Skew Schur functions: do their row overlaps determine their *F*-supports?

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`www.facstaff.bucknell.edu/pm040/`

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Preview

Conjecture. For skew shapes A and B,

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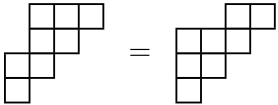
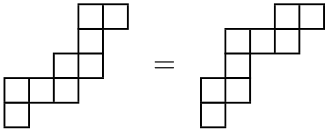
s_A: the skew Schur function for the skew shape A.

Wide Open Question. When is $s_A = s_B$?
Determine necessary and sufficient conditions on shapes of A and B.





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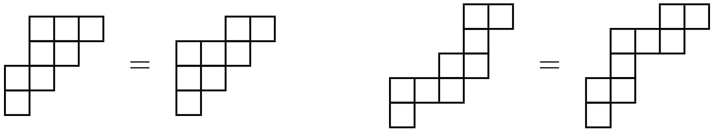

- Lou Billera, Hugh Thomas, Steph van Willigenburg (2004)
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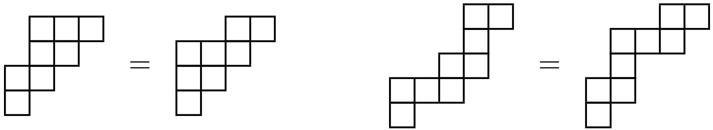

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But this is not the problem I want to talk about....

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Definition [RSvW]. For a skew shape A, let $\text{overlap}_k(i)$ be the number of columns occupied in common by rows $i, i+1, \ldots, i+k-1$.

Then $\text{rows}_k(A)$ is the weakly decreasing rearrangement of $(\text{overlap}_k(1), \text{overlap}_k(2), \ldots)$.

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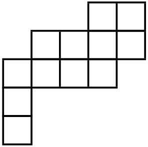
$A =$ 

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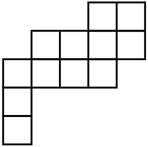
$A =$ 

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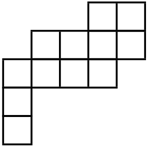
$A =$ 

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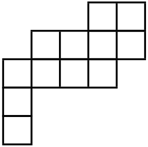
$A =$ 

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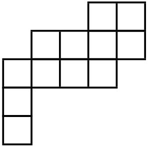
$A =$ 

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- $\text{rows}_k(A) = \emptyset$ for $k > 3$.

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Theorem [RSvW, 2006]. Let A and B be skew shapes.
If $s_A = s_B$, then

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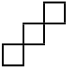
Example. $A =$ 

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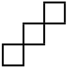
Example. $A =$ 

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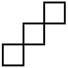
Example. $A =$ 

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Converse is definitely not true.

Main interest: inequalities

Skew Schur functions are Schur-positive:

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In fact, it suffices to assume that $\text{supp}_s(A) \supseteq \text{supp}_s(B)$.

Summary

$s_A - s_B$ is Schur-pos. $\Rightarrow$ $\text{supp}_s(A) \supseteq \text{supp}_s(B)$ $\Rightarrow$

$\text{rows}_k(A) \preccurlyeq \text{rows}_k(B)\ \forall k$

Equivalent choices:

$\text{cols}_\ell(A) \preccurlyeq \text{cols}_\ell(B)\ \forall \ell$

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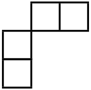
$A =$ 

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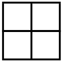
$B =$ 

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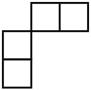
$A =$ 

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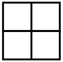
$B =$ 

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Real Goal: Find weaker algebraic conditions on A and B that imply the overlap conditions.

What algebraic conditions are being encapsulated by the overlap conditions?

The quasisymmetric perspective

Theorem [Gessel & Stanley].
s_A: nice expansion in Gessel's fundamental quasisymmetric basis F.

Theorem [McN., 2013].

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$\Downarrow$		$\Downarrow$		
$s_A - s_B$ is F-positive	$\Rightarrow$	$\mathrm{supp}_{\boldsymbol{F}}(A) \supseteq \mathrm{supp}_{\boldsymbol{F}}(B)$	$\Rightarrow$	$\mathrm{rows}_k(A) \preccurlyeq \mathrm{rows}_k(B)\ \forall k$ $\mathrm{cols}_\ell(A) \preccurlyeq \mathrm{cols}_\ell(B)\ \forall \ell$ $\mathrm{rects}_{k,\ell}(A) \leq \mathrm{rects}_{k,\ell}(B)\ \forall k, \ell$

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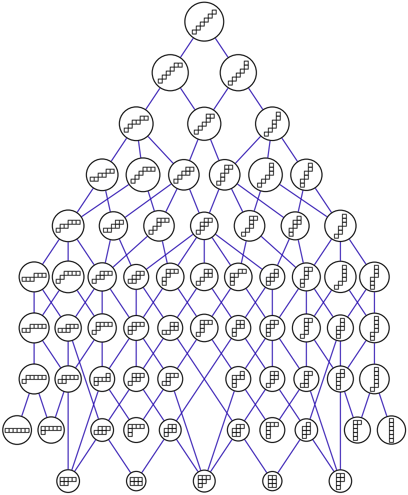
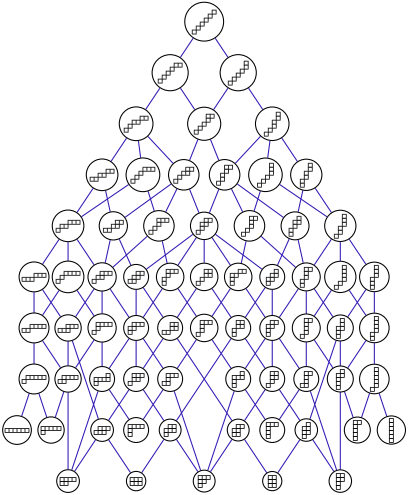
Conjecture. The rightmost implication is if and only if.

$n = 6$ example

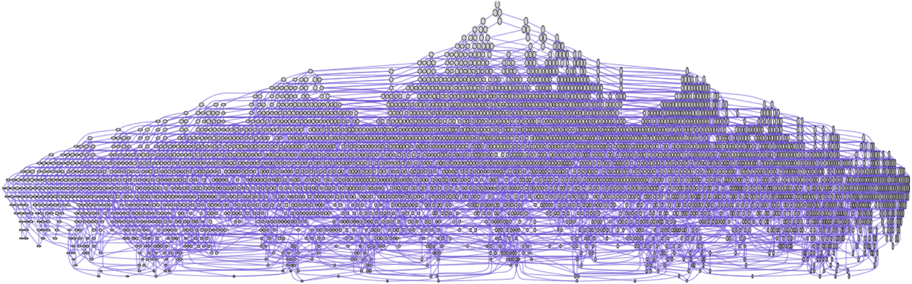


F-support containment



Dual of row overlap dominance

$n = 12$ case has 12,042 edges



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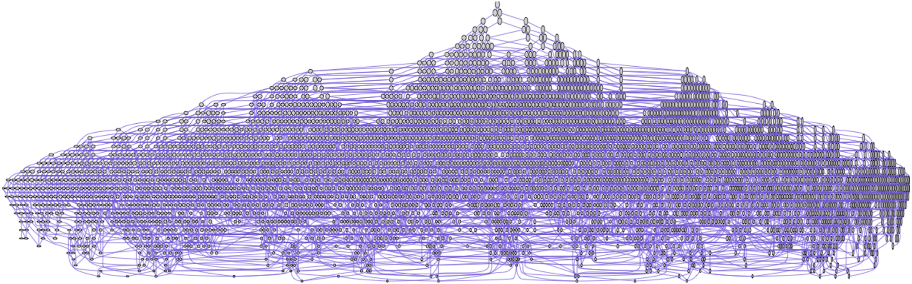

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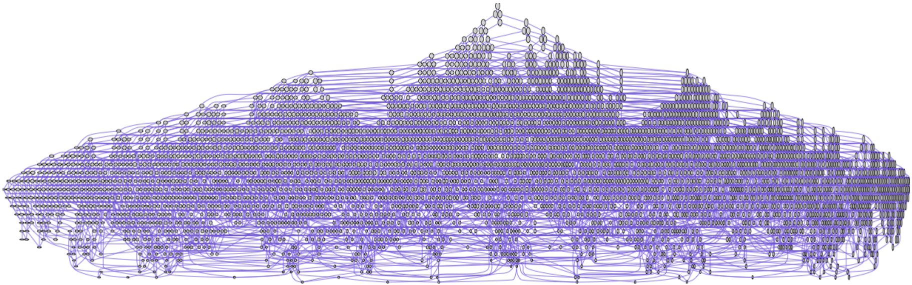

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Conjecture [McN., Morales]. A quasisym skew Saturation Theorem:

$$\mathrm{supp}_F(A) \supseteq \mathrm{supp}_F(B) \iff \mathrm{supp}_F(nA) \supseteq \mathrm{supp}_F(nB).$$

Adding other bases

$s_A - s_B$ is D-positive

$\Downarrow$

$s_A - s_B$ is Schur-pos.
$s_A - s_B$ is S-positive
$\Rightarrow$
$\mathrm{supp}_D(A) \supseteq \mathrm{supp}_D(B)$
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$s_A - s_B$ is M-positive $\Rightarrow$ $\mathrm{supp}_M(A) \supseteq \mathrm{supp}_M(B)$