h-polynomials of triangulations of flow polytopes

Karola Mészáros

(Cornell University)

h-polynomials of triangulations of flow polytopes

(and of reduction trees)

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Reduction trees and reduced forms

Reduced forms generalize h-polynomials of triangulations

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Flow polytopes

$\mathcal{F}_{K_5}(1, 0, 0, 0, -1)$

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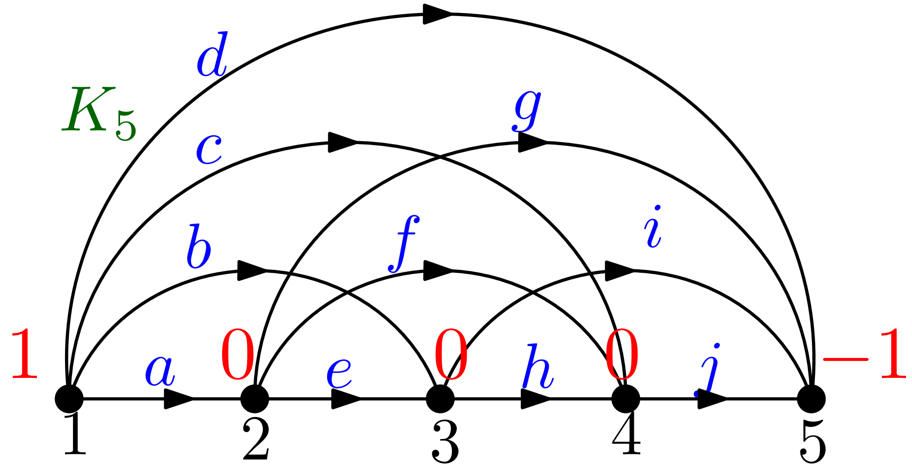
$\mathcal{F}_{K_5}(1,0,0,0,-1)$

$$1 = a+b+c+d$$
$$0 = e+f+g-a$$
$$0 = h+i-b-e$$
$$0 = j-c-f-h$$



$$a,b,c,d,e,f,g,h,i,j \geq 0$$

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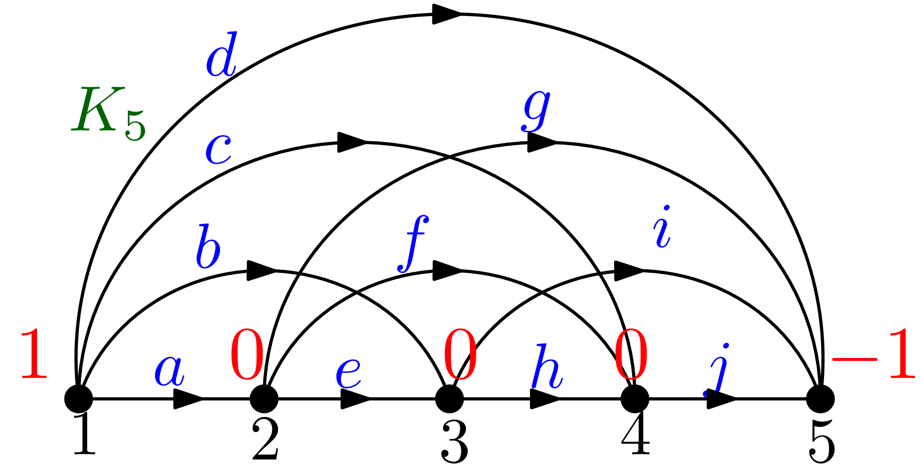
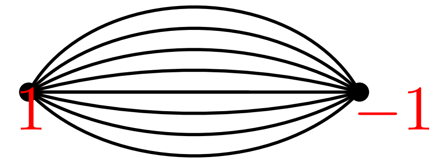
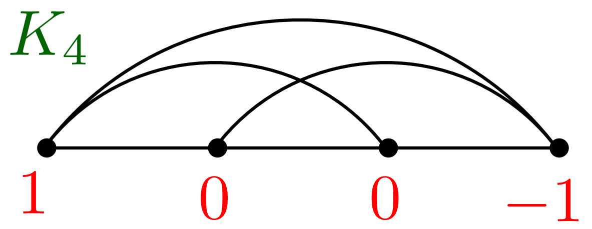


For a general graph G on the vertex set $[n]$, with net flow $\mathbf{a} = (1,0,\ldots,0,-1)$, the **flow polytope** of G, denoted $\mathcal{F}_G$, is the set of **flows** $f : E(G) \to \mathbb{R}_{\geq 0}$ such that the total flow going in at vertex 1 is one, and there is flow conservation at each of the inner vertices.

Examples of flow polytopes



simplex



An intriguing theorem

Theorem [Postnikov-Stanley]:
For a graph G on the vertex set $\{1, 2 \ldots, n\}$ we have

$$\mathrm{vol}\,(\mathcal{F}_G(1, 0, \ldots, 0, -1)) = K_G(0, d_2, \ldots, d_{n-1}, -\sum_{i=2}^{n-1} d_i),$$

where $d_i = (\text{indegree of } i) - 1$ and K_G is the Kostant partition function.

Some interesting examples of flow polytopes

Theorem [Zeilberger 99]:

$$\mathrm{vol}(\mathcal{F}_{K_{n+1}}) = Cat(1)Cat(2)\cdots Cat(n-2).$$

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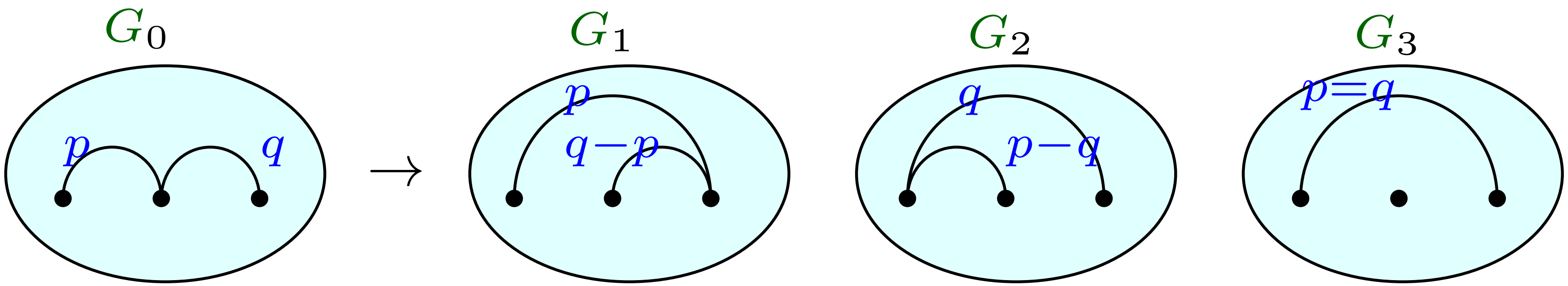
volumes given by nice product formulas.

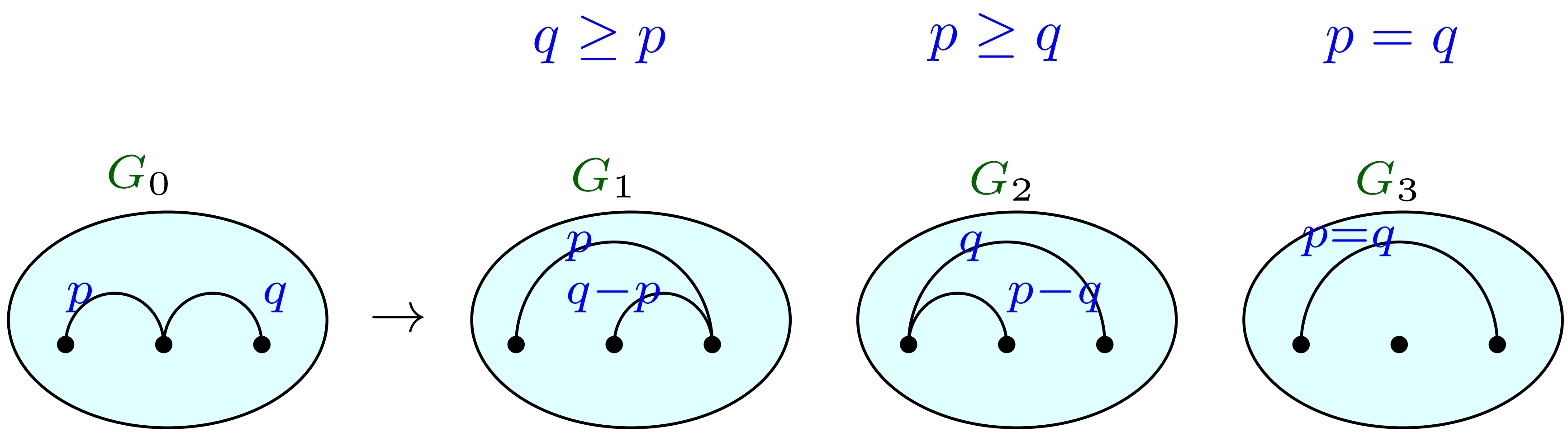
(Think $\prod_{i=m+1}^{m+n-1} \frac{1}{2i+1}\binom{m+n+i+1}{2i}$.)

Triangulating $\mathcal{F}_G$

$q \geq p$ $\quad$ $p \geq q$ $\quad$ $p = q$



Triangulating $\mathcal{F}_G$



Proposition:

$$\mathcal{F}_{G_0} = \mathcal{F}_{G_1} \cup \mathcal{F}_{G_2}, \;\; \mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} = \mathcal{F}_{G_3}.$$

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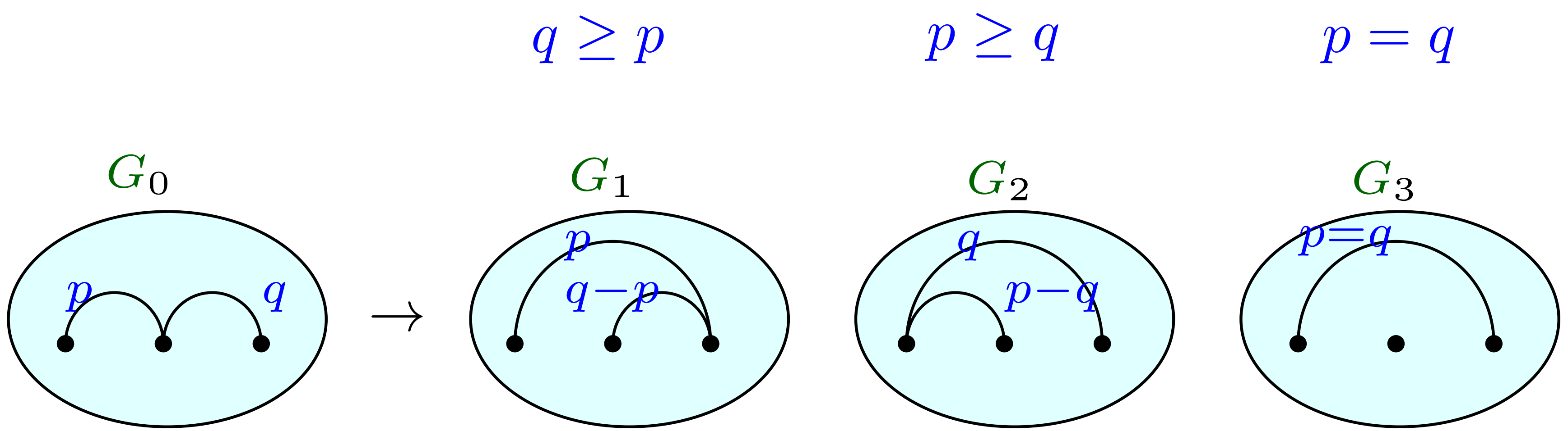

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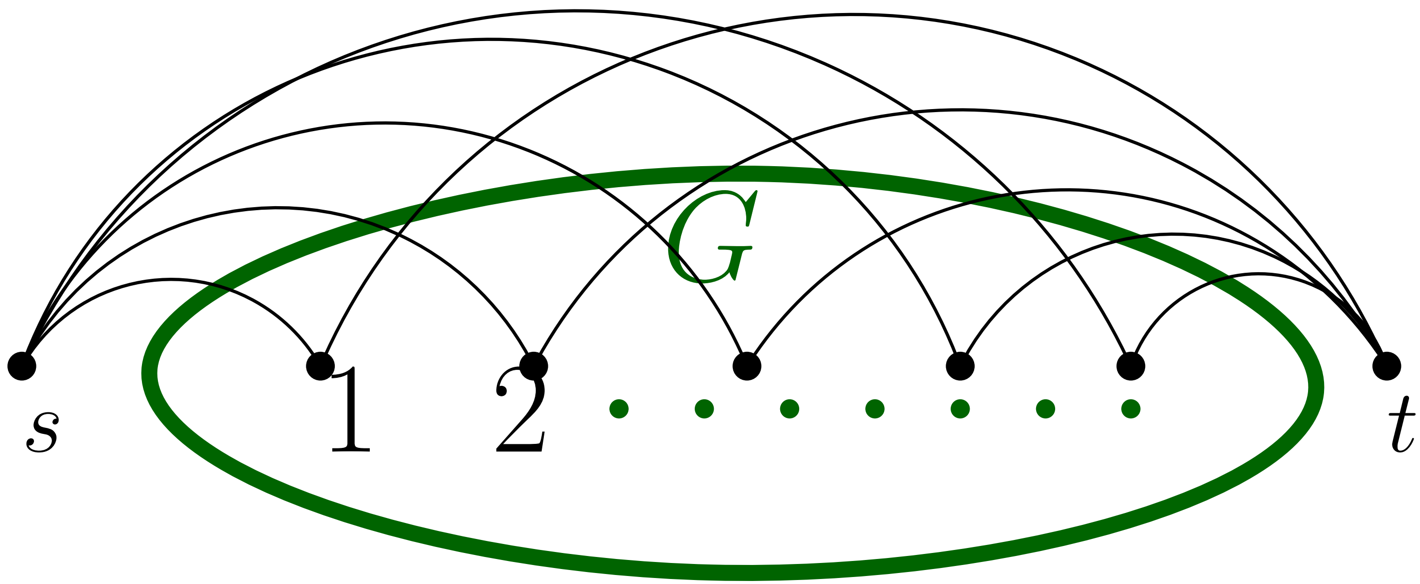
$\mathcal{F}_{G_1}$ or $\mathcal{F}_{G_2}$ could be empty.

$\widetilde{G} = G$ with s and t



Purpose: we can simply do the reductions on G

and at the end arrive to a triangulation of $\mathcal{F}_{\widetilde{G}}$.

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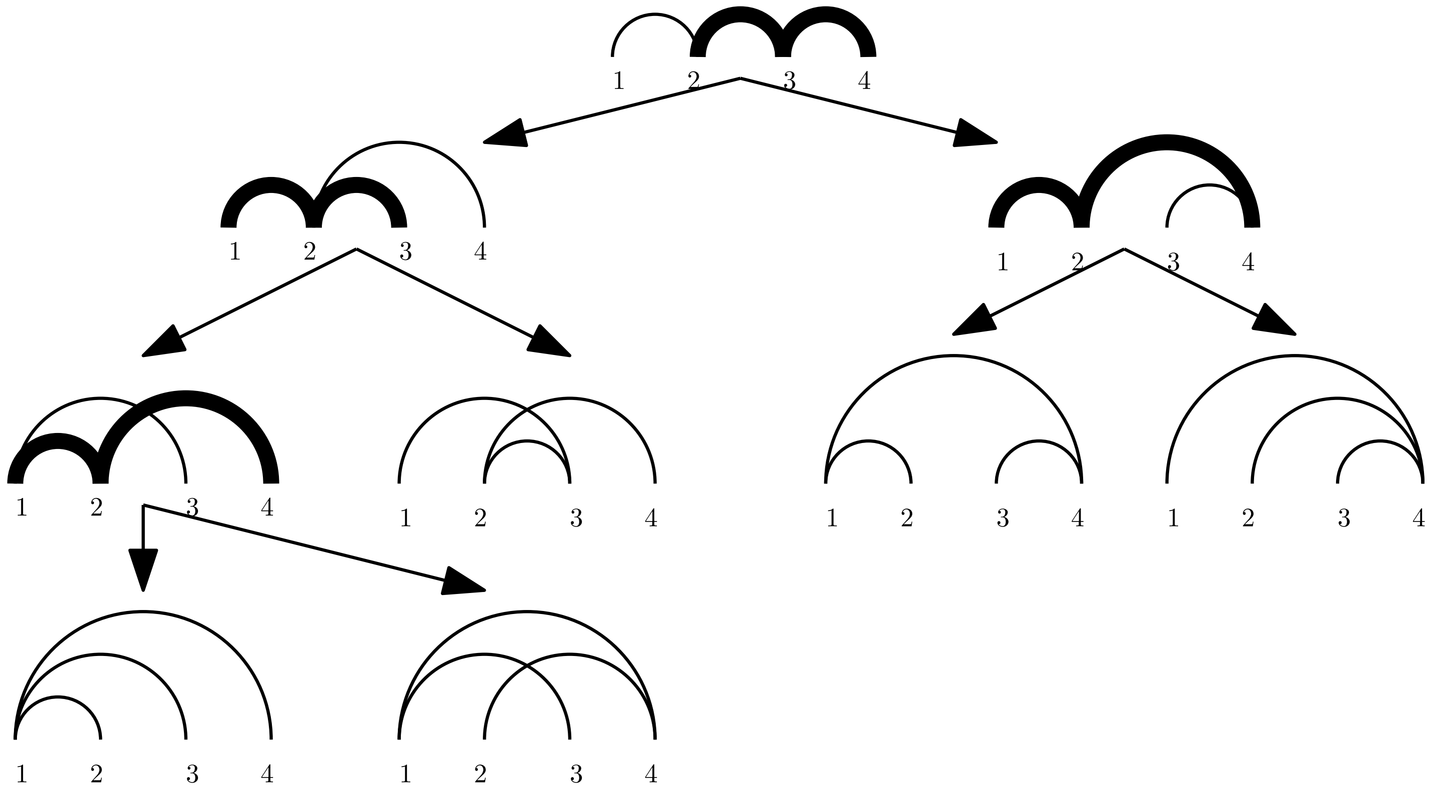
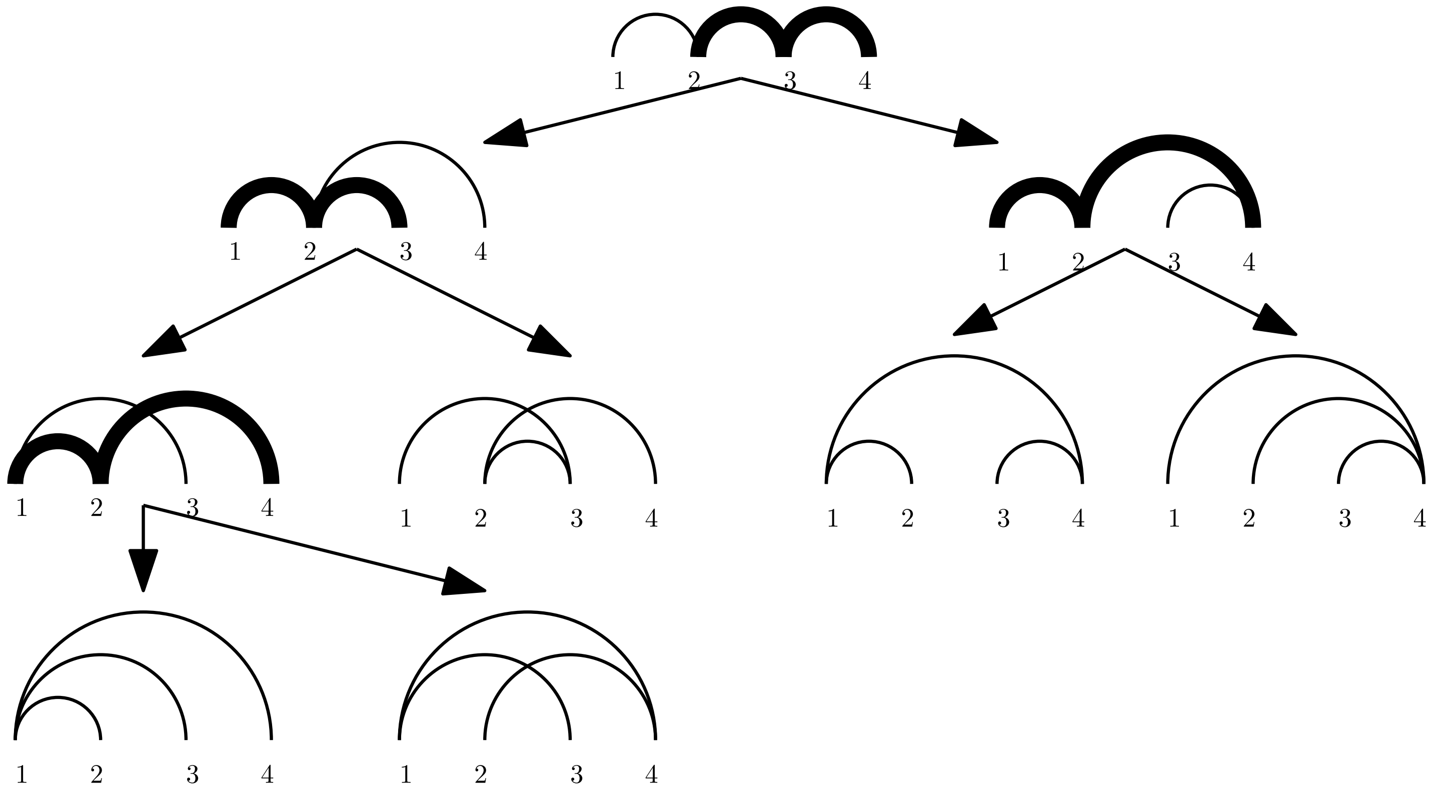
Reduction tree $\mathcal{T}(G)$



A reduction tree of $G = ([4], \{(1,2),(2,3),(3,4)\})$ with five leaves. The edges on which the reductions are performed are in bold.

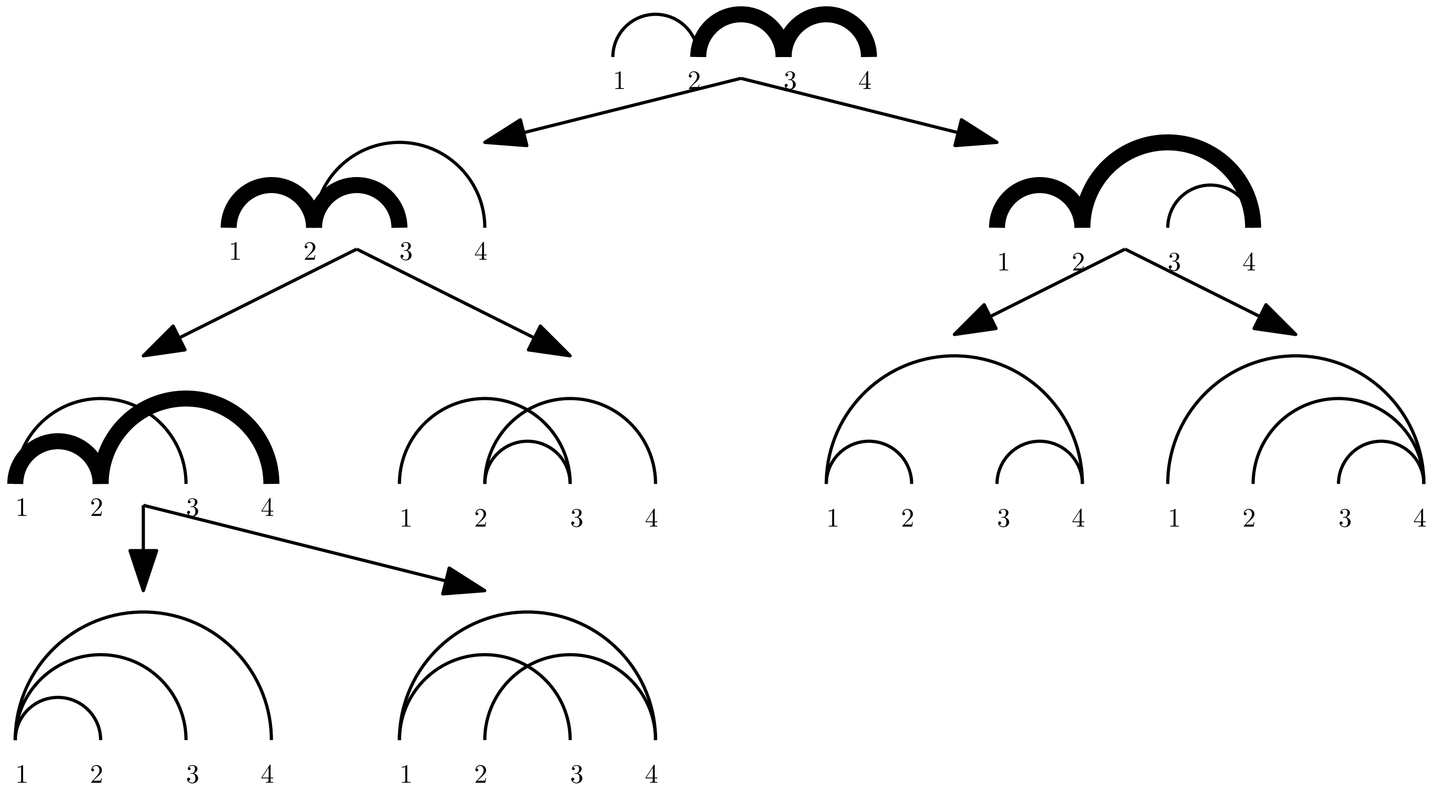
Reduction tree $\mathcal{T}(G)$



Lemma.

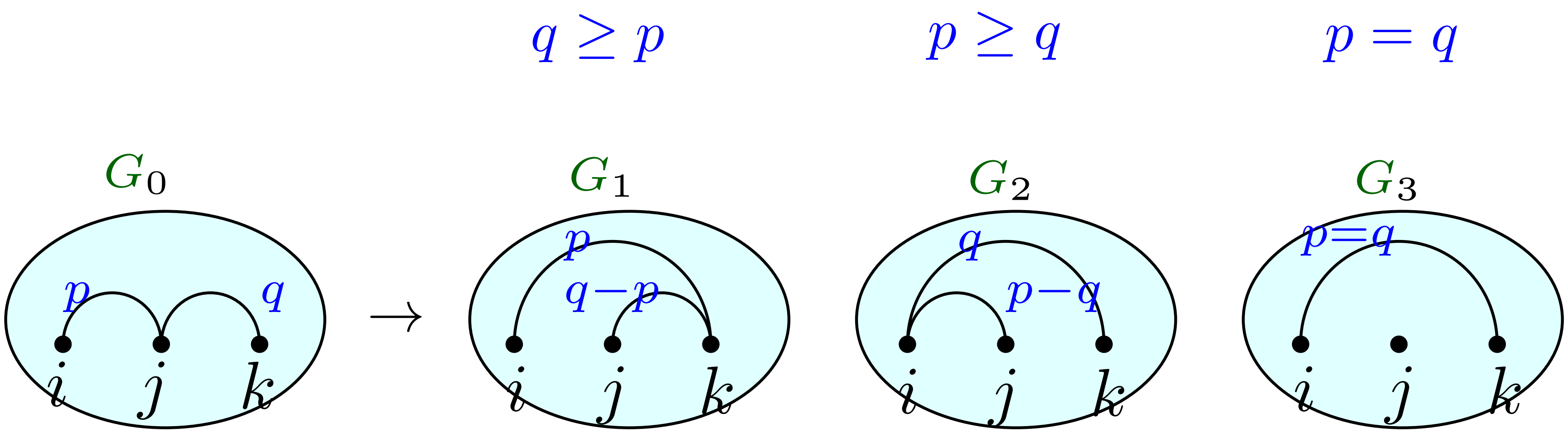
If the leaves are labeled by graphs $H_1, , H_k$ then the flow polytopes $\mathcal{F}_{\widetilde{H_1}}, \ldots, \mathcal{F}_{\widetilde{H_k}}$ are simplices.

Reduction tree $\mathcal{T}(G)$



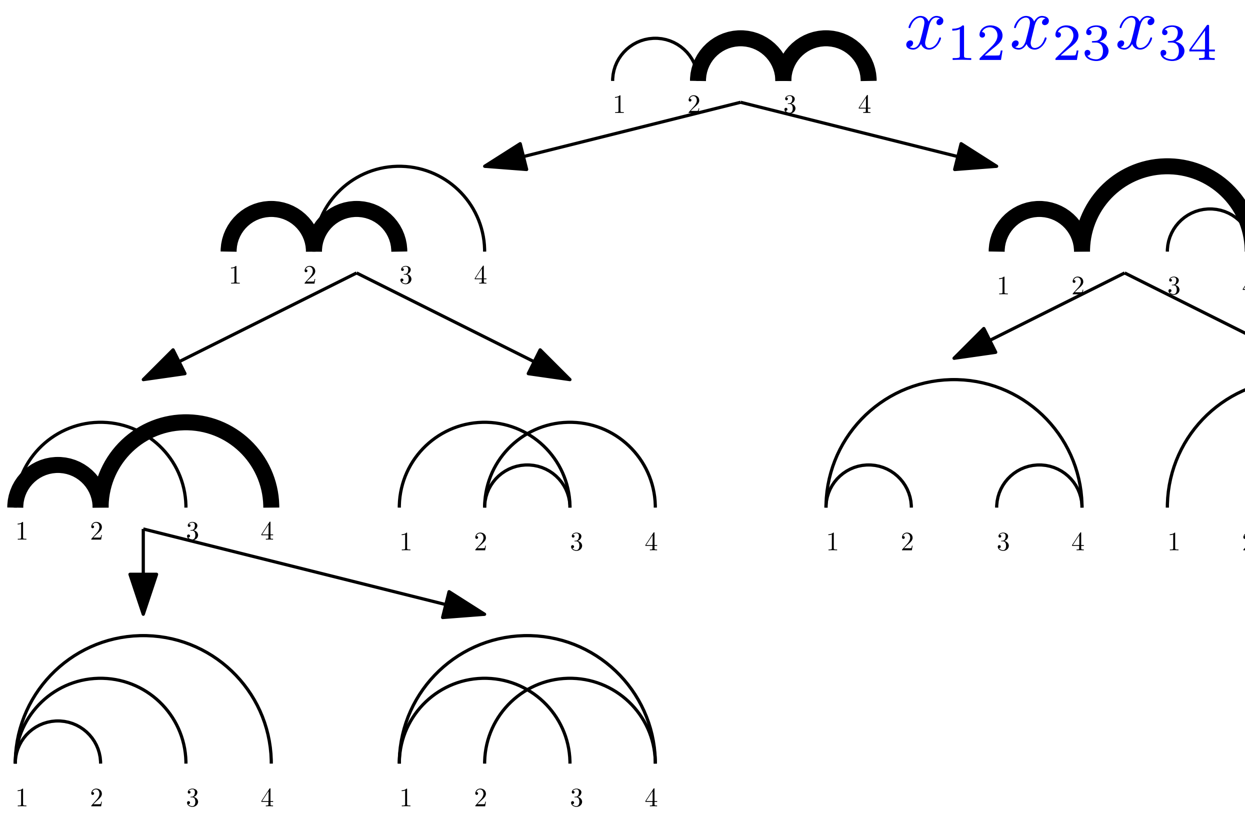
Lemma. The normalized volume of $\mathcal{F}_{\widetilde{G}}$ is equal to the number of leaves in a reduction tree $\mathcal{T}(G)$.

Reductions in variables



$$x_{ij}x_{jk} \to x_{jk}x_{ik} + x_{ik}x_{ij} + \beta x_{ik}$$

Reduced form



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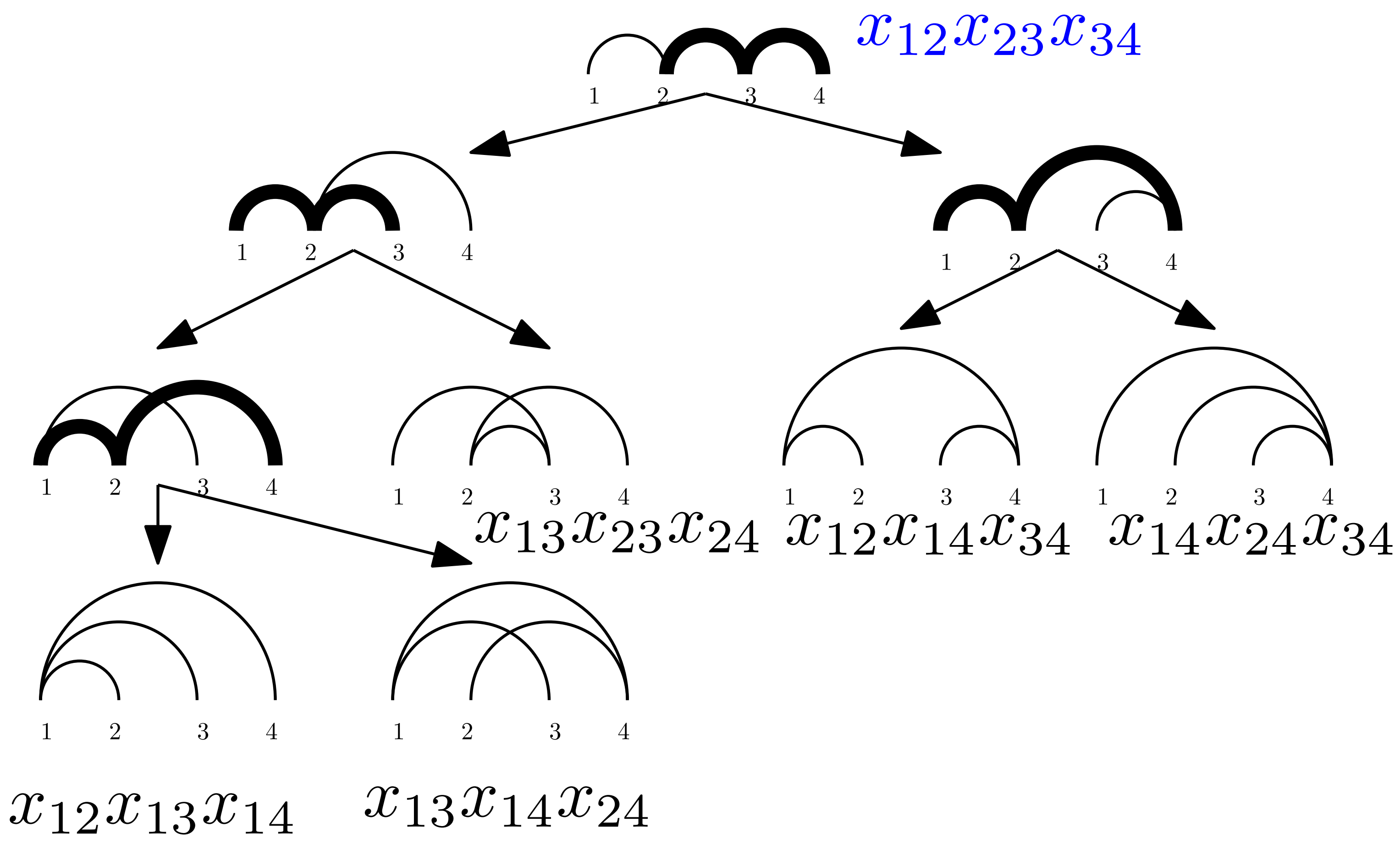

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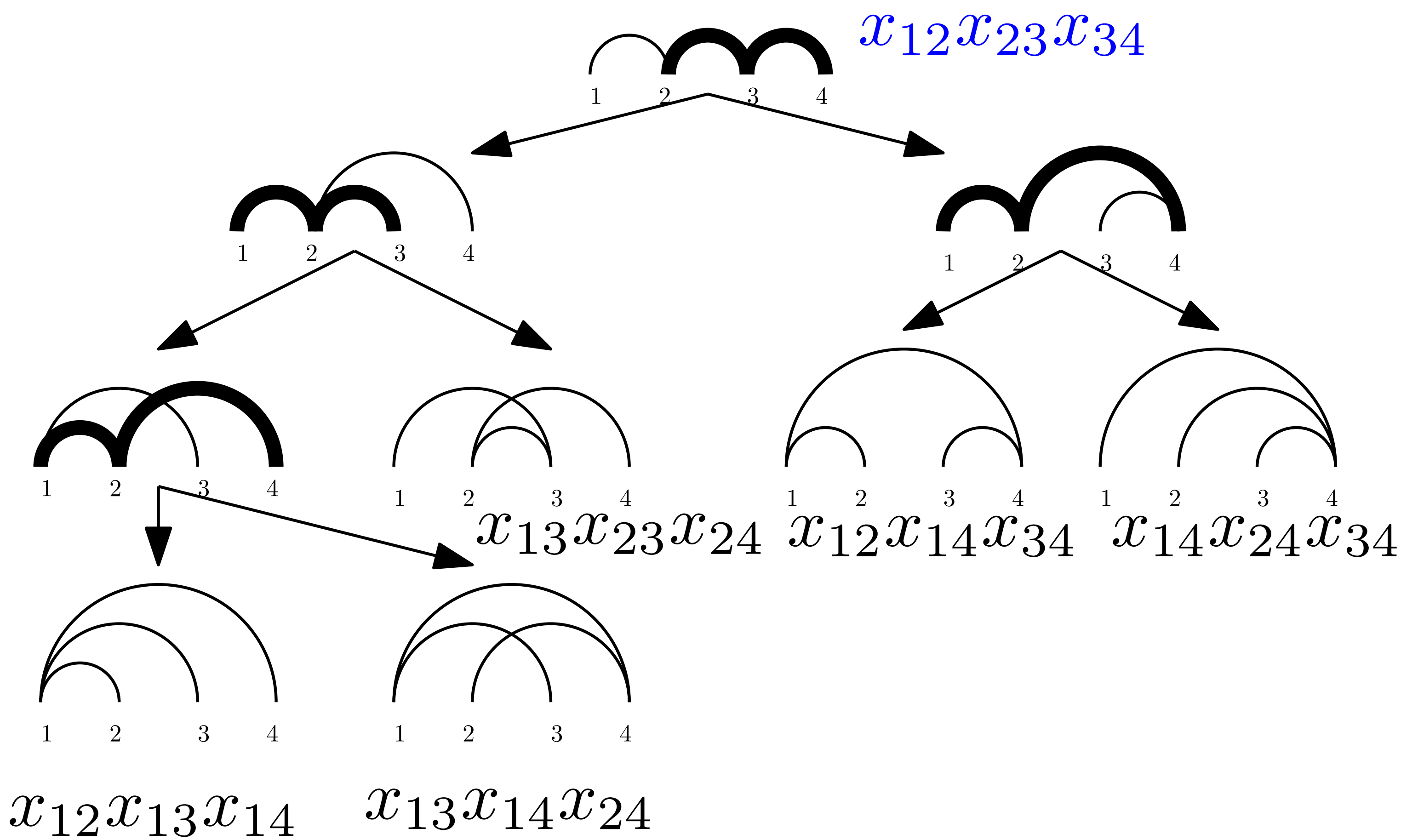

$$x_{12}x_{13}x_{14} + x_{13}x_{14}x_{24} + x_{13}x_{23}x_{24} + x_{12}x_{14}x_{34} + x_{14}x_{24}x_{34}$$

$(\beta = 0)$

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(where $\mathcal{T}$ is a "triangulation" of $\mathcal{F}_{\tilde{G}}$ obtained via the game)

In particular the coefficients of $Q_G(\beta - 1)$ are nonnegative.

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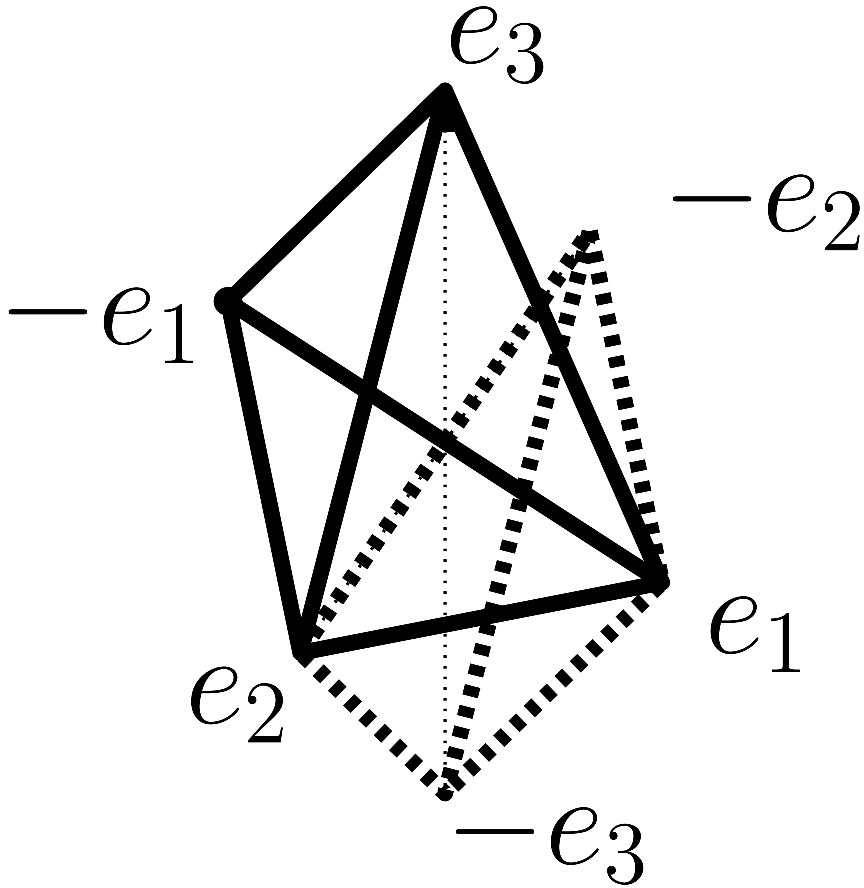

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Nevertheless, the notions of f-vectors and h-vectors still make sense.

Still, we wonder:

Is there a way to play the game and get a triangulation?

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Yes, triangulation!

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A triangulation is said to be shellable, if we can order the top dimensional simplices $F_1, \ldots, F_k$, so that F_i, $1 < i$, attaches to the preceeding simplices $F_1, \ldots, F_{i-1}$, on a union of its facets (at least one of them).

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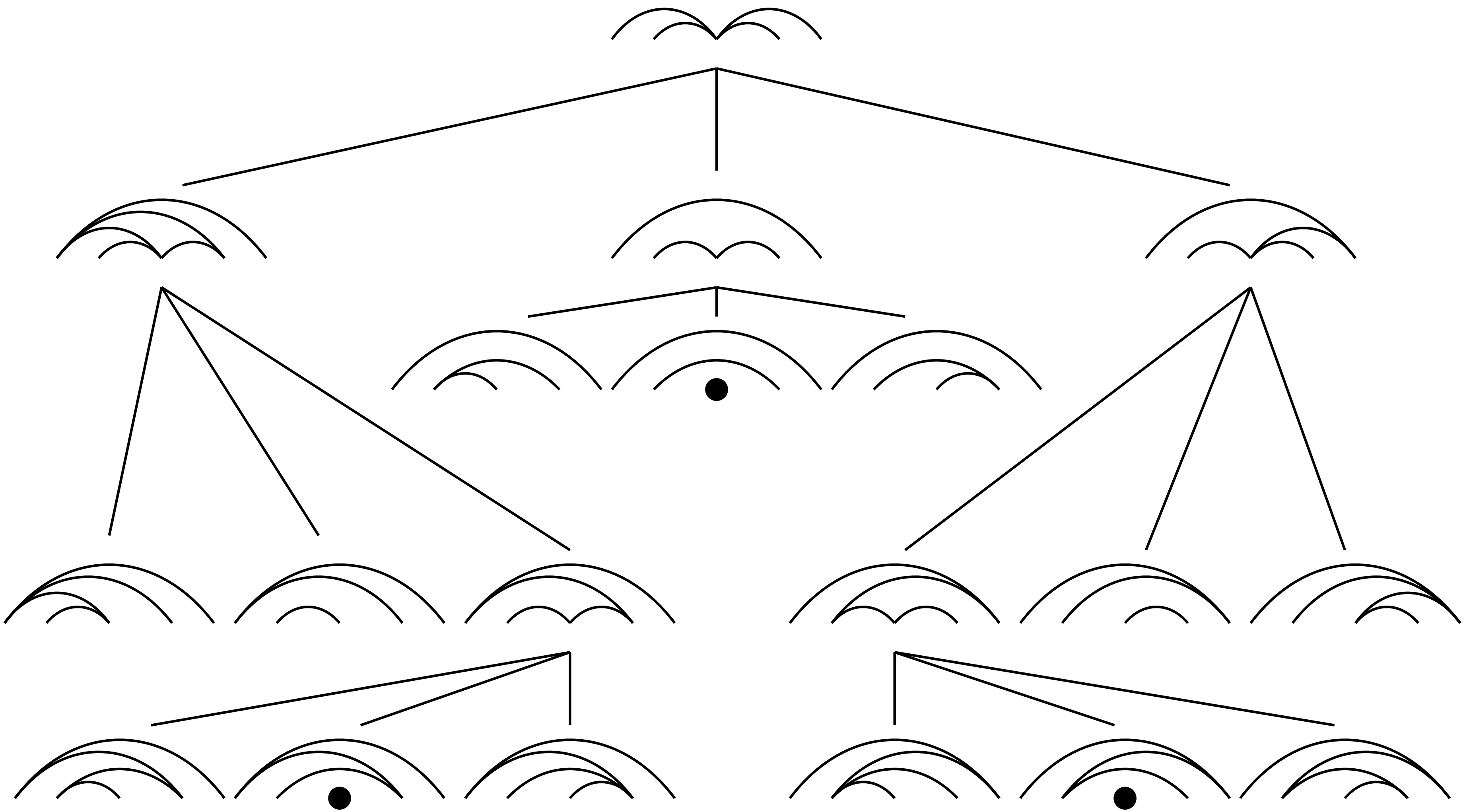
The key is to use a special reduction order. Namely, do the reductions from left to right and always on the topmost edges. We call this special order $\mathcal{O}$.

Reduction tree $R_G^{\mathcal{O}}$



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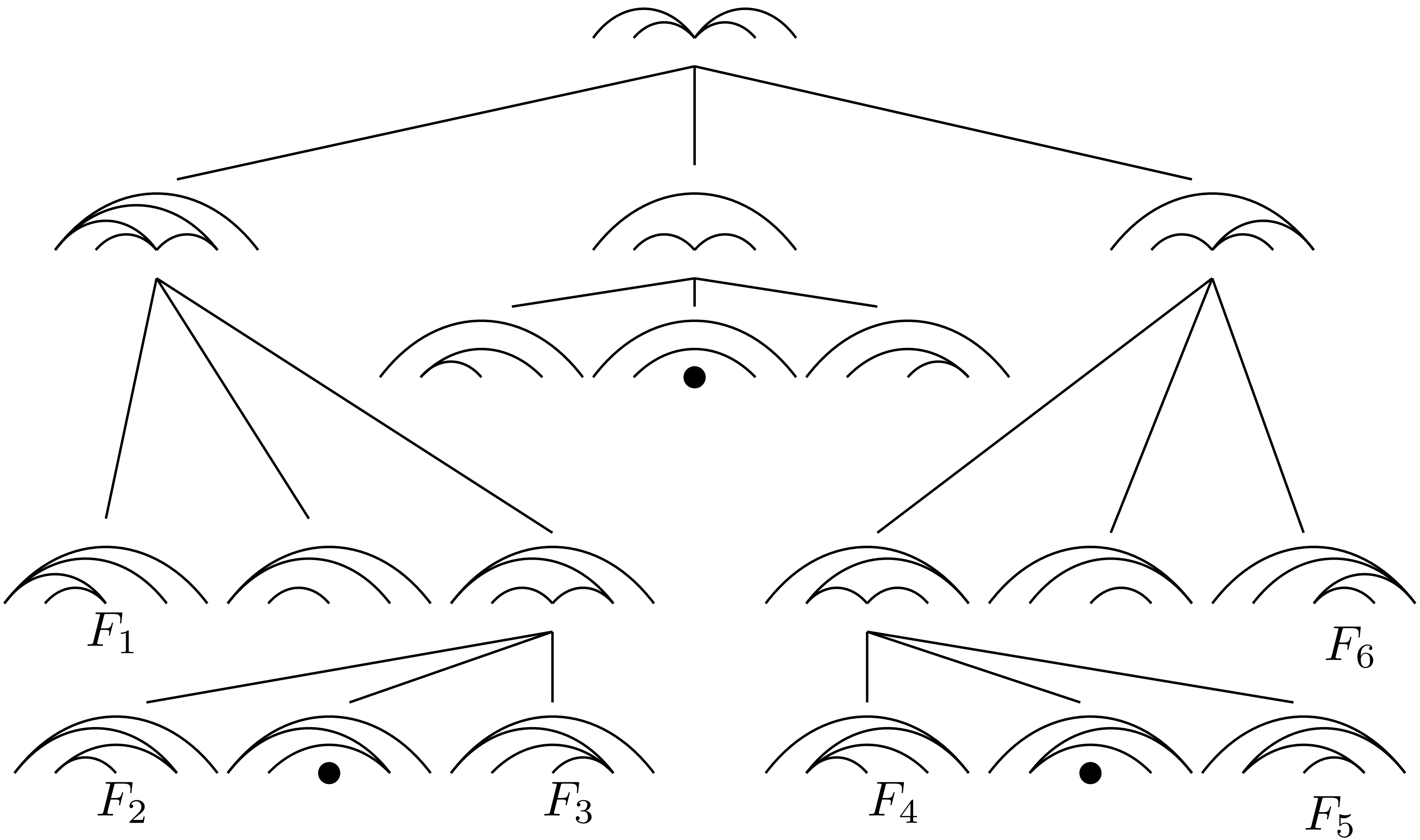

Shelling $\mathcal{T}^{\mathcal{O}}$

Let $F_1, \ldots, F_l$ be the full-dimensional leaves of $R_G^{\mathcal{O}}$ ordered by depth-first search order.

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$\mathcal{F}_{\widetilde{F_1}}, \ldots, \mathcal{F}_{\widetilde{F_l}}$ is a shelling order of the triangulation $\mathcal{T}^{\mathcal{O}}$ of $\mathcal{F}_{\widetilde{G}}$.

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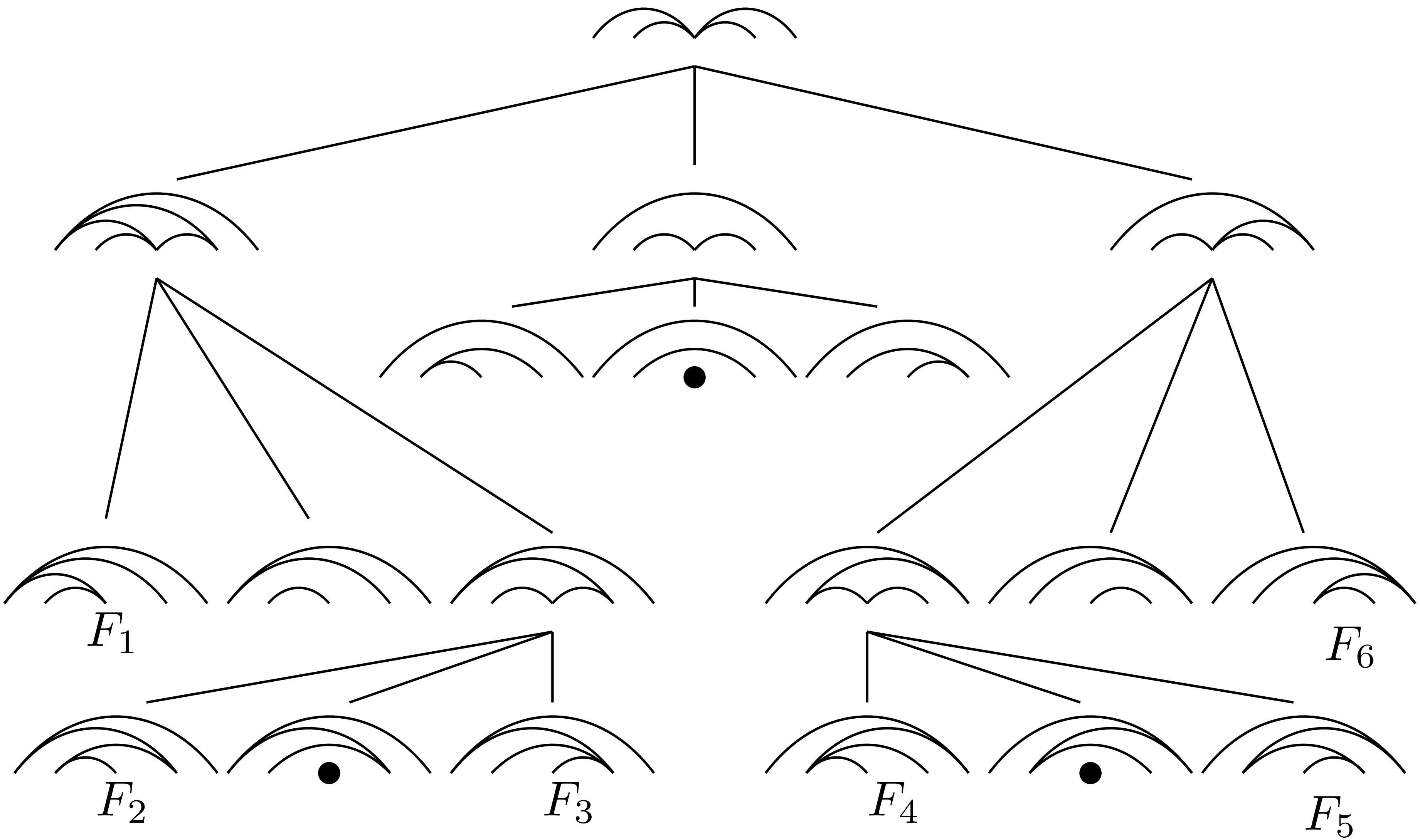
The idea of proof is weak embeddability of reduction trees.

Weak embeddable reduction tree $R_G^{\mathcal{O}}$



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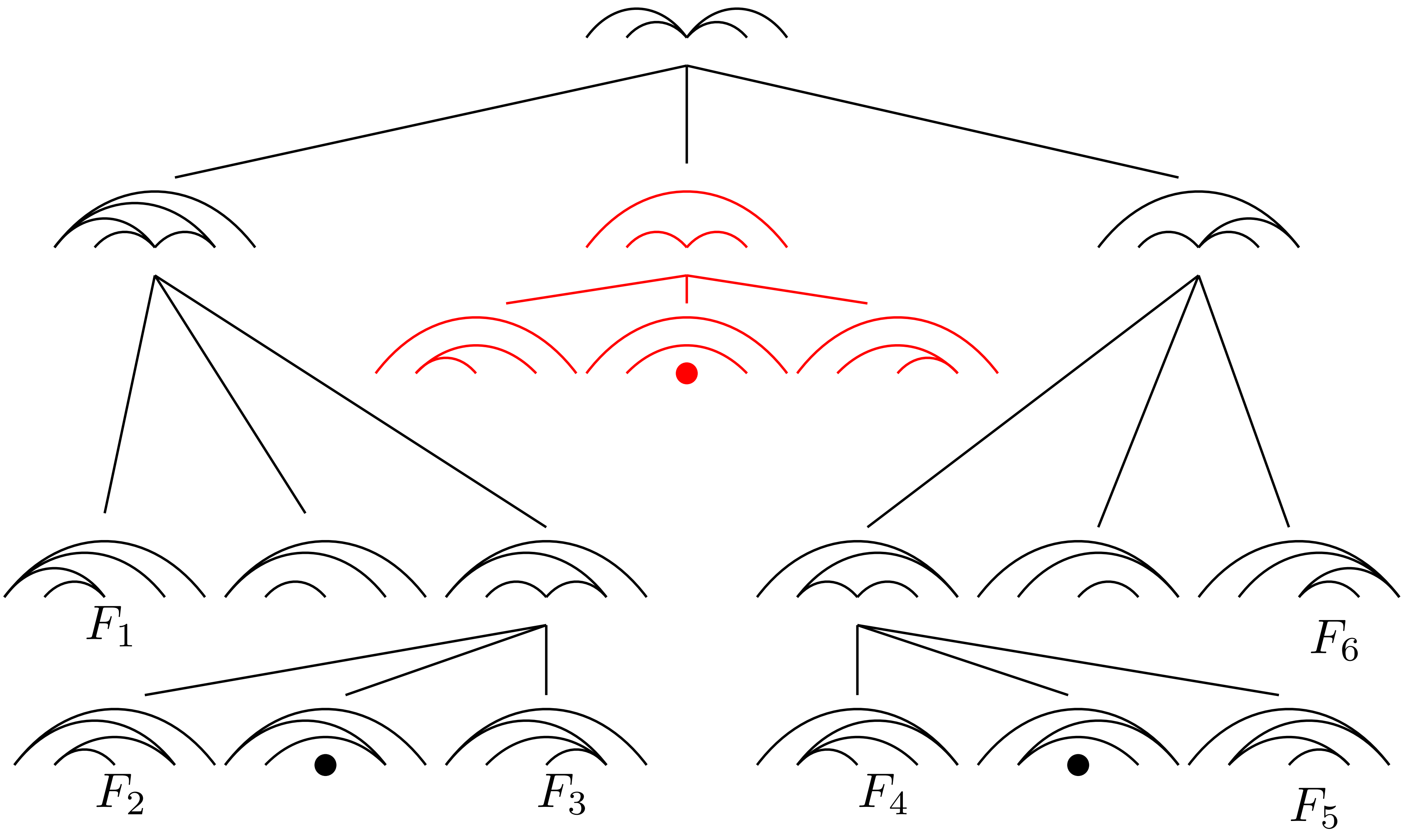

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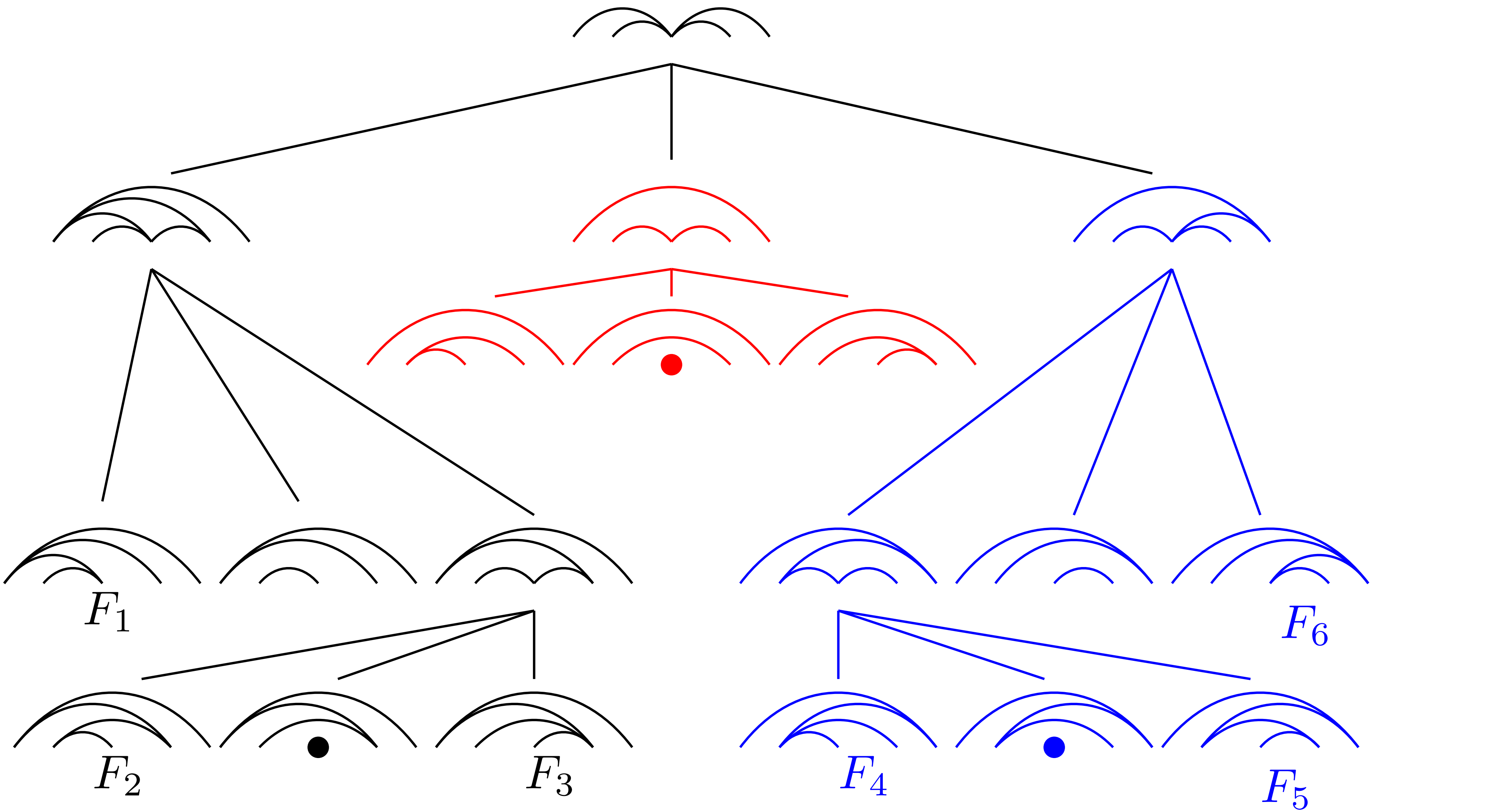

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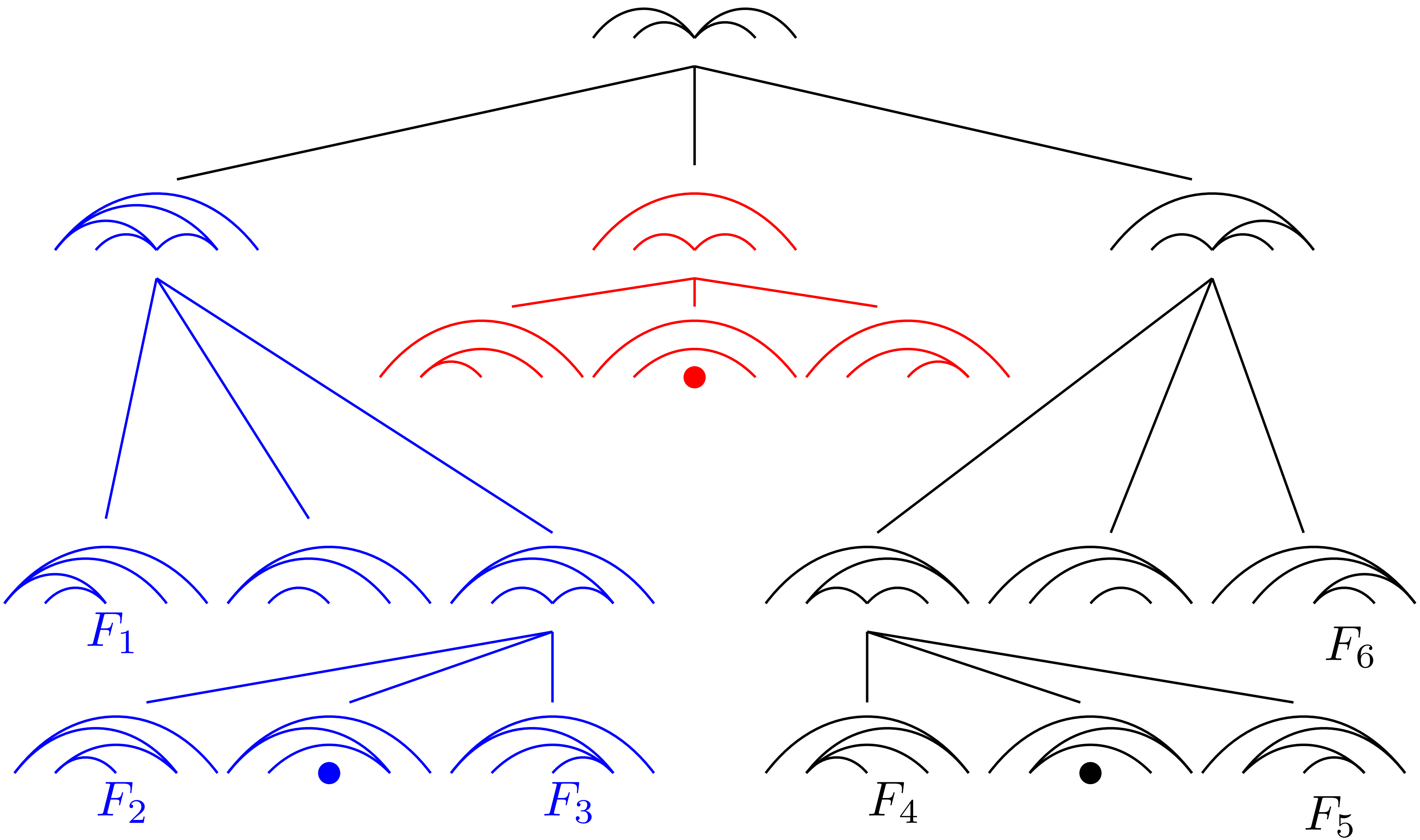

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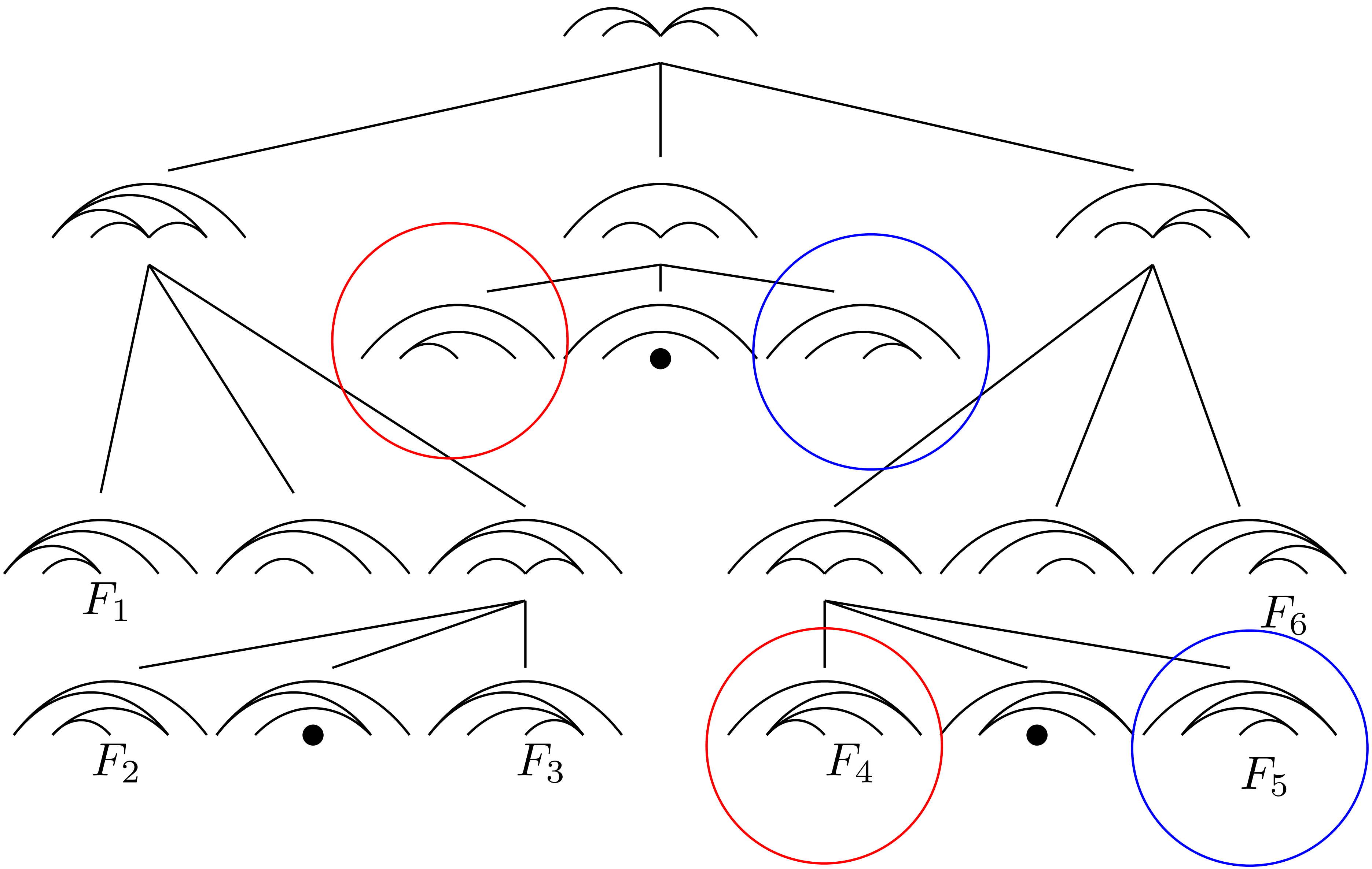

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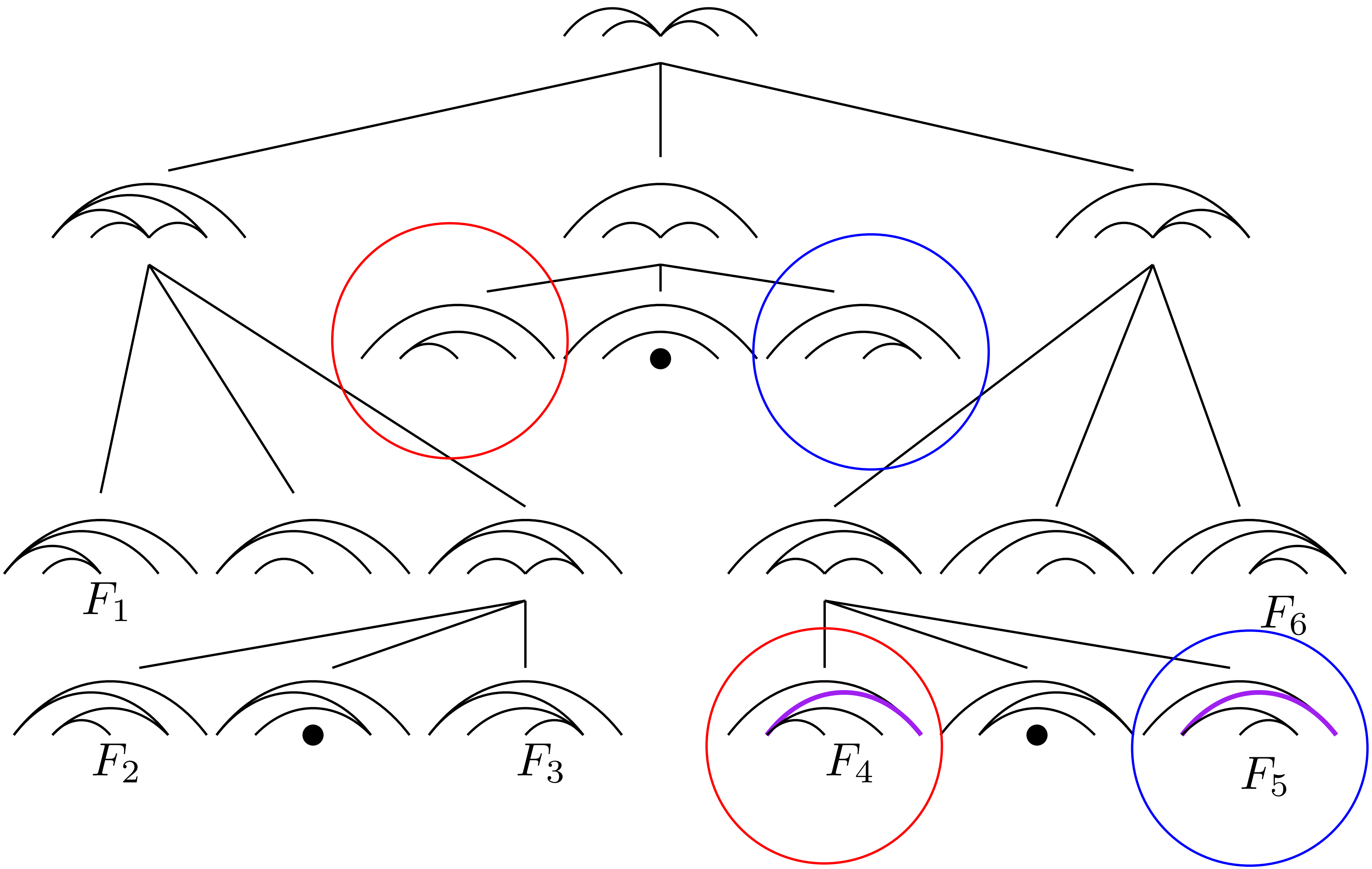

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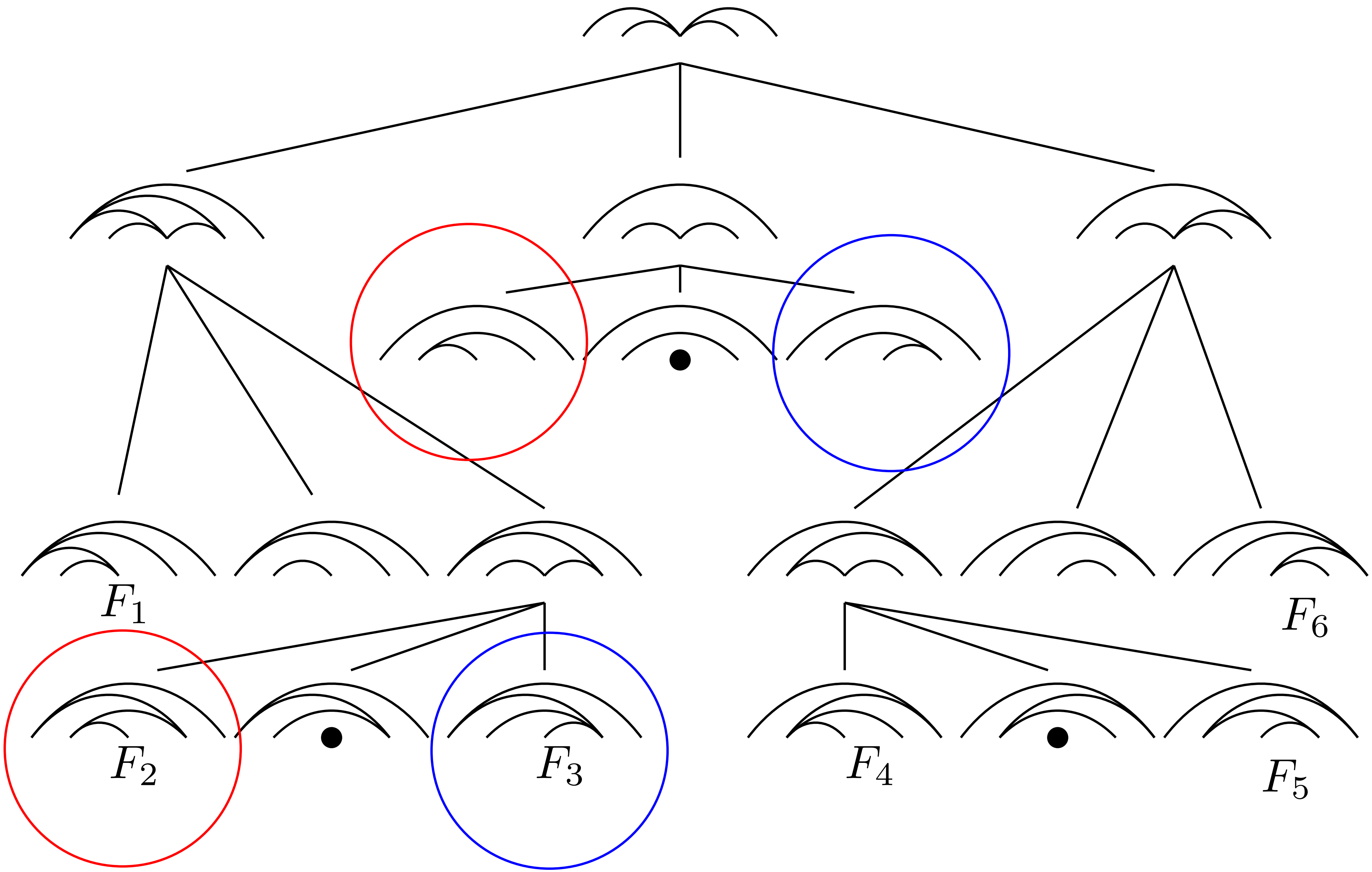

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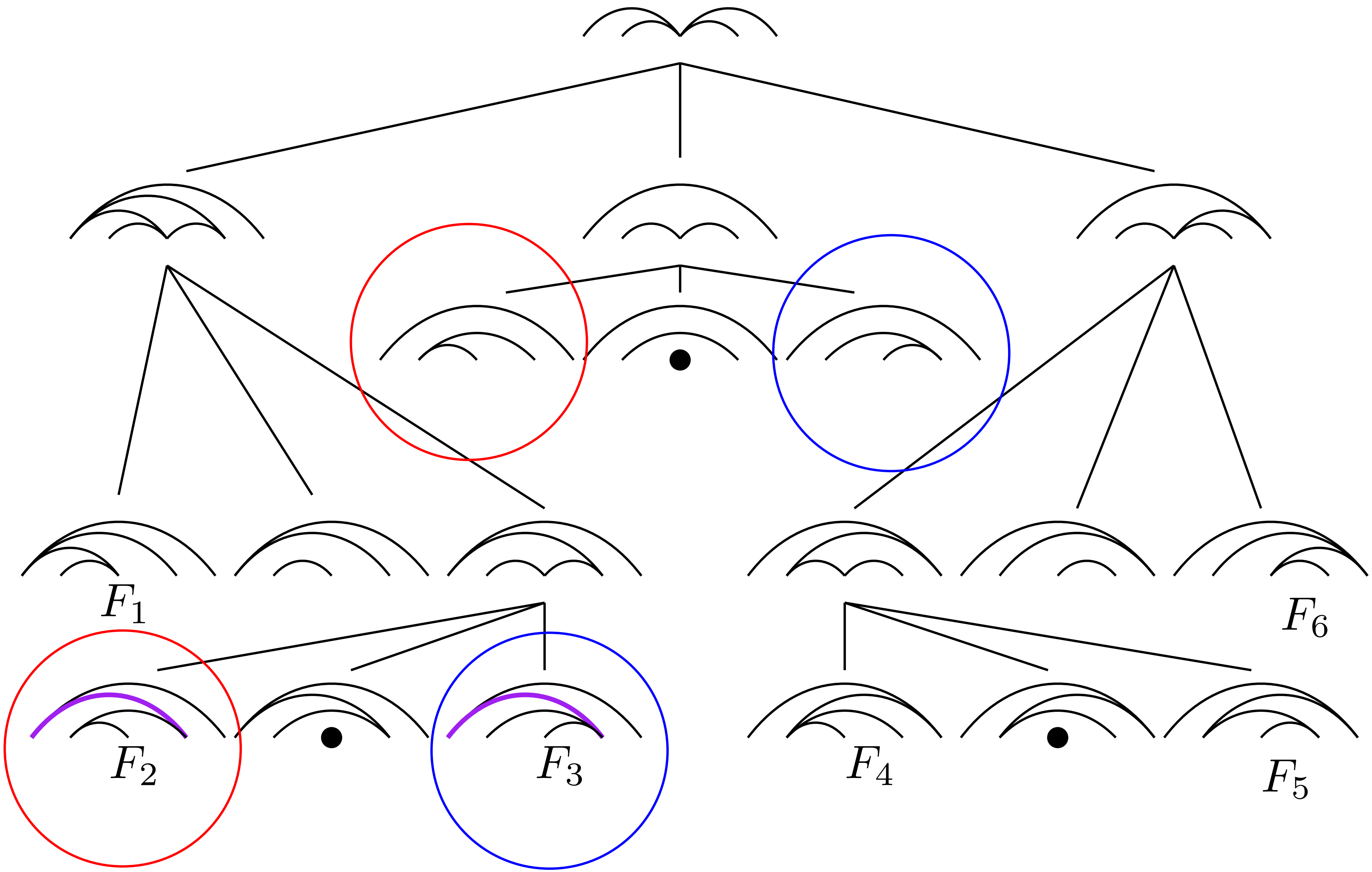
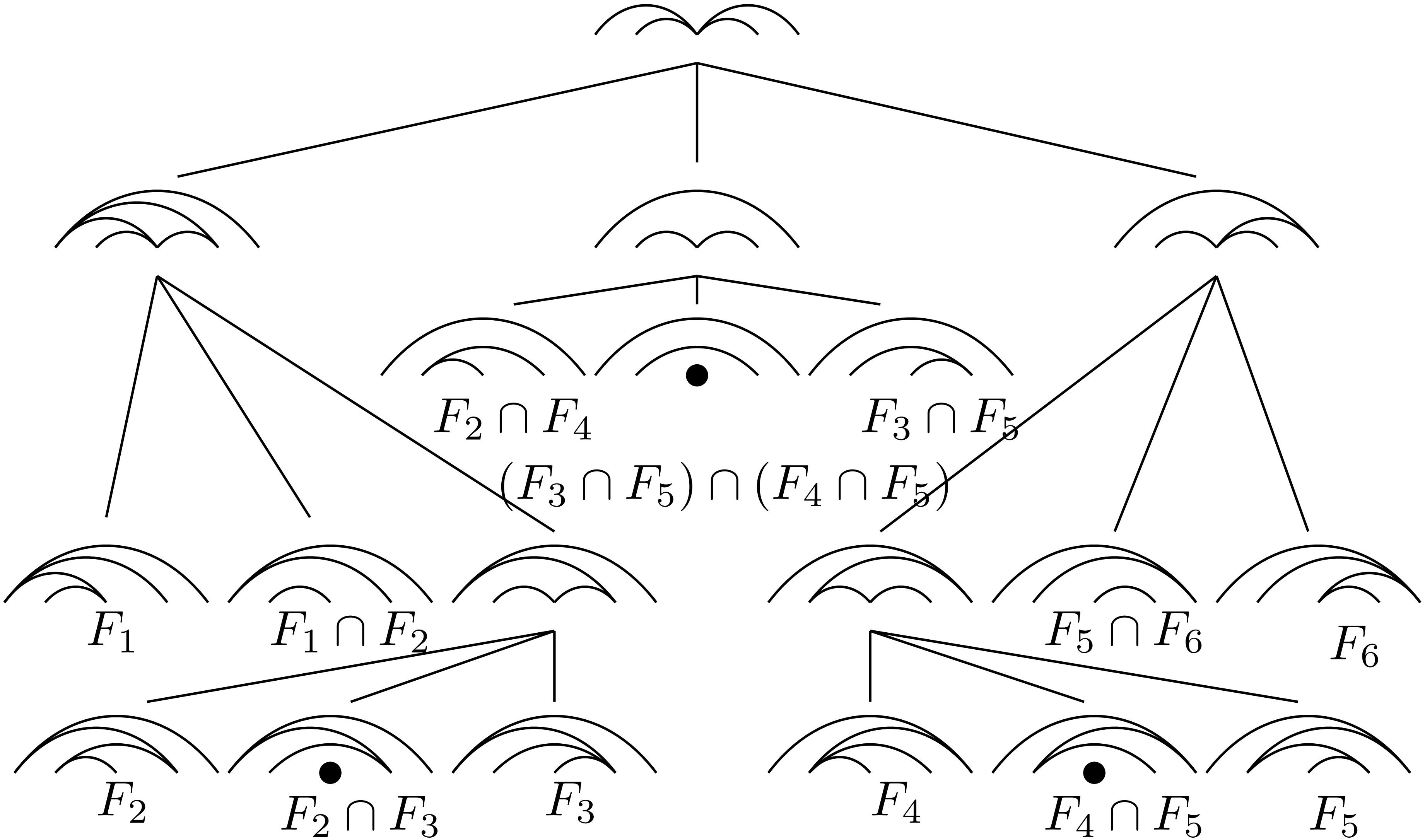


Leaves of $R_G^{\mathcal{O}}$



Leaves of $R_G^{\mathcal{O}}$

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Let $F_1, \ldots, F_l$ be the full-dimensional leaves of $R_G^{\mathcal{O}}$ ordered by depth-first search order.
Let

$$\{Q_1^i, \ldots, Q_{f(i)}^i\} = \{F_i \cap F_j \mid 1 \leq j < i, |E(F_i \cap F_j)| = |E(F_i)| - 1\}.$$

Then

$$\sum_{i=1}^{l} \prod_{j=1}^{f(i)} (F_i + Q_j^i)$$

is the formal sum of the set of the leaves of $R_G^{\mathcal{O}}$, where the product of graphs is their intersection. If $f(i) = 0$ we define $\prod_{j=1}^{f(i)} (F_i + Q_j^i) = F_i$.

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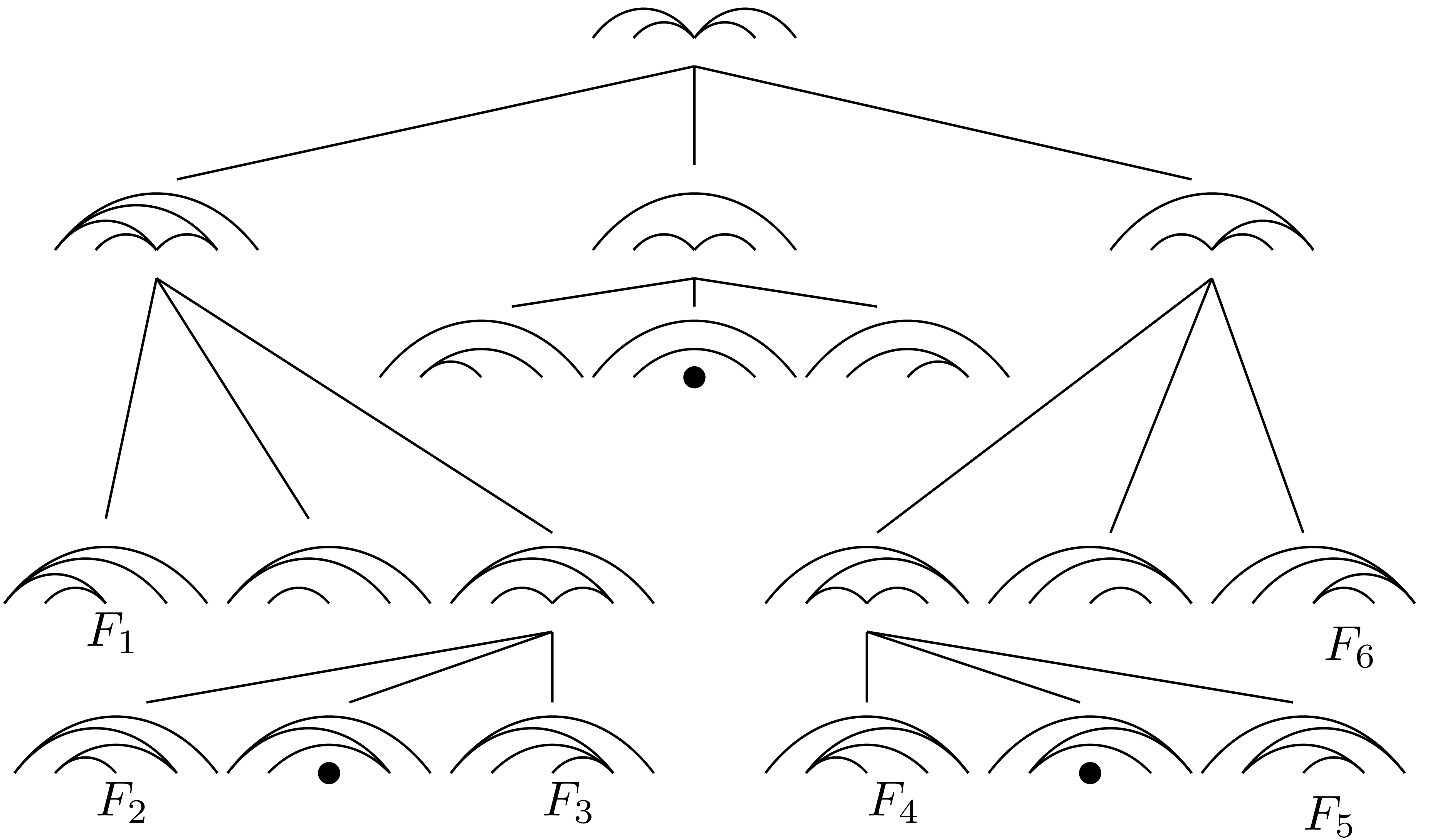
3. $*$

Strong embeddable reduction tree $R_G^{\mathcal{O}}$



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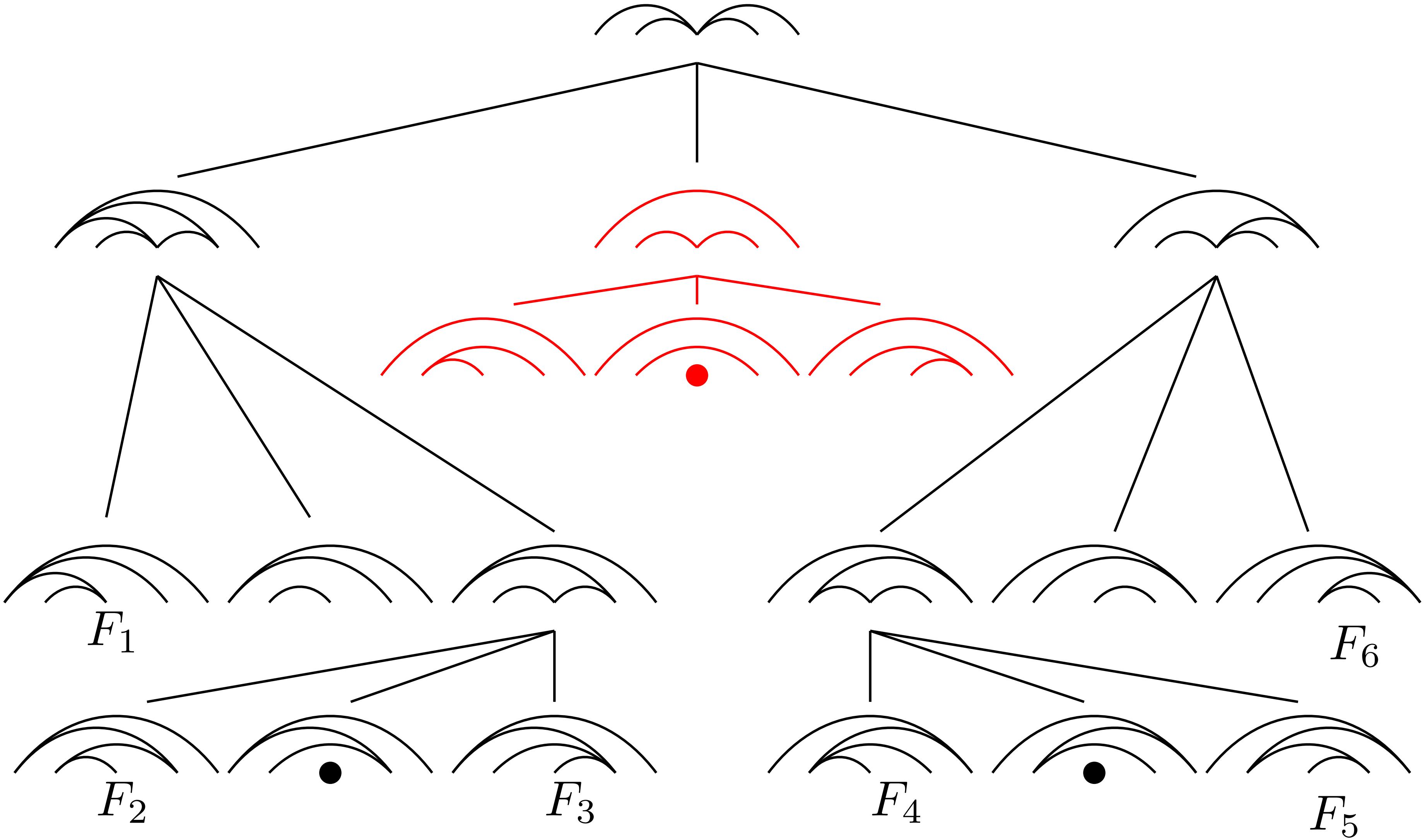

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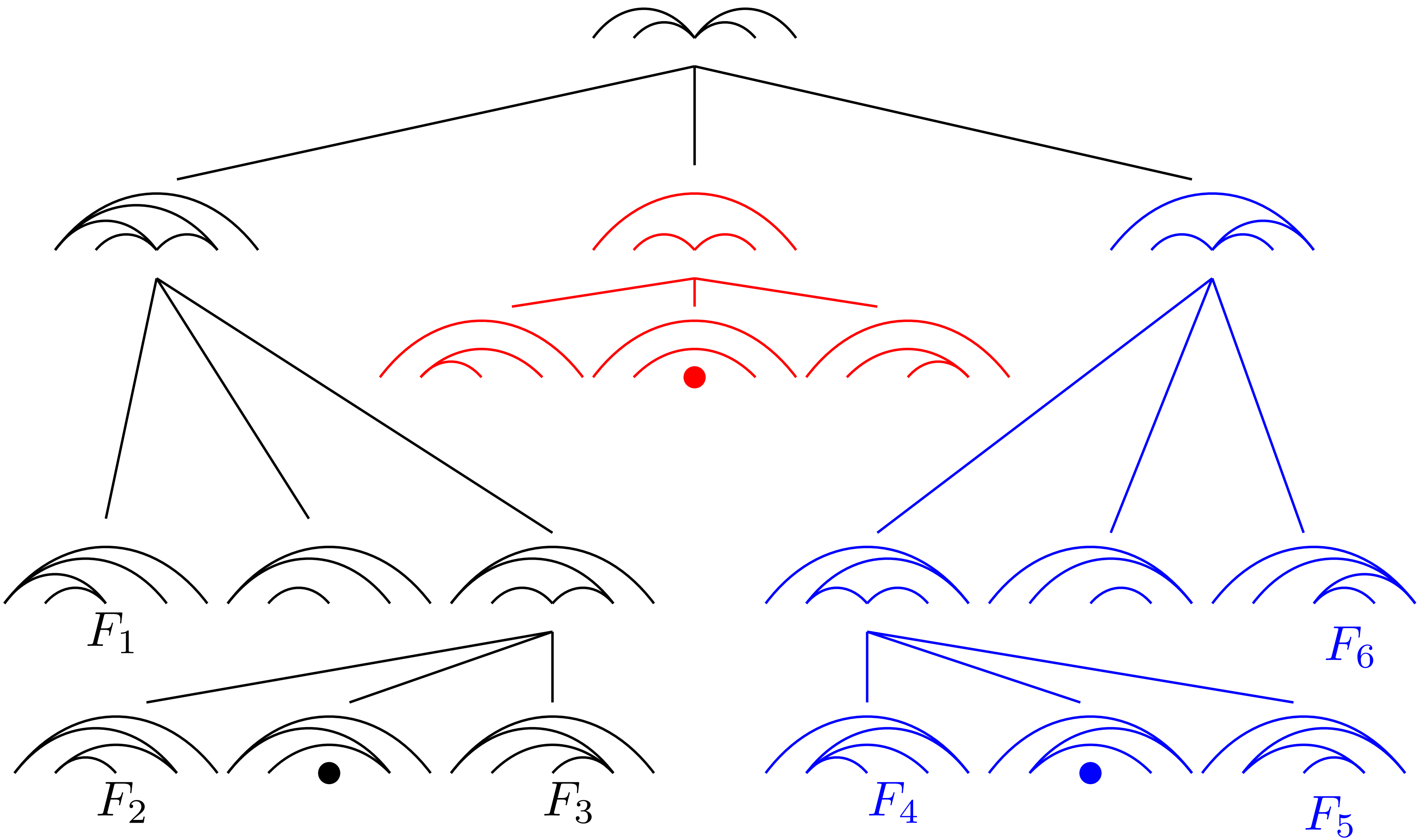

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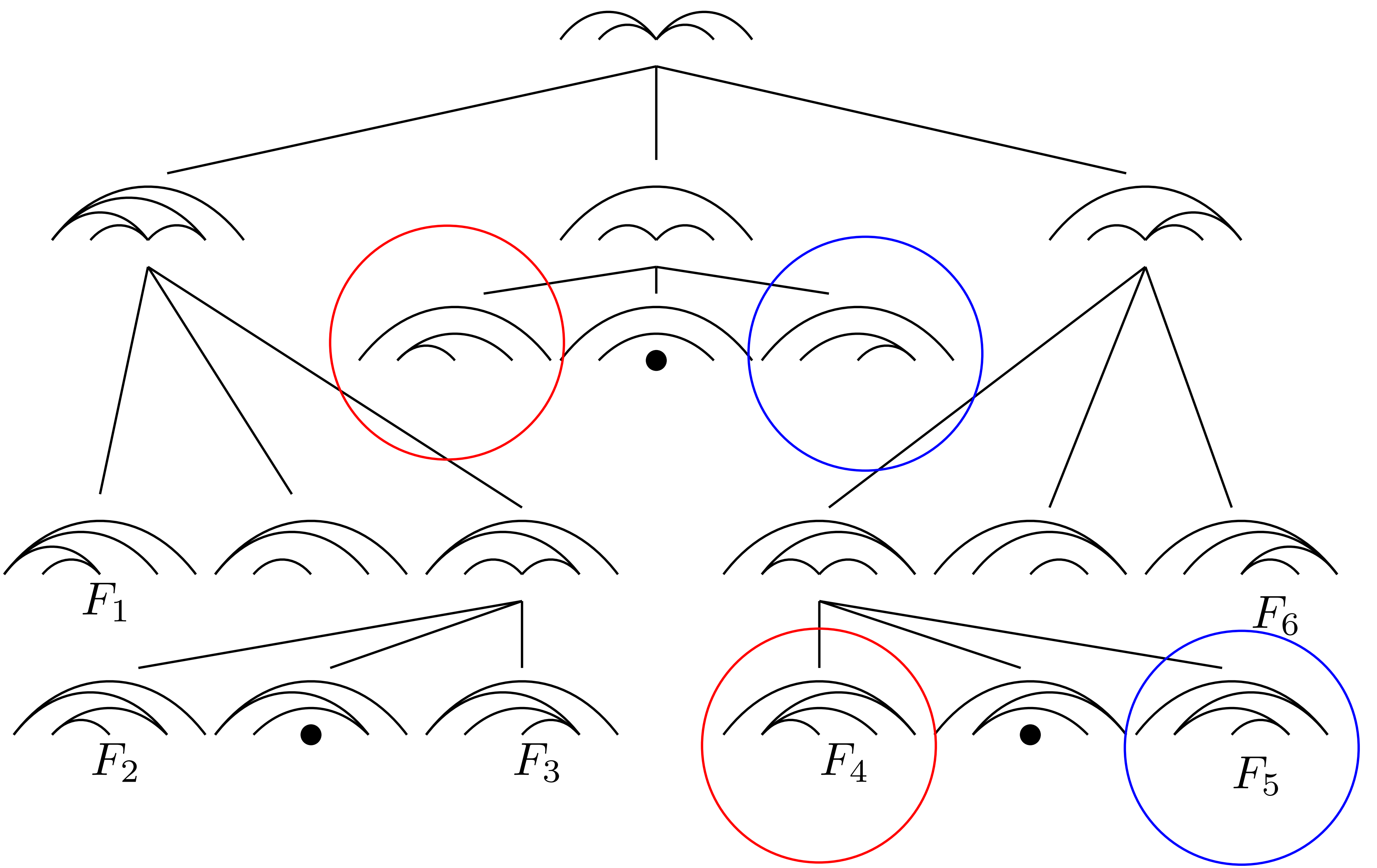

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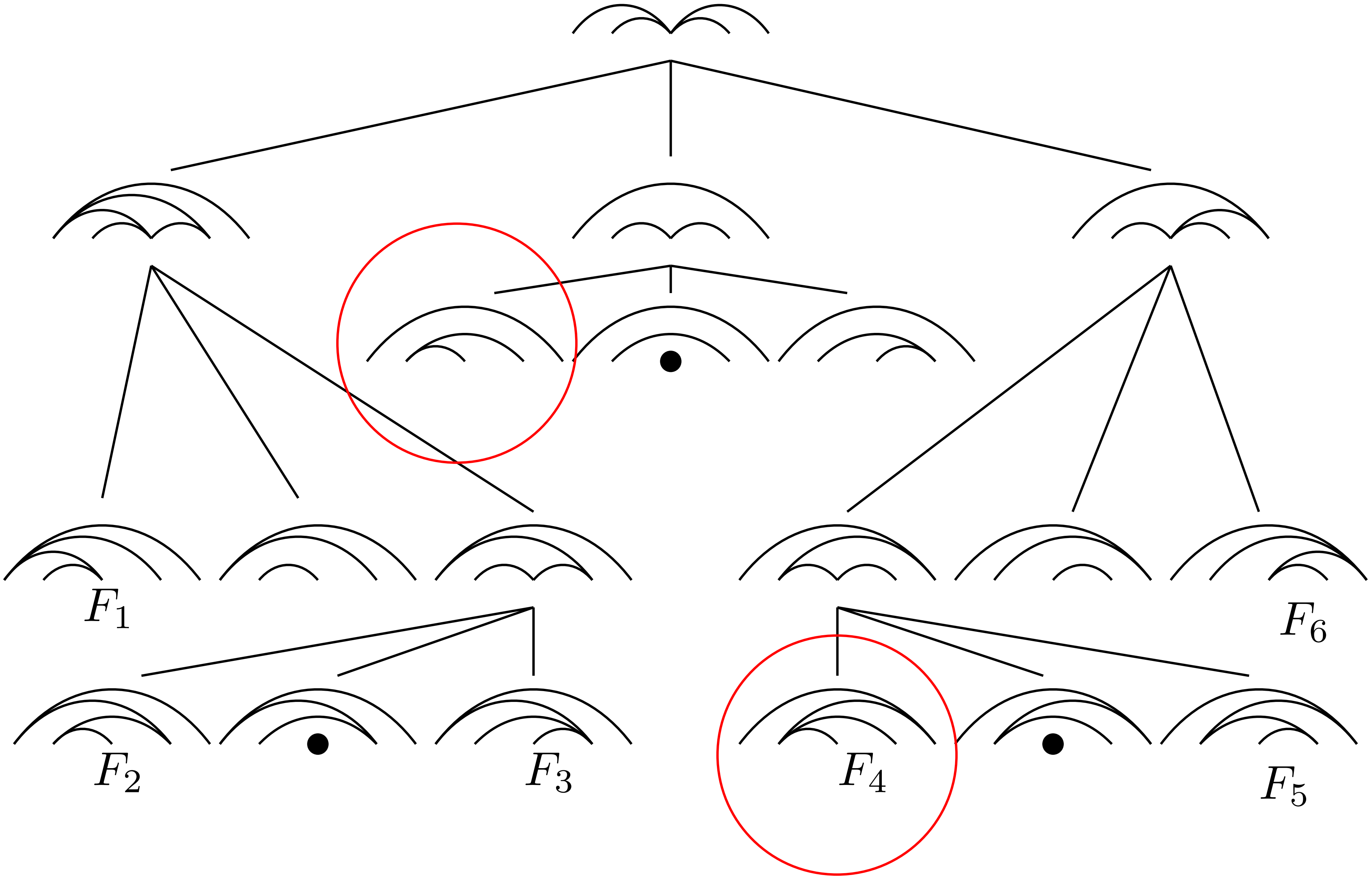

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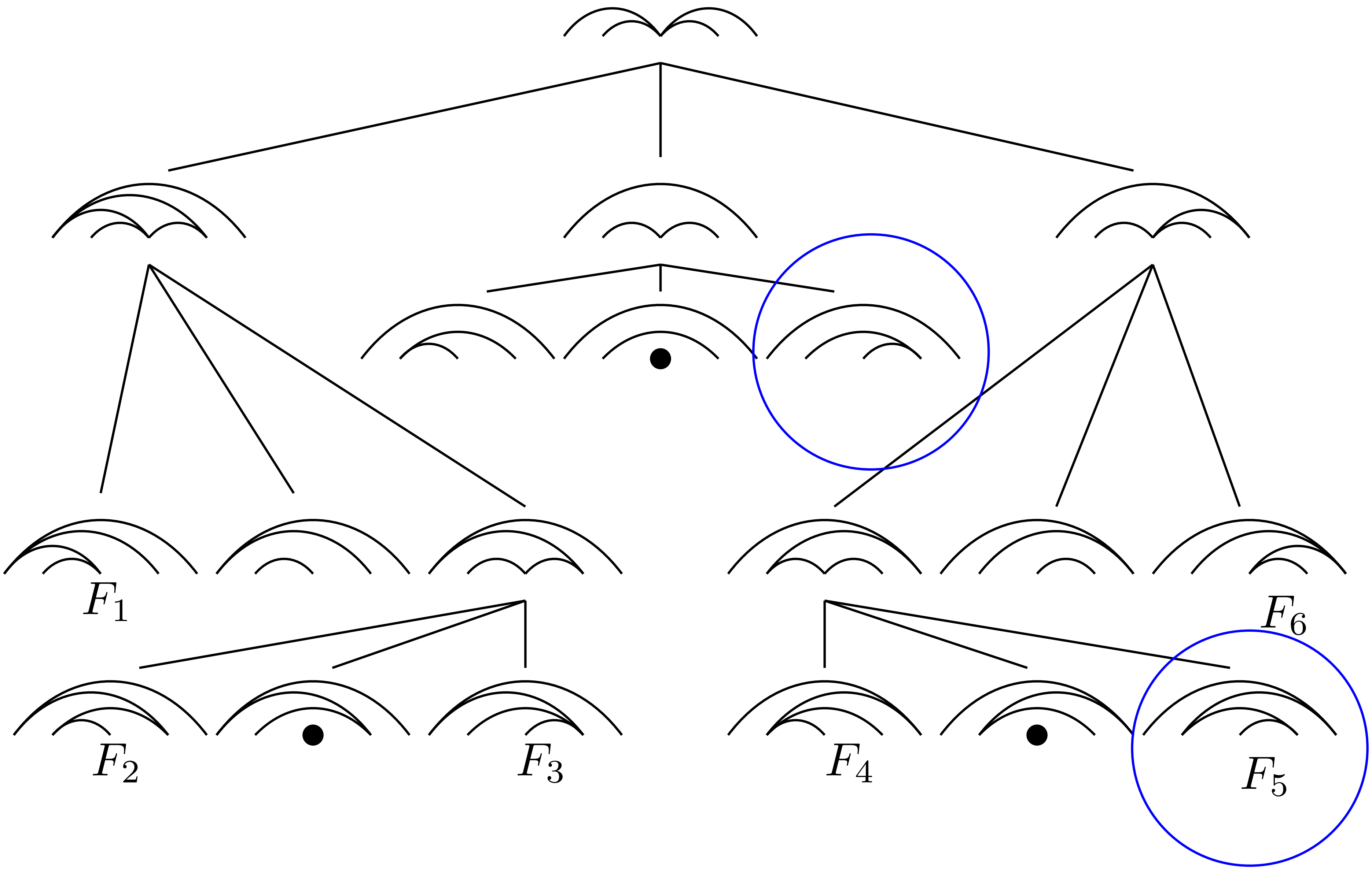

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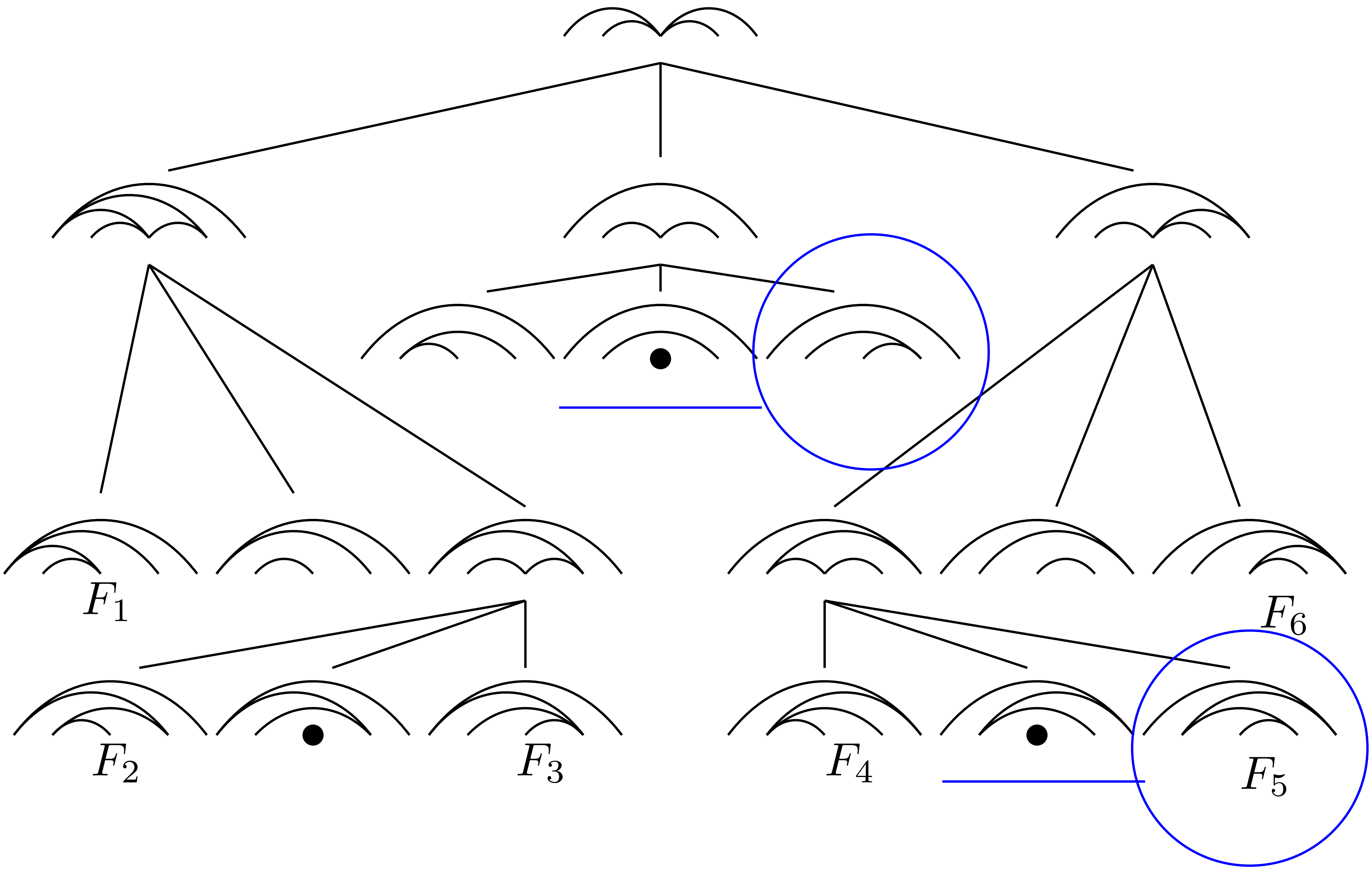

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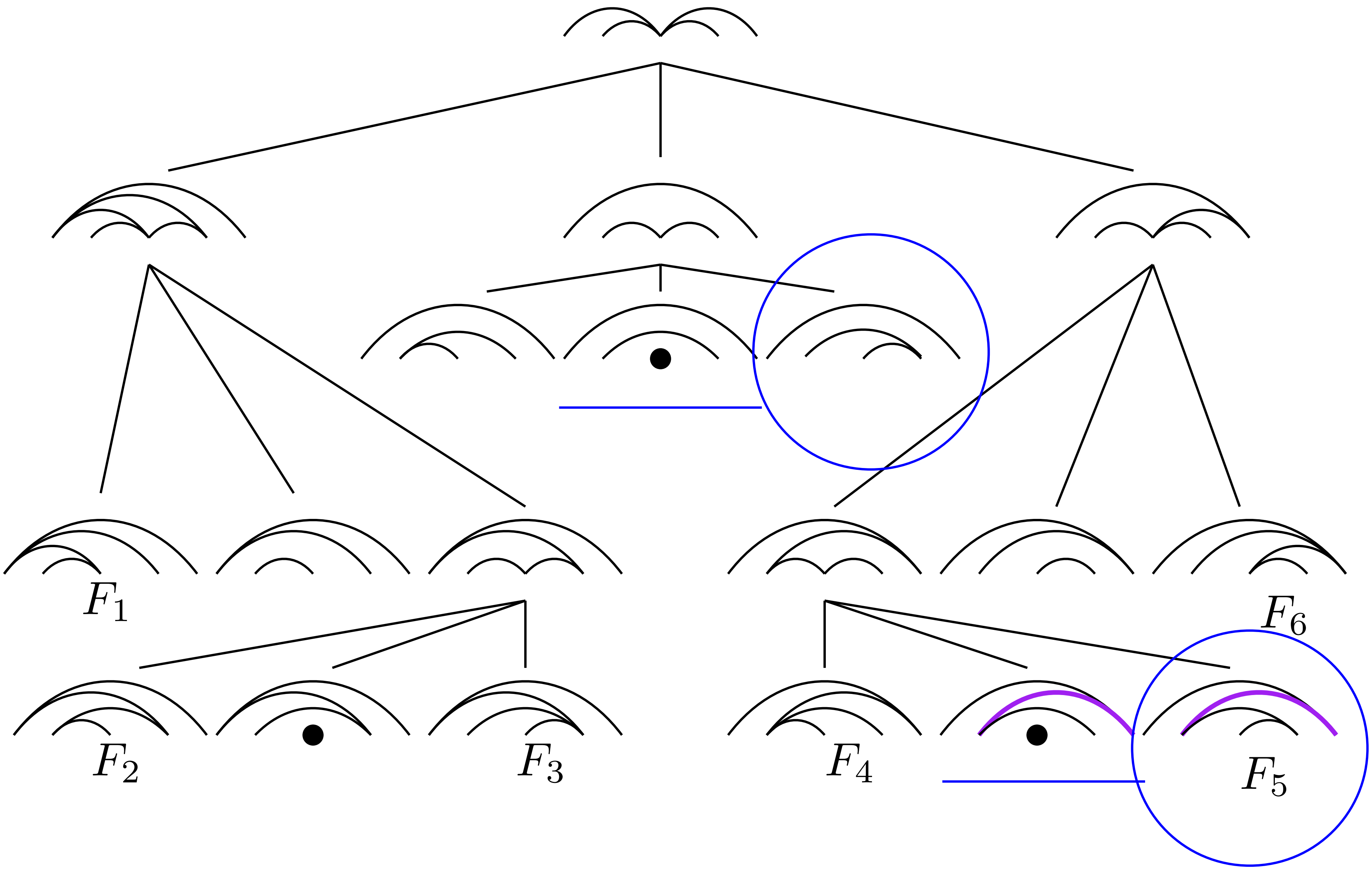

h-polynomials of reduction trees

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Define the h-polynomial of a reduction tree R_G as

$$h(R_G, \beta) = \sum_{i=0}^{\infty} s_i \beta^i,$$

where s_i is the number of full dimensional leaves L of R_G with exactly i preceeding facets.

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Theorem. (M, 2014) For strong embeddable R_G^p we have

$$Q_{R_G^p}(\beta - 1) = h(R_G^p, \beta)$$

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Reduced forms generalize h-polynomials of triangulations

Canonical triangulations of flow polytopes

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Nonnegativity results on reduced forms

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Reduced forms are shifted h-polynomials

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Generalizations of the above theorem and corollary can be used to address a nonnegativity conjecture of Kirillov.

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If a triangulation is shellable...

Recall that the motivation for the definitions of weak and strong embeddability was the shellable triangulation $\mathcal{T}^{\mathcal{O}}$ obtained from $R_G^{\mathcal{O}}$.

If a triangulation is shellable...

...one wonders if it is regular.

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(I am not sure about that, but... I know something else)

The something else

Theorem. (M, 2014) There are ways to play the game and obtain regular and flag triangulations of the flow polytope $\mathcal{F}_{\tilde{G}}$.

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Theorem. (M, 2014) There are ways to play the game and obtain regular and flag triangulations of the flow polytope $\mathcal{F}_{\tilde{G}}$.

This result builds on work of Danilov-Karzanov-Koshevoy.

Happy birthday, Richard!