Cutting polytopes

Nan Li

June 24, 2014 @ Stanley 70

Cutting polytopes

Plan of the talk:

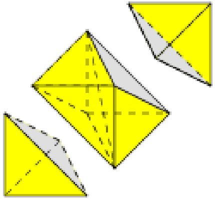
1. first example: hypersimplices (slices of the cube):
   - volume,
   - Ehrhart h-vector,
   - f-vector;
2. second example: edge polytopes;
3. general cutting-polytope framework.

Hypersimplex

The (k, n)th hypersimplex $(0 \leq k < n)$ is

$$\Delta_{k,n} = \{\mathbf{x} \in [0,1]^n \mid k \leq x_1 + \cdots + x_n \leq k+1\}.$$

For example: $\Delta_{k,3}$



For any n-dimensional polytope $\mathcal{P}$, its **normalized volume**: $\text{nvol}(\mathcal{P}) = n!\,\text{vol}(\mathcal{P})$. E.g., the unit cube $C = [0,1]^n$ has $\text{nvol}(C) = n!$.

Normalized volume of $\Delta_{k,n}$

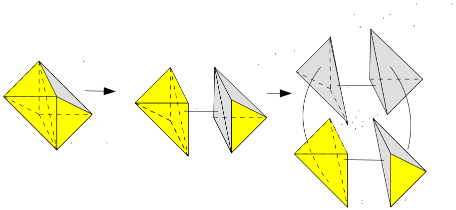
Theorem (Laplace)

$\operatorname{nvol}\Delta_{k,n} = \#\{w \in \mathfrak{S}_n \mid \operatorname{des}(w) = k\}$, *which provides a refinement of* $\operatorname{nvol}([0,1]^n)$.

Stanley gave a bijective proof in 1977 (the shortest paper).

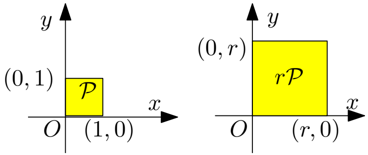
Example

$\operatorname{nvol}(\Delta_{1,3}) = 4$, and $S_3 = \{123, 213, 312, 132, 231, 321\}$.



Ehrhart h-vector

$\mathcal{P} \subset \mathbb{R}^N$: an n-dimensional integral polytope. E.g., for the unit square, we have $\#(r\mathcal{P} \cap \mathbb{Z}^2) = (r+1)^2$, for $r \in \mathbb{P}$.



- **Ehrhart polynomial**: $i(P, r) = \#(rP \cap \mathbb{Z}^N)$.

$$\sum_{r \geq 0} i(\mathcal{P}, r) t^r = \frac{h(t)}{(1-t)^{n+1}}.$$

- **h-polynomial**: $h(t) = h_0 + h_1 t + \cdots + h_n t^n$
- **h-vector**: $(h_0, \ldots, h_n)$. $h_i \in \mathbb{Z}_{\geq 0}$ (Stanley).

$$\sum_{i=0}^{n} h_i = \mathrm{nvol}(\mathcal{P}).$$

Ehrhart h-vector

Ehrhart h-vector of $\mathcal{P}$ provides a refinement of its normalized volume. For example,

- for the unit cube $[0,1]^n$,$h_i = \#\{w \in \mathfrak{S}_n \mid \text{des}(w) = i\}$;
- for the hypersimplex $\text{nvol}\, \Delta_{k,n} = \#\{w \in \mathfrak{S}_n \mid \text{des}(w) = k\}$. $h_i =?$

Key point (Stanley): study the half-open hypersimplex instead of the hypersimplex.

Definition

The **half-open hypersimplex** $\Delta'_{k,n}$ is defined as: $\Delta'_{1,n} = \Delta_{1,n}$ and if $k > 1$,

$$\Delta'_{k,n} = \{\mathbf{x} \in [0,1]^n \mid k < x_1 + \cdots + x_n \leq k+1\}.$$

Ehrhart h-vector of the half-open hypersimplex

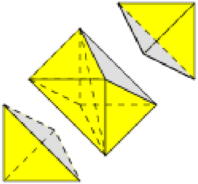
Let $\operatorname{exc}(w) = \#\{i \mid w(i) > i\}$, for any $w \in \mathfrak{S}_n$. For $\Delta'_{k,n}$,

Theorem (L. 2012, conjectured by Stanley)

$h_i = \#\{w \in \mathfrak{S}_n \mid \operatorname{exc}(w) = k \text{ and } \operatorname{des}(w) = i\}$.

Example

w	123	132	213	231	312	321
des	0	1	1	1	1	2
exc	0	1	1	2	1	1



- for $\Delta'_{0,3}$, $k = 0$, $h(t) = 1$;
- for $\Delta'_{1,3}$, $k = 1$, $h(t) = 3t + t^2$;
- for $\Delta'_{2,3}$, $k = 2$, $h(t) = t$.

Ehrhart h-vector of the half-open hypersimplex

Equivalently, the h-polynomial of $\Delta'_{k,n}$ is

$$\sum_{\substack{w \in \mathfrak{S}_n \\ \mathrm{exc}(w)=k}} t^{\mathrm{des}(w)}.$$

Two proofs:

- generating functions, based on a result by Foata and Han;
- by a unimodular shellable triangulation, and

Theorem (Stanley, 1980)

Assume an integral $\mathcal{P}$ has a shellable unimodular triangulation Γ. For each simplex $\alpha \in \Gamma$, let $\#(\alpha)$ be its shelling number. Then h-polynomial of $\mathcal{P}$ is

$$\sum_{\alpha \in \Gamma} t^{\#(\alpha)}.$$

f-vector of the half-open hypersimplex

Let $f_j'^{(n,k)}$ denote the number of j-faces of $\Delta'_{n,k}$.

Property (Hibi, L. and Ohsugi, 2013)

The sum of f-vectors for the half-open hypersimplex (also the f-vector of the hypersimplical decomposition of the unit cube) is

$$\sum_{k=0}^{n-1} f_j'^{(n,k)} = j \cdot 2^{n-j-1} \frac{n+j+2}{n+1} \cdot \binom{n+1}{j+1}.$$

Question

Connection with Chebyshev polynomials?

Fix $j = 2$, $\frac{1}{j} \sum_{k=0}^{n-1} f_j'^{(n,k)} = 1, 7, 32, 120, 400, 1232, 3584, \ldots,$ appears in the triangle table of coefficients of Chebyshev polynomials of the first kind (by OEIS).

General framework

For a polytope $\mathcal{P}$ (assume convex and integral),

1. decomposability can we cut it into two integral subpolytopes with the same dimension by a hyperplane (called separating hyperplane);
2. inheritance do the subpolytopes have the same nice properties as $\mathcal{P}$;
3. equivalence can we count or classify all the different decompositions?

Cutting edge polytopes

Definition

Let G be a connected finite graph with n vertices and edge set $E(G)$. Then define the edge polytope for G to be

$$P_G = \text{conv}\{e_i + e_j \mid (i,j) \in E(G)\}.$$

Combinatorial and algebraic properties of P_G are studied by Ohsugi and Hibi. Based on their results, we study the following question.

Question

Is P_G decomposable or not; can we classify all the separating hyperplanes?

Decomposble edge polytopes

Property (Hibi, L. and Zhang, 2013)

Any separating hyperplanes of edge polytopes have one the following two forms: $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, *with* $a_i \in \{-1, 0, 1\}$, *and for each pair of edge* (i, j), (a_i, a_j) *either*

1. *type I:* $(1, 1)$,$(-1, 1)$ *or* $(-1, -1)$*;*
2. *or type II:* $(1, 0)$,$(0, 0)$ *or* $(-1, 0)$*.*

Property (Funato, L. and Shikama, 2014)

- *Infinitely many graphs in each case: 1) type I not II, 2) type II not I, 3) both type I and II, 4) neither type I nor II.*
- *For bipartite graphs* G*, type I and II are equivalent.*

Decomposable edge polytopes

If P_G is decomposable via a separating hyperplane H, then

- $P_G = P_{G_+} \cup P_{G_-}$ where $G = G_+ \cup G_-$;
- $P_G \cap H = P_{G_+} \cap P_{G_-} = P_{G_0}$ where $G_0 = G_+ \cap G_-$.

Property (Funato, L. and Shikama, 2014)

Characterization of decomposable G in terms of G_0:

- *if G biparitite (both type I and type II), then G_0 has two connected components, both bipartite;*
- *if G not bipartite, then*
  1. *if G is type I, then G_0 is a connected bipartite graph;*
  2. *if G is type II, then G_0 has two connected components, one bipartite, the other not.*

Normal edge polytopes

Definition

We call an integral polytope $P \subset \mathbb{R}^d$ **normal** if, for all positive integers N and for all $\beta \in NP \cap \mathbb{Z}^d$, there exist $\beta_1, \ldots, \beta_N$ belonging to $P \cap \mathbb{Z}^d$ such that $\beta = \sum_i \beta_i$.

Theorem (Hibi, L. and Zhang, 2013)

If P_G can be decomposed into $P_{G_+} \cup P_{G_-}$, then P_G is normal if and only if both P_{G_+} and P_{G_-} are normal.

General framework

Let $\mathcal{P}$ be a convex and integral polytope and not a simplex.

1. Can we cut it into two integral subpolytopes? E.g.,
   - edge polytopes;
   - *order polytopes, chain polytopes (Yes);
   - *Birkhoff polytopes (No).
2. Do the subpolytopes have the same nice properties as $\mathcal{P}$?
   - Algebraic properties: normality, quadratic generation of toric ideals;
   - combinatorial properties: volume, f-vector, h-vector.
3. Can we count or classify all the decomposations? E.g.,
   - *cutting cubes by two hyperplanes;
   - *order polytopes and chain polytopes for some special posets.

* In a recent work with Hibi.