Growth of ideals in subword posets
Notes for talk at conference Stanley@70, June 26, 2014
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1. Combinatorics

Let X be a finite set.
$X^\star = \{w = w_1 w_2 \cdots w_n \mid w_i \in X\}$ words on X.
Poset structures $\le$ on $X^\star$:

(1) $w \le_I w'$ if w is a subword of w', i.e., there exists a function $f\colon [\ell(w)] \to [\ell(w')]$ with $f(1) < f(2) < \cdots < f(\ell(w))$ and $w_i = w'_{f(i)}$.
(2) $w \le_{II} w'$ if $w \le_I w'$ and furthermore, we can choose f as above such that for all j, there exists $f(i) \le j$ such that $w'_{f(i)} = w'_j$.

Example 1.1. $X = [2]$. $112 \le_I 1212$ but $112 \not\le_{II} 1212$. □

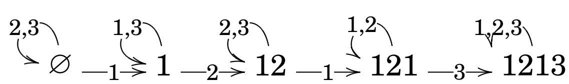
$I \subseteq X^\star$ is an **ideal** if $x \in I$ and $y \ge x$ implies $y \in I$.
Hilbert series of I:

$$\mathrm{H}_I(t) = \sum_{w \in I} t^{\ell(w)}$$

Proposition 1.2. *In both cases,* $\mathrm{H}_I(t)$ *is a rational function* $f(t)/g(t)$ *where* $g(t) = \prod_i (1 - a_i t)$ *and* $a_i \in \{1, 2, \dots, |X|\}$.

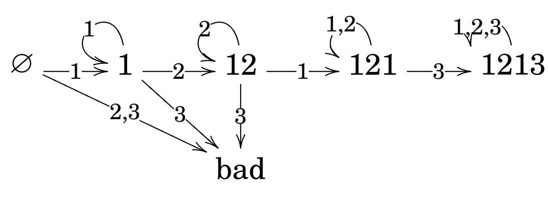
Proof idea:

- Generating function of regular language is rational: translate the elements in an ideal into walks in directed graph and use transfer-matrix method (roots of denominator are inverses of eigenvalues of adjacency matrix)
- For principal ideal $I_w = \{w' \ge w\}$, use graph which "tracks progress."
  Example: case 1, $X = [3]$ and $w = 1213$:



  The labeled walks from ∅ to 1213 are exactly those words that contain 1213 as a subword under $\le_I$.

  For case 2, we introduce a sink "bad" which shows the difference between the two definitions:



  Still, labeled walks from ∅ to 1213 give all words that contain 1213 as a subword under $\le_{II}$.

  Eigenvalues are integers of desired form.

  For finite unions of principal ideals, can use similar ideas (but notation is more cumbersome).
- $X^\star$ has no infinite antichains, so every ideal is a finite union of principal ideals.
  Case 1: Higman's lemma
  Case 2: we prove

2. QUESTIONS

Poset $\Pi = \bigcup_{n\geq 0} S_n$ (S_n is nth symmetric group) of permutations: $\sigma \leq \sigma'$ if σ is a subword of σ' (when rewritten in relative order). This is definition used in pattern avoidance (σ' contains σ as a subpattern).

Poset $\mathscr{M} \subset \Pi$ of perfect matchings (i.e., fixed-point free involutions).

What can be said about Hilbert series of ideals in these posets? There are infinite antichains in $\mathscr{M}$ (and hence Π), so might want to restrict to the finitely generated ones (i.e., union of finitely many principal ideals). Might expect them to be D-finite, but should have more refined statements.

3. ALGEBRA

Motivation: (algebraic structures for) infinite-dimensional combinatorial commutative algebra (Segre, Veronese varieties, etc.)

A surj-**module** is graded vector space $M = \bigoplus_{n\geq 0} M_n$ such that for every surjective function $f\colon [n] \to [m]$ have operator $M(f)\colon M_m \to M_n$ (opposite!) and $M(f \circ g) = M(g) \circ M(f)$.

Submodule is graded subspace closed under all operations.

M is **finitely generated** if there exists finite $S \subset M$ such that smallest submodule containing S is M.

Hilbert series of M:

$$\mathrm{H}_M(t) = \sum_{n\geq 0} \dim_{\mathbf{k}}(M_n) t^n.$$

Proposition 3.1. *If M is finitely generated in degree $\leq D$, then $\mathrm{H}_M(t)$ is a rational function $f(t)/g(t)$ where $g(t) = \prod_i (1 - a_i t)$ and $a_i \in \{1, 2, \ldots, D\}$.*

Proof idea:

- Define Gröbner basis theory for surj-modules to reduce to “monomial modules”
- Monomial modules are identified with ideals in $(X^\star, \leq_{II})$ with $|X| \leq D$

Remark 3.2.

- Analogous theory for $(X^\star, \leq_I)$ and (decorated) injective functions
- q-analogues of surj-modules with sets replaced by $\mathbf{F}_q$-vector spaces (related to study of unstable modules over Steenrod algebra) □