Lattice-point counting and cyclic sieving,

or,

Ehrhart theory and cyclic sieving: two great tastes that taste great together?

James Propp

June 24, 2014

Slides at http://jamespropp.org/polytope-csp.pdf

Ehrhart theory

Given a polytope Π in $\mathbb{R}^d$ with vertices in $\mathbb{Z}^d$, there is a polynomial P such that the number of lattice points in the dilated polytope $N\Pi$ is $P(N)$ for all non-negative integers N.

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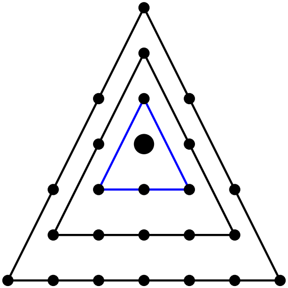
E.g., for Π in $\mathbb{R}^2$ with vertices $(-1,-1)$, $(0,1)$, and $(1,-1)$, we have $\#(N\Pi \cap \mathbb{Z}^2) = 2N^2 + 2N + 1$ for all $N \geq 0$:



What if the vertices aren't lattice points?

For polytopes with vertices in $\mathbb{Q}^d$, we need quasipolynomials; that is, powers of roots of unity get involved.

E.g., if Π is the polytope in $\mathbb{R}^1$ with vertices $-\frac{1}{2}$ and $\frac{1}{2}$, we have $\#(N\Pi \cap \mathbb{Z}^2) = N + \frac{1}{2} + \frac{1}{2}(-1)^N$ for all $N \geq 0$.

Cyclic sieving

Given a set S and a map $L : S \to S$ satisfying $L^n = \mathrm{Id}_S$ giving rise to an action of the cyclic group $\mathbb{Z}/n\mathbb{Z}$ on S, there is (in many cases) a "natural" polynomial $p(\cdot)$ such that the number of fixed-points of L^k is $|p(\zeta^k)|$ for all integers k (where ζ is a primitive nth root of unity).

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More specifically, we often have $\#\mathrm{Fix}(L^k) = |p(\zeta^k)|$ with

$$p(t) = \sum_{s \in S} t^{f(s)},$$

where $f : S \to \mathbb{Z}$ is some function that reflects the structure of S and L.

Why does this make sense?

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For general k, we have

$$|p(\zeta^k)| = |\sum_{s \in S} \zeta^{kf(s)}| \leq \sum_{s \in S} |\zeta^{kf(s)}| = |S|.$$

What am I trying to do?

I've been trying to combine Ehrhart Theory and cyclic sieving, letting S be $N\Pi$ with L some linear map from $\mathbb{R}^d$ to itself (restricted to $N\Pi \cap \mathbb{Z}^d$).

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Motivation: Let's see what kind of geometrical situations lead to cyclic sieving.

Then we can go back to combinatorial contexts and see if that kind of geometry is latent there.

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Richard's role in founding the Cambridge Combinatorics Coffee Club

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Here as in many other Ehrhart-ish situations, one should instead STACK the dilated polytopes to form a polyhedral cone in a space with 1 extra dimension.

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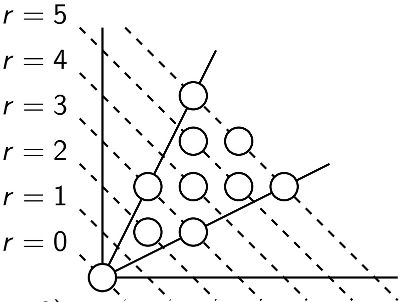
a linear function r from $\mathbb{Z}^d$ to $\mathbb{Z}$ (a "rank function") whose level sets $\{x : x \in C_{\mathbb{Z}},\ r(x) = N\}$ are empty when $N < 0$ and finite otherwise;
we say a function f from $C_{\mathbb{Z}}$ to $\mathbb{Z}$ is sieving when for all integers k and for all non-negative integers N, the number of $x \in C_{\mathbb{Z}}$ with $r(x) = N$ and $L^k x = x$ equals $\left|\sum_{x \in C_{\mathbb{Z}},\ r(x)=N} \zeta^{kf(s)}\right|$, where ζ is a primitive nth root of 1.

A two-dimensional example

Example 1: Take $C = \langle (2,1),(1,2) \rangle$, $L : (x,y) \mapsto (y,x)$, $n = 2$, $r(i,j) = i + j$.



Total points ($k = 0$):	1	0	1	2	1	2	3
Fixed points ($k = 1$):	1	0	1	0	1	0	1

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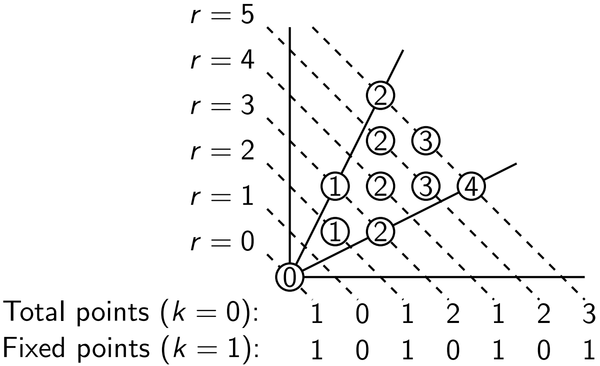

Then $f(i,j) = i$ is sieving.

Some higher-dimensional examples you can try at home

Example 2: Take $C = \langle (1,0,0),(0,1,0),(0,0,1) \rangle$, $L : (x,y,z) \mapsto (y,z,x)$, $n = 3$, $r(i,j,k) = i + j + k$.

Then $f(i,j,k) = j + 2k$ is sieving. (To generalize to higher dimensions, write this as $0i + 1j + 2k$. This example encodes the prototypical CSP for the $\mathbb{Z}/(a+b)\mathbb{Z}$ rotation action on a-element subsets of an $a+b$-element set.)

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Example 3: Take $C = \langle (1,0,1),(-1,0,1),(0,1,1),(0,-1,1) \rangle$,
$L : (x,y,z) \mapsto (y,-x,z)$, $n = 4$, $r(i,j,k) = k$.

Then $f(i,j,k) = i + 2j$ is sieving.

My old conjecture

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I found a counterexample: Let $C_{\text{bad}} \subset \mathbb{R}^3$ be the cone generated by lattice vectors $(2, 1, 1)$, $(-1, 2, 1)$, $(-2, -1, 1)$, and $(1, -2, 1)$, with rank function $r(i, j, k) = k$, and let L be 90-degree rotation counterclockwise about the z-axis.

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Case analysis shows that no linear sieving function f exists.

But: We can find a sieving function f that, while not linear, comes close.

My new conjecture

Conjecture: Given C, L, and r as above, there exists a sieving function f such that for each generating vector v_i, there is a constant c_i such that $f(x + v_i) - f(x) = c_i$ for all x in $C_{\mathbb{Z}}$.

E.g., returning to Example 1 (with $v_1 = (2, 1)$ and $v_2 = (1, 2)$) the following f would qualify (with $c_1 = 1$ and $c_2 = 0$):

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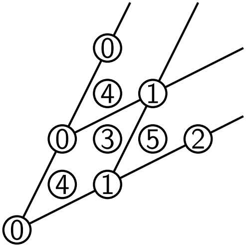

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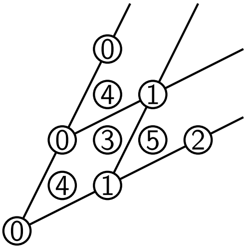

C_{bad} is not a counterexample to my new conjecture; computer search finds many suitable f's (too many!).

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