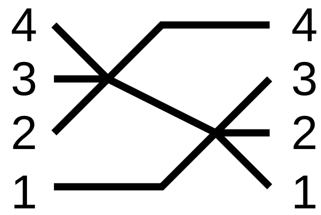
$$\eta_q^{211}(q^{\frac{\ell(w)}{2}} C'_w) = 1 + 3q + 7q^2 + 7q^3 + 3q^4 + q^5,$$

$$\eta_q^{1111}(q^{\frac{\ell(w)}{2}} C'_w) = 1 + 3q + 8q^2 + 8q^3 + 3q^4 + q^5.$$

Cylindrical tableaux

Call an F-tableau *cylindrical* if each row has the form

$$(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1).$$

Example: Two such tableaux of shape 31 for $F =$ 

are

3,3		
1,2	2,4	4,1

,

4,4		
2,3	3,1	1,2

.

For each tableau U above, $U_2 \circ U_1$ is equal to

3,3	1,2	2,4	4,1

,

4,4	2,3	3,1	1,2

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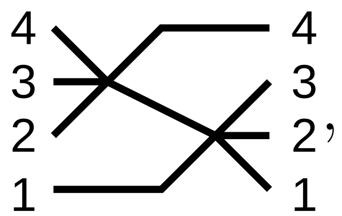
Inversions in F-tableaux

Call intersecting paths (i, w_i) and (j, w_j) a *(left) inversion* in an F-tableau if $i > j$ and (i, w_i) appears earlier than (j, w_j).

Let $\textsc{inv}(U)$ denote the number of inversions in U.

The previous tableaux satisfy $\textsc{inv}(U_2 \circ U_1) = 2, 4$, respectively:

$F =$ 

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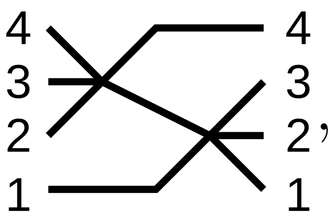

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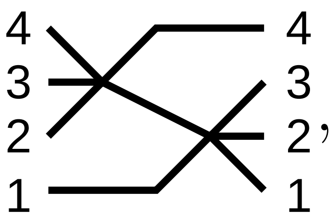
$F =$ 

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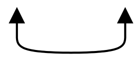

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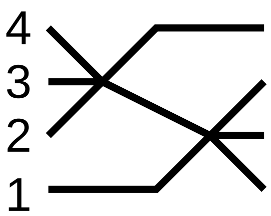
$F =$ 

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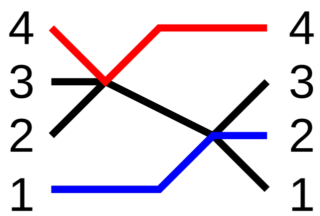
$F =$ 

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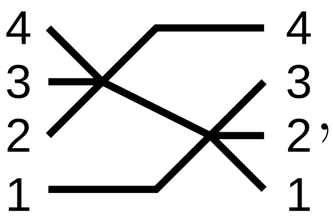
$F =$ 

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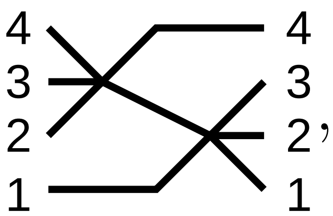
$F =$ 

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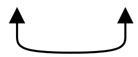

Power sum traces

Conj: (CHSS '13) For w avoiding 3412, 4231, we have

$$\psi_q^{\lambda}(q^{\frac{\ell(w)}{2}} C'_w) = \sum_U q^{\mathrm{INV}(U_r \circ \cdots \circ U_1)},$$

where F corresponds to w, and the sum is over all cylindrical F-tableaux of shape $\lambda = (\lambda_1, \ldots, \lambda_r)$.

Ex: For previous network F, we have $w = 3421$ and

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Open questions

Shareshian-Wachs conjectured interpretations of coefficients of p_λ in $X_{P,q}$. By main theorem, these should be equal to the conjectured evaluations of ψ_q^λ.

Question: Are the conjectures equivalent?

Question: Is there a similar combinatorial interpretation of $\phi_q^\lambda(q^{\frac{\ell(w)}{2}} C'_w)$ when w avoids 3412, 4231?

Conj: (SS '93) We have $\phi^\lambda(C'_w(1)) \geq 0$ when w avoids 3412, 4231.

Conj: (H '92) We have $\phi^\lambda(q^{\frac{\ell(w)}{2}} C'_w) \in \mathbb{N}[q]$ for all w.

Conj: (SW '12) We have $\phi^\lambda(q^{\frac{\ell(w)}{2}} C'_w) \in \mathbb{N}[q]$ when w avoids 3412, 4231.