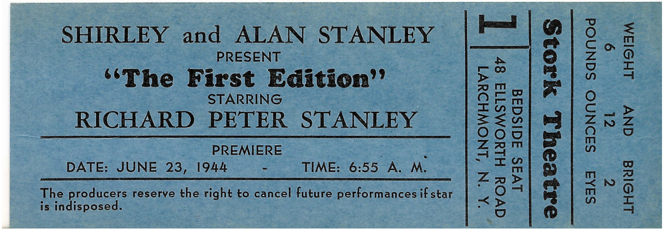
Richard Stanley: The Legend
Part I: Early Years

Curtis Greene

June 23, 2014

Happy Birthday!



Publications of Richard P. Stanley

(September 2011)

1.
Algorithmic Complexity, NASA Report No.32-999 (September 1, 1966).

2.
Zero-square rings, *Pacific J. Math.* **30** (1969), 811-824.

3.
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6.
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A chromatic-like polynomial for ordered sets, in *Proc. Second Chapel Hill Conference on Combinatorial Mathematics and Its Applications* (May, 1970), pp. 421-427.

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The following papers appeared in the Jet Propulsion Laboratory *Space Programs Summary* or *Deep Space Network* journals:

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Left: 1973, Oberwolfach Photo Archive
Right: 1976, Jay Goldman's Photo Archive

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12.

Theory and application of plane partitions, Parts 1 and 2, *Studies in Applied Math.* **50** (1971), 167-188, 259-279.

RESEARCH ANNOUNCEMENTS

The purpose of this department is to provide early announcement of significant new results, with some indications of proof. Although ordinarily a research announcement should be a brief summary of a paper to be published in full elsewhere, papers giving complete proofs of results of exceptional interest are also solicited. Manuscripts more than eight typewritten double spaced pages long will not be considered as acceptable. All papers to be communicated by a Council member should be sent directly to M. H. Protter, Department of Mathematics, University of California, Berkeley, California 94720.

STRUCTURE OF INCIDENCE ALGEBRAS AND THEIR AUTOMORPHISM GROUPS[1]

BY RICHARD P. STANLEY

Communicated by Gian-Carlo Rota, June 9, 1970

Let P be a locally finite ordered set, i.e., a (partially) ordered set for which every segment $[X, Y] = \{Z \mid X \leqq Z \leqq Y\}$ is finite. The *incidence algebra* $I(P)$ of P over a field K is defined [2] as the algebra of all functions from segments of P into K under the multiplication (convolution)

$$fg(X, Y) = \sum_{Z \in [X, Y]} f(X, Z)g(Z, Y).$$

(We write $f(X, Y)$ for $f([X, Y])$.) Note that the algebra $I(P)$ has an identity element δ given by

$$\begin{aligned} \delta(X, Y) &= 1, \quad \text{if } X = Y, \\ &= 0, \quad \text{if } X \neq Y. \end{aligned}$$

THEOREM 1. *Let P and Q be locally finite ordered sets. If $I(P)$ and $I(Q)$ are isomorphic as K-algebras, then P and Q are isomorphic.*

SKETCH OF PROOF. The idea is to show that the ordered set P can be uniquely recovered from $I(P)$. Let the elements of P be denoted X_α, where α ranges over some index set. Then a maximal set of primitive orthogonal idempotents for $I(P)$ consists of the functions e_α defined by

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AMS 1969 subject classifications. Primary 0620, 1650, 1660; Secondary 0510.
Key words and phrases. Ordered set, partially ordered set, incidence algebra, primitive orthogonal idempotents, outer automorphism group, Hasse diagram.
[1] The research was supported by an NSF Graduate Fellowship and by the Air Force Office of Scientific Research AF 44620-70-C-0079.

ON THE FOUNDATIONS OF COMBINATORIAL THEORY (VI): THE IDEA OF GENERATING FUNCTION

PETER DOUBILET, GIAN-CARLO ROTA
and
RICHARD STANLEY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1. Introduction

Since Laplace discovered the remarkable correspondence between set theoretic operations and operations on formal power series, and put it to use with great success to solve a variety of combinatorial problems, generating functions (and their continuous analogues, namely, characteristic functions) have become an essential probabilistic and combinatorial technique. A unified exposition of their theory, however, is lacking in the literature. This is not surprising, in view of the fact that all too often generating functions have been considered to be simply an application of the current methods of harmonic analysis. From several of the examples discussed in this paper it will appear that this is not the case: in order to extend the theory beyond its present reaches and develop new kinds of algebras of generating functions better suited to combinatorial and probabilistic problems, it seems necessary to abandon the notion of group algebra (or semigroup algebra), so current nowadays, and rely instead on an altogether different approach.

The insufficiency of the notion of semigroup algebra is clearly seen in the example of Dirichlet series. The functions

$$n \to 1/n^s \tag{1.1}$$

defined on the semigroup S of positive integers under multiplication, are characters of S. They are not, however, all the characters of this semigroup, nor does there seem to be a canonical way of separating these characters from the rest (see, for example, Hewitt and Zuckerman [32]). In other words, there does not seem to be a natural way of characterizing the algebra of formal Dirichlet series as a subalgebra of the semigroup algebra (eventually completed under a suitable topology) of the semigroup S. In the present theory, however, the algebra of formal Dirichlet series arises naturally from the incidence algebra (definition below) of the lattice of finite cyclic groups, as we shall see.

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- A study of Varshamov codes for asymmetric channels (with M. F. Yoder), JPL Technical Report 32-1526, *DSM*, Vol. XIV (1973), 117-123.

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10.

ABSTRACT

The concept of algorithmic complexity that was introduced by Kolmogorov and expanded by Ofman provides a quantitative means of measuring the complexity of computing a discrete function—i.e., a function with finite domain and range. To be precise in the work reported here, it is assumed that the computation is done by a special type of finite-state machine, a (p, q) automaton. After reviewing the definitions in the field of algorithmic complexity, estimates are made for the maximum possible algorithmic complexity of a discrete function that can be computed on the simplest possible (p, q) automaton, a $(2, 2)$; this allows comparison of the algorithmic complexities relative to (p, q) automata and those relative to $(2, 2)$ automata. Next, bounds are obtained on the complexity of matrix multiplication. Finally, algorithmic complexity is related to the theory of permutation groups on the domain and range of a function, and various criteria are discussed for ensuring a function's having relatively low complexity.

I. INTRODUCTION

In this report, two fundamental problems of computer design are considered theoretically—minimizing the number of components (and, therefore, the cost) of the computer, and minimizing the computation time required. We define a mathematical object called a (p,q) automaton, where p and q are integers ≥ 2, which is to be regarded as an abstract model of a computer. The theory is easily modified to handle many other models of computers. Each (p,q) automaton computes a specific function and has a well defined number of components (stages) and computation time. Our object is to obtain upper and lower bounds on the number of stages and on the computation time required to calculate various functions. The least number of stages and least time required to compute a function f on any (p,q) automaton for fixed p and q is defined to be the algorithmic complexity of f relative to (p,q) automata. A precise definition of algorithmic complexity is given below.

In Section II, we consider the largest possible algorithmic complexity that a function can have; and in Section III, we discuss the complexity of matrix multiplication [over the field $GF(2)$]. Finally, in Section IV, by using the concept of equivalence of functions under permutation groups, we obtain criteria that guarantee that two functions have approximately the same complexity, and that a function has a relatively low complexity.

We begin with the necessary definitions. Let V_p^m denote the space of m-tuples over an alphabet of p symbols. Then, to define the algorithmic complexity of a function f: $V_p^k \rightarrow V_p^n$, we must first define a (p,q) automaton that computes f.

A. Definition of (p, q) Automaton

A (p,q) automaton, with $p,q \geq 2$, is an autonomous finite-state machine built of storage elements and gates. The storage elements, or *stages*, can be in one of p states at any time, corresponding to the p symbols of the alphabet. The gates determine the next state of the stages as a function of the immediately preceding states of, at most, q stages. In digital circuit terminology, there is, at most, one level of gating, and the gates have a

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ZERO SQUARE RINGS

RICHARD P. STANLEY

A ring R for which $x^2 = 0$ for all $x \in R$ is called a *zero-square ring*. Zero-square rings are easily seen to be locally nilpotent. This leads to two problems: (1) constructing finitely generated zero-square rings with large index of nilpotence, and (2) investigating the structure of finitely generated zero-square rings with given index of nilpotence. For the first problem we construct a class of zero-square rings, called *free* zero-square rings, whose index of nilpotence can be arbitrarily large. We show that every zero-square ring whose generators have (additive) orders dividing the orders of the generators of some free zero-square ring is a homomorphic image of the free ring. For the second problem, we assume $R^n \neq 0$ and obtain conditions on the additive group R_+ of R (and thus also on the order of R). When $n = 2$, we completely characterize R_+. When $n > 3$ we obtain the smallest possible number of generators of R_+, and the smallest number of generators of order 2 in a minimal set of generators. We also determine the possible orders of R.

Trivially every null ring (that is, $R^2 = 0$) is a zero-square ring. From every nonnull commutative ring S we can make $S \times S \times S$ into a nonnull zero square ring R by defining addition componentwise and multiplication by

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (0, 0, x_1y_2 - x_2y_1) .$$

In this example we always have $R^3 = 0$. If S is a field, then R is an algebra over S. Zero-square algebras over a field have been investigated in [1].

2. **Preliminaries.** Every zero-square ring is anti-commutative, for $0 = (x+y)^2 = x^2 + xy + yx + y^2 = xy + yx$. From anti-commutativity we get $2R^3 = 0$, for $yzx = y(-xz) = -(yx)z = xyz$ and $(yz)x = -x(yz)$, so $2xyz = 0$ for all $x, y, z \in R$. It follows that a zero-square ring R is commutative if and only if $2R^2 = 0$.

If R is a zero-square ring with n generators, then any product of $n+1$ generators must contain two factors the same. By applying anti-commutativity we get a square factor in the product; hence $R^{n+1} = 0$. In particular, every zero-square ring is locally nilpotent.

If G is a finitely generated abelian group, then by the fundamental theorem on abelian groups we have

$$(1) \qquad G = C_{a_1} \oplus \cdots \oplus C_{a_n},\ a_i \mid a_{i+1} \text{ for } 1 \leqq i \leqq k-1 ,\quad a_{k+1} = \cdots = a_n = \infty ,$$

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1964	William Zame	Faculty, SUNY Buffalo
1965	Michael Aschbacher Richard P. Stanley	CIT Faculty, Won Cole Prize in Algebra 1980 Shaler Arthur Hanish Professor of Mathematics Faculty, MIT Fairchild Scholar at Caltech 1986
1966	(no award)	
1967	James Maiorana Alan J. Schwenk	Inst. For Defense Analyses (Princeton)

		Faculty, Western Michigan University
1968	Michael Fredman	Faculty, Appl. Physics & Info. Sci. (UCSD)
1969	Robert E. Tarjan	N.E.C. Research Institute & Dept of Computer Science, Princeton, NJ
1970	(no award)	
1971	(no award)	
1972	Daniel J. Rudolph	Faculty, University of Maryland
1973	Bruce Reznick	Faculty, University of Illinois, Urbana-Champaign
1974	David S. Dummit	Faculty, University of Vermont (Burlington)
1975	James B. Shearer	IBM
1976	John Gustafson Albert Wells, Jr. Hugh Woodin	Appl. Spec. Floating Pt. Syst. (Portland, OR) Yale Law School Faculty, Berkeley Presidential Young Investigator Award 1985 Carp Prize in 1988

1977	Thos. G. Kennedy	Faculty, University of Arizona
1978	(no award)	
1979	(no award)	
1980	Eugene Y. Loh John Stembridge Robert Weaver	Sun Microsystems Faculty, University of Michigan Ohio State University
1981	Daniel Gordon Peter Shor	C.C.R., La Jolla, CA AT&T Bell Labs
1982	Forrest Quinn Thiennu H. Vu	MIT, Cambridge, MA University of California, San Francisco, San Francisco General Hospital
1983	Mark Purtill Vipul Perival	Texas A&M, Kingsville Princeton, NJ
1984	Bradley Brock Alan Murray	C.C.R., Princeton, NJ Super Computing Research Center, MD
1985	Charles Nainan	University of Illinois, Champaign, IL

1986	Arthur Duval Everette Howe	Faculty, Dept. of Math Sci., University of Texas, El Paso Berkeley, CA
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1988	Laura Anderson Eric Babson	Gahanna, OH MIT, Cambridge, MA
1989	James Coykendall, IV	Gatlinburg, TN
1990	(no award)	
1991	Allen Knutson	UC Berkeley
1992	Robert Southworth Michael Maxwell	
1993	(no award)	
1994	Julian Jamison	Kellogg School of Management, Northwestern Univ.
1995	(no award)	
1996	Winston Yang	Clarke College, Dubuque, Iowa
1997	Marc A. Coram Mason Alexander Porter	University of Chicago Georgia Institute of Technology
1998	(no award)	
1999	Scott Carnahan Melvin Boon-Tiong Leok	UC Berkeley
2000	Keshav Dani	UC Berkeley

	Peter Gerdes	UC Berkeley
2001	(no award)	
2002	Suhas Nayak	Stanford University
2003	(no award)	
2004	Ameera Chowdhury Po-Shen Loh	 Cambridge University
2005	Patrick Hummel Timothy Nguyen Trevor Wilson	 MIT Berkeley
2006	Timothy Nguyen	MIT
2007	David Renshaw Jed Yang	Carnegie Mellon University UCLA
2008	Phillip Perepelitsky	UC Santa Cruz
2009	Jeffrey Kuan Ila Varma	 University of Leiden (Fullbright Scholar)
2010	Domenic Denicola	
2011	Jeffrey Lin	
2012	Alexandra Mustat	
2013	Andrew Zucker	

JOURNAL OF COMBINATORIAL THEORY **10**, 74–79 (1971)

On the Number of Open Sets of Finite Topologies

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Received March 26, 1969

ABSTRACT

Recent papers of Sharp [4] and Stephen [5] have shown that any finite topology with n points which is not discrete contains $\leqslant (3/4)2^n$ open sets, and that this inequality is best possible. We use the correspondence between finite T_0-topologies and partial orders to find all non-homeomorphic topologies with n points and $\geqslant (7/16)2^n$ open sets. We determine which of these topologies are T_0, and in the opposite direction we find finite T_0 and non-T_0 topologies with a small number of open sets. The corresponding results for topologies on a finite set are also given.

If X is a finite topological space, then X is determined by the minimal open sets U_x containing each of its points x. X is a T_0-space if and only if $U_x = U_y$ implies $x = y$ for all points x, y in X. If X is not T_0, the space $\hat{X}$ obtained by identifying all points $x, y \in X$ such that $U_x = U_y$, is a T_0-space with the same lattice of open sets as X. Topological properties of the operation $X \to \hat{X}$ are discussed by McCord [3]. Thus for the present we restrict ourselves to T_0-spaces.

If X is a finite T_0-space, define $x \leqslant y$ for $x, y \in X$ whenever $U_x \subseteq U_y$. This defines a partial ordering on X. Conversely, if P is any partially ordered set, we obtain a T_0-topology on P by defining $U_x = \{y/y \leqslant x\}$ for $x \in P$. The open sets of this topology are the *ideals* (also called semi-ideals) of P, i.e., subsets Q of P such that $x \in Q$, $y \leqslant x$ implies $y \in Q$.

Let P be a finite partially ordered set of order p, and define $\omega(P) = j(P)\,2^{-p}$, where $j(P)$ is the number of ideals of P. If Q is another finite partially ordered set, let $P + Q$ denote the disjoint union (direct sum) of P and Q. Then $j(P + Q) = j(P)j(Q)$ and $\omega(P + Q) = \omega(P)\,\omega(Q)$. Let H_p denote the partially ordered set consisting of p disjoint points, so $\omega(H_p) = 1$.

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JOURNAL OF COMBINATORIAL THEORY (A) 14, 53–65 (1973)

The Conjugate Trace and Trace of a Plane Partition

RICHARD P. STANLEY*

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Communicated by the Late Theodore S. Motzkin

Received November 13, 1970

The *conjugate trace* and *trace* of a plane partition are defined, and the generating function for the number of plane partitions π of n with $\leqslant r$ rows and largest part $\leqslant m$, with conjugate trace t (or trace t, when $m = \infty$), is found. Various properties of this generating function are studied. One consequence of these properties is a formula which can be regarded as a q-analog of a well-known result arising in the representation theory of the symmetric group.

1. INTRODUCTION

A *plane partition* π of n is an array of non-negative integers,

$$\begin{matrix} n_{11} & n_{12} & n_{13} & \cdots \\ n_{21} & n_{22} & n_{23} & \cdots \\ \vdots & \vdots & \vdots & \end{matrix} \tag{1}$$

for which $\sum_{i,j} n_{ij} = n$ and the rows and columns are in non-increasing order:

$$n_{ij} \geqslant n_{(i+1)j}, \qquad n_{ij} \geqslant n_{i(j+1)}, \qquad \text{for all} \quad i, j \geqslant 1.$$

The non-zero entries $n_{ij} > 0$ are called the *parts* of π. If there are λ_i parts in the i-th row of π, so that, for some r,

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r > \lambda_{r+1} = 0,$$

then we call the partition $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_r$ of the integer $p = \lambda_1 + \cdots + \lambda_r$ the *shape* of π, denoted by λ. We also say that π has

* The research was partially supported by an NSF Graduate Fellowship at Harvard University and by the Air Force Office of Scientific Research AF 44620-70-C-0079.

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Theory and Application of Plane Partitions: Part 1

By Richard P. Stanley

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Theory and Application of Plane Partitions. Part 2

By Richard P. Stanley

IV. Enumeration of column-strict plane partitions

14. *Part restrictions*

We are now ready to apply our theory of Schur functions to the enumeration of plane partitions. The first such results were obtained by MacMahon [9], using an entirely different technique.

If p_n is the number of plane partitions of n with a certain property, we say that the *generating function* for these plane partitions is the (formal) power series

$$\Sigma p_n x^n. \tag{46}$$

We will regard the plane partitions counted by (46) to be *enumerated* if an explicit expression can be found for (46). Only in rare cases can an explicit expression be found for p_n itself.

We will employ the notation

$$\begin{aligned}(\boldsymbol{k}) &= 1 - x^k \\ (\boldsymbol{k})! &= (\boldsymbol{1})(\boldsymbol{2})\dots(\boldsymbol{k})\end{aligned} \tag{47}$$

For instance, the generating function for plane partitions with ≤ 1 row (i.e., ordinary partitions) is $\prod_{n=1}^{\infty}(\boldsymbol{n})^{-1}$, a well-known result of Euler (see Hardy and Wright [6, Ch. 19]). The generating function for plane partitions with ≤ 1 row and ≤ 2 columns is $1/(\boldsymbol{2})!$, and here we have the explicit expression $p_n = \frac{1}{4}(2n + 3 + (-1)^n)$. In these examples, the generating functions can be determined by "inspection." For more general types of plane partitions, the generating functions still have a simple form, but there appears to be no "obvious" reason why this is so.

14.1. THEOREM. (Bender and Knuth [18]). *Let S be any subset of the positive integers. The generating function for column-strict plane partitions whose parts all lie in S is*

$$\prod_{i \in S}(\boldsymbol{i})^{-1} \prod_{\substack{i,j \in S \\ i<j}} (\boldsymbol{i}+\boldsymbol{j})^{-1}.$$

ORDERED STRUCTURES AND PARTITIONS*

by

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SUPERSOLVABLE LATTICES[1])

R. P. STANLEY

1. Introduction

We shall investigate a certain class of finite lattices which we call *supersolvable lattices* (for a reason to be made clear shortly). These lattices L have a number of interesting combinatorial properties connected with the counting of chains in L, which can be formulated in terms of Möbius functions. I am grateful to the referee for his helpful suggestions, which have led to more general results with simpler proofs.

1.1. DEFINITION. Let L be a finite lattice and Δ a maximal chain of L. If, for every chain K of L, the sublattice generated by K and Δ is distributive, then we call Δ an *M-chain* of L; and we call (L, Δ) a *supersolvable lattice* (or *SS-lattice*).

Sometimes, by abuse of notation, we refer to L itself as an *SS*-lattice, the *M*-chain Δ being tacitly assumed.

A wide variety of examples of *SS*-lattices is given in the next section. In this section, we define two fundamental concepts associated with *SS*-lattices, viz., the *rank-selected Möbius invariant* and the *set of Jordan-Holder permutations.* We shall outline their connection with each other, together with some consequences. Proofs will be given in later sections.

If L is an *SS*-lattice whose *M*-chain Δ has length n (or cardinality $n+1$), then every maximal chain K of L has length n since all maximal chains of the distributive lattice generated by Δ and K have the same length. Hence if $\hat{0}$ denotes the bottom element and $\hat{1}$ the top element of L, then L has defined on it a unique *rank function* $r: L \to \{0, 1, 2, \dots n\}$ satisfying $r(\hat{0})=0$, $r(\hat{1})=n$, $r(y)=r(x)+1$ if *y covers x* (i.e., $y>x$ and no $z \in L$ satisfies $y>z>x$). Let S be any subset of the set $\mathbf{n-1}$, where we use the notation

$$\mathbf{k} = \{1, 2, \dots, k\}.$$

We will also write $S=\{m_1, m_2, \dots, m_s\}^<$ to signify that $m_1<m_2<\dots<m_s$. Define $\alpha(S)$ to be the number of chains

$$\hat{0} = y_0 < y_1 < \dots < y_s < \hat{1}$$

in L such that $r(y_i)=m_i$, $i=1, 2, \dots, s$. In particular, if $S=\{m\}$, then $\alpha(S)$ is the number

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FINITE LATTICES AND JORDAN-HÖLDER SETS[1)]

RICHARD P. STANLEY

1. Introduction

In this paper we extend some aspects of the theory of 'supersolvable lattices' [3] to a more general class of finite lattices which includes the upper-semimodular lattices. In particular, all conjectures made in [3] concerning upper-semimodular lattices will be proved. For instance, we will prove that if L is finite upper-semimodular and if L' denotes L with any set of 'levels' removed, then the Möbius function of L' alternates in sign. Familiarity with [3] will be helpful but not essential for the understanding of the results of this paper. However, many of the proofs are identical to the proofs in [3] (once the machinery has been suitably generalized) and will be omitted.

2. Admissible labelings

Let L be a finite lattice with bottom $\hat{0}$ and top $\hat{1}$, such that every maximal chain of L has the same length n. Hence L has a rank function ϱ satisfying $\varrho(\hat{0})=0$, $\varrho(\hat{1})=n$, and $\varrho(y)=1+\varrho(x)$ whenever y covers x in L. We call L a *graded* lattice.

Let I denote the set of join-irreducible elements of L. A *labeling* ω of L is any map $\omega: I \to \mathbf{P}$, where $\mathbf{P}$ denotes the positive integers. A labeling ω is said to be *natural* if $z, z' \in I$ and $z \leq z'$ implies $\omega(z) \leq \omega(z')$. If $x<y$ in L and ω is a fixed labeling of L, define

$$\gamma(x, y) = \min\{\omega(z) \mid z \in I, x < x \vee z \leq y\}.$$

Thus, $\gamma(x, y)$ is the least label of a join-irreducible which is less than or equal to y but not less than or equal to x. Note that $\gamma(x, y)$ is always defined since y is a join of join-irreducibles. We are now able to make the key definition of this paper. A labeling ω is said to be *admissible* if whenever $x<y$ in L, there is a *unique* unrefinable chain $x=x_0<x_1<\cdots<x_m=y$ between x and y (so $m=\varrho(y)-\varrho(x)$) such that

$$\gamma(x_0, x_1) \leq \gamma(x_1, x_2) \leq \cdots \leq \gamma(x_{m-1}, x_m). \tag{1}$$

We then call the pair (L, ω) an *admissible* lattice. Our motivation for this definition is that admissibility seems to be the weakest condition for which Theorem 3.1 holds.

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ACYCLIC ORIENTATIONS OF GRAPHS*

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Abstract. Let G be a finite graph with p vertices and χ its chromatic polynomial. A combinatorial interpretation is given to the positive integer $(-1)^p \chi(-\lambda)$, where λ is a positive integer, in terms of acyclic orientations of G. In particular, $(-1)^p \chi(-1)$ is the number of acyclic orientations of G. An application is given to the enumeration of labeled acyclic digraphs. An algebra of full binomial type, in the sense of Doubilet–Rota–Stanley, is constructed which yields the generating functions which occur in the above context.

1. The chromatic polynomial with negative arguments

Let G be a finite graph, which we assume to be without loops or multiple edges. Let $V = V(G)$ denote the set of vertices of G and $X = X(G)$ the set of edges. An edge $e \in X$ is thought of as an unordered pair $\{u, v\}$ of two distinct vertices. The integers p and q denote the cardinalities of V and X, respectively. An *orientation* of G is an assignment of a direction to each edge $\{u, v\}$, denoted by $u \to v$ or $v \to u$, as the case may be. An orientation of G is said to be *acyclic* if it has no directed cycles.

Let $\chi(\lambda) = \chi(G, \lambda)$ denote the chromatic polynomial of G evaluated at $\lambda \in \mathbf{C}$. If λ is a non-negative integer, then $\chi(\lambda)$ has the following rather unorthodox interpretation.

Proposition 1.1. $\chi(\lambda)$ *is equal to the number of pairs* $(\sigma, \mathcal{O})$, *where* σ *is any map* $\sigma : V \to \{1, 2, ..., \lambda\}$ *and* $\mathcal{O}$ *is an orientation of G, subject to the two conditions*:

(a) *The orientation* $\mathcal{O}$ *is acyclic.*

(b) *If* $u \to v$ *in the orientation* $\mathcal{O}$, *then* $\sigma(u) > \sigma(v)$.

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LINEAR HOMOGENEOUS DIOPHANTINE EQUATIONS AND MAGIC LABELINGS OF GRAPHS

RICHARD P. STANLEY

1. Introduction. Let G be a finite graph allowing loops and multiple edges. Hence G is a *pseudograph* in the terminology of [10]. We shall denote the set of vertices of G by V, the set of edges by E, the number $|V|$ of vertices by p, and the number $|E|$ of edges by q. Also if an edge e is incident to a vertex v, we write $v \in e$. Any undefined graph-theoretical terminology used here may be found in [10]. A *magic labeling of G of index r* is an assignment $L : E \to \{0, 1, 2, \cdots\}$ of a nonnegative integer $L(e)$ to each edge e of G such that for each vertex v of G the sum of the labels of all edges incident to v is r (counting each loop at v once only). In other words,

$$\sum_{e : v \in e} L(e) = r, \qquad \text{for all } v \in V. \tag{1}$$

For each edge e of G let z_e be an indeterminate and let z be an additional indeterminate. For each vertex v of G define the homogeneous linear form

$$P_v = z - \sum_{e : v \in e} z_e , \qquad v \in V, \tag{2}$$

where the sum is over all e incident to v. Hence by (1) a magic labeling L of G corresponds to a solution of the system of equations

$$P_v = 0, \qquad v \in V, \tag{3}$$

in nonnegative integers (the value of z is the index of L). Thus the theory of magic labelings can be put into the more general context of *linear homogeneous diophantine equations*. Many of our results will be given in this more general context and then specialized to magic labelings.

It may happen that there are edges e of G that are always labeled 0 in any magic labeling. If this is the case, then these edges may be ignored in so far as studying magic labelings is concerned; so we may assume without loss of generality that for any edge e of G there is a magic labeling L of G for which $L(e) > 0$. We then call G a *positive* pseudograph. If in a magic labeling L of G every edge receives a positive label, then we call L a *positive* magic labeling. If L_1 and L_2 are magic labelings, we define their *sum* $L = L_1 + L_2$ by $L(e) = L_1(e) + L_2(e)$ for every edge e of G. Clearly if L_1 and L_2 are of index r_1 and r_2, then L is magic of index $r_1 + r_2$. Now note that every positive pseudograph G possesses a positive magic labeling L, e.g., for each edge e of G let L_e be a magic labeling positive on e, and let $L = \sum L_e$.

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MAGIC LABELINGS OF GRAPHS, SYMMETRIC MAGIC SQUARES, SYSTEMS OF PARAMETERS, AND COHEN-MACAULAY RINGS

RICHARD P. STANLEY

1. Introduction.

Let Γ be a finite graph allowing loops and multiple edges, so that Γ is a *pseudograph* in the terminology of [5]. Let $E = E(\Gamma)$ denote the set of edges of Γ and $\mathbf{N}$ the set of non-negative integers. A *magic labeling of* Γ *of index* r is an assignment $L : E \to \mathbf{N}$ of a non-negative integer $L(e)$ to each edge e of Γ such that for each vertex v of Γ, the sum of the labels of all edges incident to v is r (counting each loop at v once only). We will assume that we have chosen some fixed ordering $e_1, e_2, \cdots, e_q$ of the edges of Γ; and we will identify the magic labeling L with the vector $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_q) \in \mathbf{N}^q$, where $\alpha_i = L(e_i)$.

Let $H_\Gamma(r)$ denote the number of magic labelings of Γ of index r. It may happen that there are edges e of Γ that are always labeled 0 in any magic labeling. If these edges are removed, we obtain a pseudograph Δ satisfying the two conditions: (i) $H_\Gamma(r) = H_\Delta(r)$ for all $r \in \mathbf{N}$, and (ii) some magic labeling L of Δ satisfies $L(e) > 0$ for every edge e of Δ. We call a pseudograph Δ satisfying (ii) a *positive pseudograph*. By (i) and (ii), in studying the function $H_\Gamma(r)$ it suffices to assume that Γ is positive. A magic labeling L of Γ satisfying $L(e) > 0$ for all edges $e \in E(\Gamma)$ is called a *positive magic labeling*. Any undefined graph theory terminology used in this paper may be found in any textbook on graph theory, e.g., [5].

In [14] the following two theorems were proved.

THEOREM 1.1. [14, Thm. 1.1]. *Let* Γ *be a finite pseudograph. Then either* $H_\Gamma(r) = \delta_{0r}$ *(the Kronecker delta), or else there exist polynomials* $P_\Gamma(r)$ *and* $Q_\Gamma(r)$ *such that* $H_\Gamma(r) = P_\Gamma(r) + (-1)^r Q_\Gamma(r)$ *for all* $r \in \mathbf{N}$.

THEOREM 1.2 [14, Prop. 5.2]. *Let* Γ *be a finite positive pseudograph with at least one edge. Then* $\deg P_\Gamma(r) = q - p + b$, *where* q *is the number of edges of* Γ, p *the number of vertices, and* b *the number of connected components which are bipartite.*

For reasons which will become clear shortly, we define the *dimension* of Γ, denoted $\dim \Gamma$, by $\dim \Gamma = 1 + \deg P_\Gamma(r)$. In [14, p. 630], the problem was raised of obtaining a reasonable upper bound on $\deg Q_\Gamma(r)$. It is trivial that $\deg Q_\Gamma(r) \leq \deg P_\Gamma(r)$, and [14, Cor. 2.10] gives a condition for $Q_\Gamma(r) = 0$. Empirical evidence suggests that if Γ is a "typical" pseudograph, then $\deg Q_\Gamma(r)$

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Combinatorial Reciprocity Theorems*

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CONTENTS

A *combinatorial reciprocity theorem* is a result which establishes a kind of duality between two related enumeration problems. This rather vague concept will become clearer as more and more examples of such theorems are given. We will begin with simple, known results and see to what extent they can be generalized. The culmination of our efforts will be the "Monster Reciprocity Theorem" of Section 10,

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From *Chicago (movie version, 2002)*, "Mama's Good To You" (excerpt) Sung by Queen Latifah.

Ask any of the chickies in my pen
They'll tell you I'm the biggest Mutha. . . .Hen
I love them all and all of them love me -
Because the system works;
the system called reciprocity!

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Mama's good to you!

Let's all stroke together
Like the Princeton crew -
When you're strokin' Mama
Mama's strokin' you!

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HAPPY BIRTHDAY RICHARD!