Truncated Stanley symmetric functions and amplituhedron cells

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Reduced words

The symmetric group S_n is generated by $s_1, s_2, \ldots, s_{n-1}$ with relations

$$\begin{aligned} s_i^2 &= 1 \\ s_i s_j &= s_j s_i && \text{if } |i-j| \geq 2 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \end{aligned}$$

A reduced word $\mathbf{i}$ for $w \in S_n$ is a sequence

$$\mathbf{i} = i_1 i_2 \cdots i_\ell \in \{1, 2, \ldots, n-1\}^\ell$$

such that

$$w = s_{i_1} s_{i_2} \cdots s_{i_\ell}$$

and $\ell = \ell(w)$ is minimal.

Stanley symmetric functions

Let $R(w)$ denote the set of reduced words of $w \in S_n$.

Definition (Stanley symmetric function)

$$F_w(x_1, x_2, \ldots) := \sum_{\mathbf{i}=i_1 i_2 \cdots i_\ell \in R(w)} \sum_{\substack{1 \le a_1 \le a_2 \le \cdots \le a_\ell \\ i_j < i_{j+1} \implies a_{j+1} > a_j}} x_{a_1} x_{a_2} \cdots x_{a_\ell}$$

The coefficient of $x_1 x_2 \cdots x_\ell$ in F_w is $|R(w)|$.

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Example

$n = 3$ and $w = w_0 = 321$. We have $R(w) = \{121, 212\}$, so

$$\begin{aligned} F_w &= (x_1 x_2^2 + x_1 x_2 x_3 + \cdots) + (x_1^2 x_2 + x_1 x_2 x_3 + \cdots) \\ &= m_{21} + 2m_{111} \\ &= s_{21} \end{aligned}$$

Symmetry and Schur-positivity

Theorem (Stanley)

F_w is a symmetric function.

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Let $w_0 = n(n-1)\cdots 1$ be the longest permutation in S_n. Then

$$|R(w_0)| = \frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)^1}$$

Theorem (Edelman-Greene, Lascoux-Schützenberger)

F_w is Schur-positive.

Affine Stanley symmetric functions

The affine symmetric group $\tilde{S}_n$ is generated by $s_0, s_1, s_2, \ldots, s_{m-1}$ with relations

$$\begin{aligned} s_i^2 &= 1 & \\ s_i s_j &= s_j s_i & \text{if } |i-j| \geq 2 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} & \end{aligned}$$

where indices are taken modulo n.

The affine Stanley symmetric function $\tilde{F}_w$ is defined by introducing a notion of cyclically decreasing factorizations for $\tilde{S}_n$.

Theorem (L.)

1. *$\tilde{F}_w$ is a symmetric function.*
2. *$\tilde{F}_w$ is "affine Schur"-positive.*

Postnikov's TNN Grassmannian

Take integers $1 \leq k \leq n$. The Grassmannian $\mathrm{Gr}(k, n)$ is the set of k-dimensional subspaces of $\mathbb{C}^n$.

$$X = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \end{pmatrix}$$

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Definition (Totally nonnegative Grassmannian)

The totally nonnegative Grassmannian $\mathrm{Gr}(k,n)_{\geq 0}$ is the locus in the real Grassmannian representable by X such that all $k \times k$ minors are nonnegative.

Also studied by Lusztig, with a different definition.

$\mathrm{Gr}(k, n)_{\geq 0}$ is like a simplex

Let $k = 1$. Then $\mathrm{Gr}(1, n) = \mathbb{P}^{n-1}$ and

$\mathrm{Gr}(1, n)_{\geq 0} = \{(a_1, a_2, \ldots, a_n) \neq \mathbf{0} \mid a_i \in \mathbb{R}_{\geq 0}\}$ modulo scaling by $\mathbb{R}_{>0}$

which can be identified with the simplex

$\Delta_{n-1} := \{(a_1, a_2, \ldots, a_n) \mid a_i \in [0, 1] \text{ and } a_1 + a_2 + \cdots + a_n = 1\}.$

Polytopes and amplituhedra

A convex polytope in $\mathbb{R}^d$ with vertices $v_1, v_2, \ldots, v_n$ is the image of a simplex

$$\Delta_n = \mathrm{conv}(e_1, e_2, \ldots, e_n) \subset \mathbb{R}^{n+1}$$

under a projection map $Z : \mathbb{R}^n \to \mathbb{R}^d$ where

$$Z(e_i) = v_i.$$

Definition (Arkani-Hamed and Trnka's amplituhedron)

An amplituhedron $A(k, n, d)$ in $\mathrm{Gr}(k, d)$ is the image of $\mathrm{Gr}(k, n)_{\geq 0}$ under a (positive) projection map $Z : \mathbb{R}^n \to \mathbb{R}^d$ inducing $Z_{\mathrm{Gr}} : \mathrm{Gr}(k, n) \to \mathrm{Gr}(k, d)$.

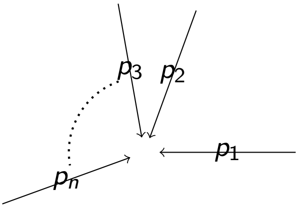
(Caution: Z_{Gr} is not defined everywhere.)

Scattering amplitudes

Arkani-Hamed and Trnka assert that the scattering amplitude (at tree level) in $N = 4$ super Yang-Mills is the integral of a "volume form" ω_{SYM} of an amplituhedron (for $d = k + 4$), and that this form can be calculated by studying "triangulations" of $A(k, n, d)$:

$$\omega_{SYM} = \sum_{\text{cells } Y_f \text{ in a triangulation of } A(k,n,d)} \omega_{Y_f}$$

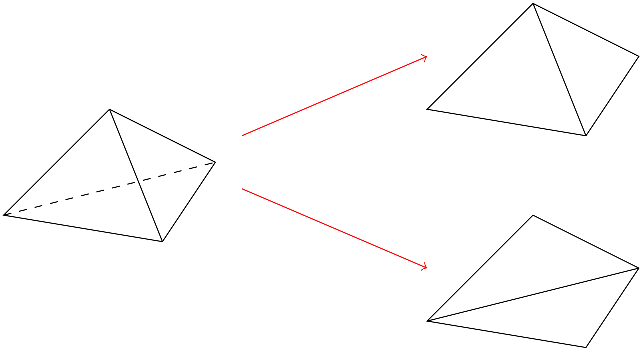
where ω_{Y_f}'s can be considered known.



Scattering amplitude $= A(p_1, p_2, \ldots, p_n)$ "=" $\int \omega_{SYM}$

Triangulating a quadrilateral

Cells of a triangulations of a polytope $Z(\Delta_n)$ can be obtained by looking at the images $Z(F)$ of lower-dimensional faces F of Δ_n.



$\mathbb{R}^3$ or $\mathbb{P}^3(\mathbb{R})$

$\mathbb{R}^2$ or $\mathbb{P}^2(\mathbb{R})$

Positroid cells

Postnikov described the facial structure of $\mathrm{Gr}(k,n)_{\geq 0}$:

$$\mathrm{Gr}(k,n)_{\geq 0} = \bigsqcup_{f \in \mathrm{Bound}(k,n)} (\Pi_f)_{>0}$$

where

$$(\Pi_f)_{>0} \simeq \mathbb{R}^d_{>0}$$

are called positroid cells and

$$\mathrm{Bound}(k,n) \subset \tilde{S}'_n$$

is the set of bounded affine permutations, certain elements in the extended affine symmetric group $\tilde{S}'_n$.

Postnikov gave many objects to index these strata: Grassmann necklaces, decorated permutations, Le-diagrams,...

Partial order

The closure partial order for positroid cells was described by Postnikov and Rietsch.

Theorem (Knutson-L.-Speyer, after Postnikov and Rietsch)

$$\overline{(\Pi_f)_{>0}} = \bigcup_{g \geq f} (\Pi_g)_{>0}$$

where $\geq$ is Bruhat order for the affine symmetric group restricted to $\mathrm{Bound}(k, n)$.

For $k = 1$, the set $\mathrm{Bound}(1, n)$ is in bijection with nonempty subsets of $[n]$, which index faces of the simplex. The partial order is simply containment of subsets.

Triangulations of the amplituhedron

Define the amplituhedron cell

$$(Y_f)_{>0} := Z_{\mathrm{Gr}}((\Pi_f)_{>0}).$$

The map Z_{Gr} exhibits some features that are not present in the polytope case:

1. Even when $Z : \mathbb{R}^n \to \mathbb{R}^d$ is generic, the image $Z_{\mathrm{Gr}}((\Pi_f)_{>0})$ may not have the expected dimension.
2. Even in the dimension-preserving case, the map

$$Z_{\mathrm{Gr}} : (\Pi_f)_{>0} \longmapsto (Y_f)_{>0}$$

can have degree greater than one.

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These questions bring us into the realm of Schubert calculus!

Cohomology of the Grassmannian

The cohomology ring $H^*(\mathrm{Gr}(k,n))$ can be identified with a quotient of the ring of symmetric functions.

$$H^*(\mathrm{Gr}(k,n)) = \bigoplus_{\lambda \subset (n-k)^k} \mathbb{Z} \cdot s_\lambda.$$

- Each irreducible subvariety $X \subset \mathrm{Gr}(k,n)$ has a cohomology class $[X]$.
- The Schur function s_λ is the cohomology classes of the Schubert variety $X_\lambda \subset \mathrm{Gr}(k,n)$.

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Cohomology classes know about:

1. dimension
2. degree (expected number of points of intersection with a generic hyperspace)

When $k = 1$, the cohomology class $[L]$ of a linear subspace $L \subset \mathrm{Gr}(1,n) = \mathbb{P}^{n-1}$ is simply its dimension.

Cohomology class of a positroid variety

The positroid variety Π_f is the Zariski-closure of $(\Pi_f)_{>0}$ in the (complex) Grassmannian $\mathrm{Gr}(k,n)$. Each Π_f is an intersection of rotated Schubert varieties:

$$\Pi_f = X_{I_1} \cap \chi(X_{I_2}) \cap \cdots \cap \chi^{n-1}(X_{I_n})$$

where χ denotes rotation.

Theorem (Knutson-L.-Speyer)

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All faces of Δ_n of the same dimension "look" the same. The faces of $\mathrm{Gr}(k, n)_{\geq 0}$ of the same dimension are abstractly homeomorphic, but don't "look" the same when considered as embedded subsets of the Grassmannian.

Truncation

Suppose

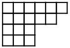
$$G = \sum_{\lambda \subset (n-k)^k} a_\lambda s_\lambda \in H^*(\mathrm{Gr}(k,n)).$$

Define the truncation

$$\tau_d(G) = \sum_{\mu \subset (d-k)^k} a_{\mu^+} s_\mu \in H^*(\mathrm{Gr}(k,d))$$

where μ^+ is obtained from μ by adding $n-d$ columns of length k to the left of μ

$\mu =$ 

$\mu^+ =$ 

An example

Example

Let $k = 2, n = 8, d = 6$. For $w = s_1 s_3 s_5 s_7$ we have

$$F_w = (x_1 + x_2 + \cdots)^4 = s_{(4)} + 3s_{(3,1)} + 2s_{(2,2)} + 3s_{(2,1,1)} + s_{(1,1,1,1)}$$

and

$$\tau_d(F_w) = 2.$$

This is the smallest "physical" example, where the amplituhedron cell is mapped onto with degree 2.

Cohomology class of an amplituhedron variety

Suppose Z is generic. Define the amplituhedron variety

$$Y_f := \overline{Z_{\mathrm{Gr}}(\Pi_f)}.$$

Say f has kinematical support if $\dim Y_f = \dim \Pi_f$.

Theorem (L.)

1. *Suppose $\tau_d(\tilde{F}_f) = 0$. Then f does not have kinematical support.*
2. *Suppose $\tau_d(\tilde{F}_f) \neq 0$. Then f has kinematical support and*

$$[Y_f] = \frac{1}{\kappa}\tau_d(\tilde{F}_f)$$

where κ is the degree of $Z_{\mathrm{Gr}}|_{\Pi_f}$.

3. *Suppose* $\dim(\Pi_f) = \mathrm{Gr}(k,d)$ *and f has kinematical support. Then $\kappa = [s_{(n-d)^k}]\tilde{F}_f$.*

We can also obtain properties of $(Y_f)_{>0}$ since $Y_f = \overline{(Y_f)_{>0}}$.

Truncated Stanley symmetric functions

Problem

Find a "monomial" description of $\tau_d(\tilde{F}_f)$.

Problem

What happens if Z is not generic?

The cyclic polytope is the image of Δ_n under a generic "positive" map.
When Z is not generic, we are replacing the analogue of the cyclic polytope, by an arbitrary polytope.

Problem

The closure partial order for Π_f is affine Bruhat order. What is the closure partial order for Y_f (and how do we define it)?

This should be some kind of "quotient" of Bruhat order.

Happy Birthday, Richard!