Two EC tidbits

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Dartmouth College

In honor of Richard Stanley's 70th birthday


sergio

Tidbit 1

A bijection for pairs of non-crossing lattice paths

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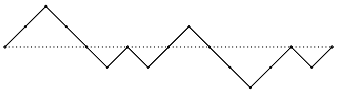
Stanley #70

Grand Dyck paths and Dyck path prefixes

We consider two kinds of lattice paths with steps $U = (1, 1)$ and $D = (1, -1)$ starting at the origin.

Grand Dyck paths end on the x-axis (or at height 1 for paths of odd length):



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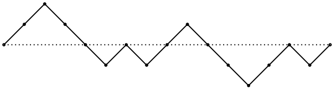

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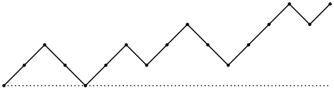
Dyck path prefixes never go below x-axis, but can end at any height:



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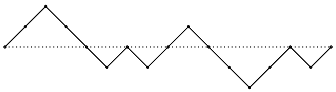

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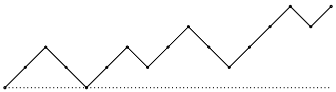

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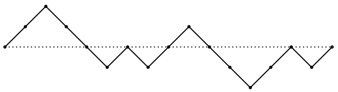

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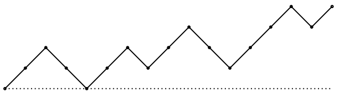

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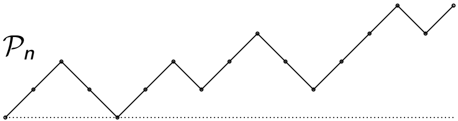
Not so trivial: $|\mathcal{P}_n| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

A classical bijection $\xi : \mathcal{P}_n \to \mathcal{G}_n$



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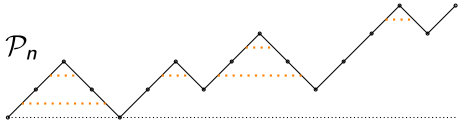

- Match Us and Ds that "face" each other.

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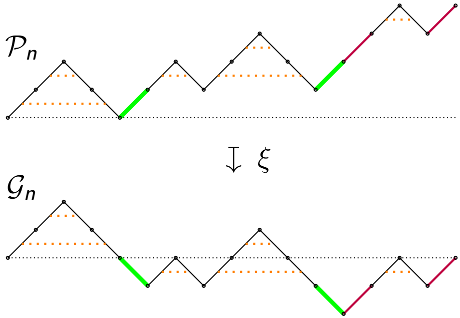

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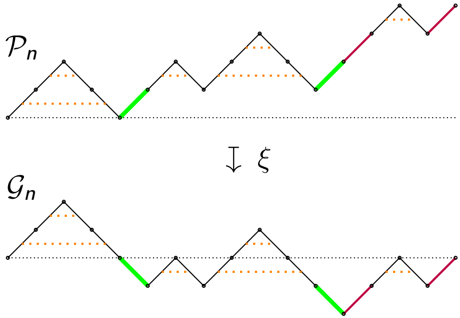

- Match Us and Ds that "face" each other.
- Among the unmatched steps (which are all Us), change the lefmost half of them into D steps.

To reverse, simply change unmatched Ds into Us.

k-tuples of non-crossing paths

For lattice paths P and Q, write $Q \leq P$ if Q is weakly below P.

$(P_1, \dots, P_k)$ is a k-tuple of **nested** paths if $P_k \leq \cdots \leq P_1$.

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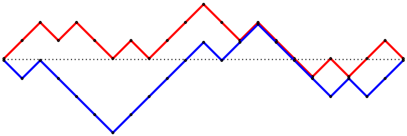

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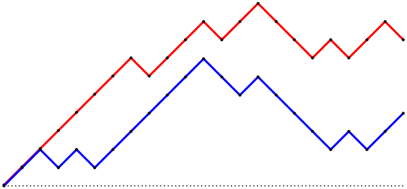

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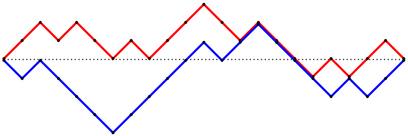

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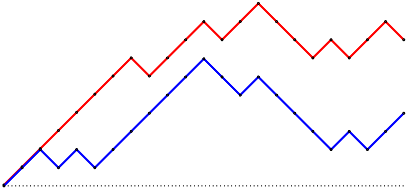
Gessel–Viennot, MacMahon:

$$|\mathcal{G}_n^{(k)}| = \det\left(\binom{n}{\lfloor \frac{n}{2} \rfloor - i + j}\right)_{i,j=1}^{k} = \prod_{i=1}^{\lceil \frac{n}{2} \rceil} \prod_{j=1}^{\lfloor \frac{n}{2} \rfloor} \prod_{l=1}^{k} \frac{i+j+l-1}{i+j+l-2}$$



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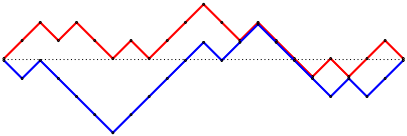

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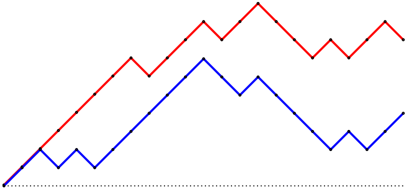

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$$|\mathcal{P}_n^{(k)}| = ?$$

Richard Stanley to the rescue

Computing the first few terms, it seems that

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[EC1, Exercise 3.47(f)]

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Prove that the following posets have the same order polynomial:

- $\mathbf{q} \times \mathbf{p}$ (product of two chains),
- pairs $\{(i,j) : 1 \le i \le j \le p+q-i, 1 \le i \le q\}$ ordered by $(i,j) \le (i',j')$ if $i \le i'$ and $j \le j'$.

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For $p = q$, this is equivalent to $|\mathcal{G}_n^{(k)}| = |\mathcal{P}_n^{(k)}|$.

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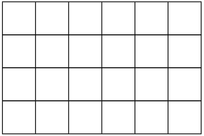
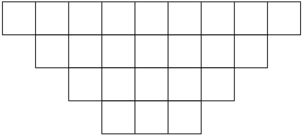
This was proved by Robert Proctor in the following form:

Theorem (Proctor '83)

 # plane partitions inside rectangle shape (p^q) with entries $\leq k$ $=$ *# shifted plane partitions inside shifted shape $[p+q-1, p+q-3, \ldots, p-q+1]$ with entries $\leq k$*





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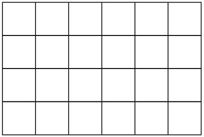
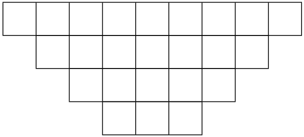


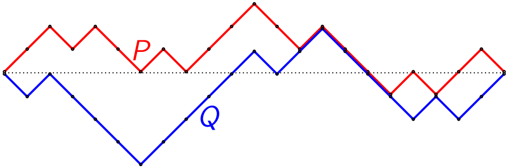


Proctor's proof uses representations of semisimple Lie algebras, and it is not bijective.

A bijective proof for $k = 2$

E. '14: Explicit bijection $\mathcal{G}_n^{(2)} \to \mathcal{P}_n^{(2)}$.

$\mathcal{G}_n^{(2)}$



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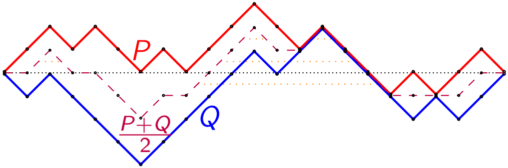

Step 1:

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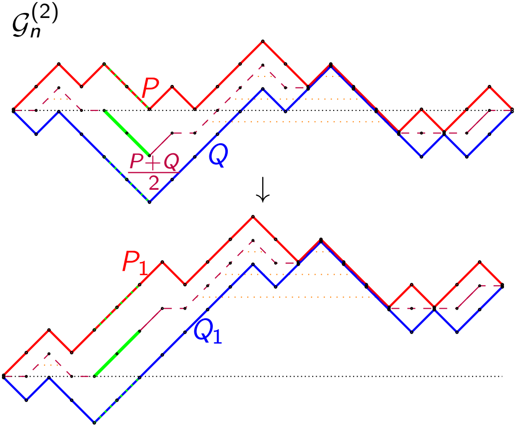

Step 1:

Consider the *average* path $\frac{P+Q}{2}$.

Find its unmatched Ds, and turn them into Us to get P_1 and Q_1.

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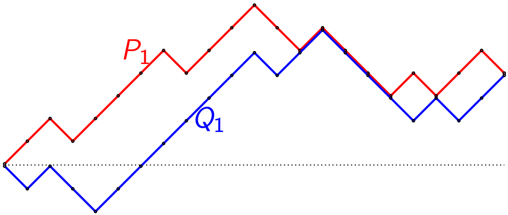

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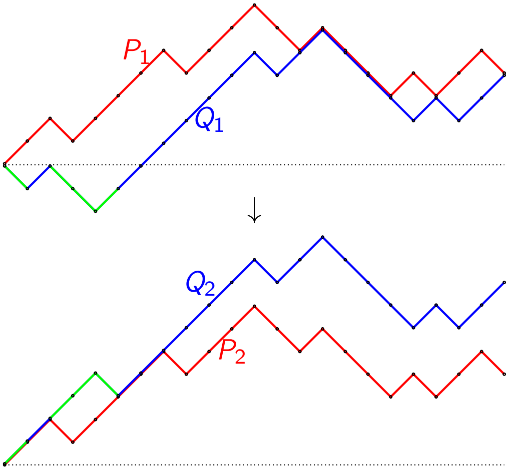

Step 2:

Let Q_2 be the path obtained by flipping the steps of Q_1 that end strictly below the x-axis.

Let $P_2 = P_1$.

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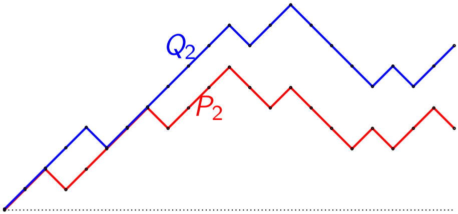

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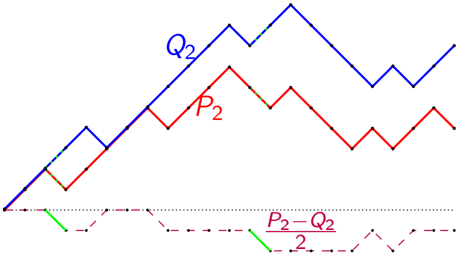

Step 3:

Find the unmatched D steps of $\frac{P_2 - Q_2}{2}$.

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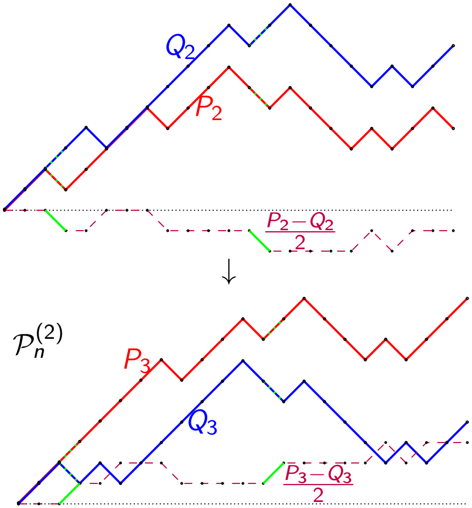

Step 3:

Find the unmatched D steps of $\frac{P_2 - Q_2}{2}$.

Let P_3 and Q_3 be the paths obtained by flipping the corresponding steps of P_2 and Q_2.

A bijective proof for $k = 2$

Theorem (E.'14)

This map is a bijection between $\mathcal{G}_n^{(2)}$ *and* $\mathcal{P}_n^{(2)}$.

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It can be generalized by allowing different endpoints for the paths.
It gives a bijective proof of Proctor's result for $k = 2$.

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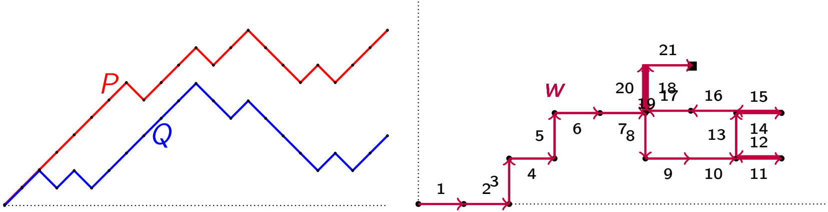
Open problem: Generalize to a bijection between $\mathcal{G}_n^{(k)}$ and $\mathcal{P}_n^{(k)}$.

The bijection in terms of walks

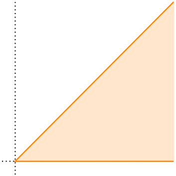
Pairs (P, Q) of lattice paths correspond to walks w in the plane with unit steps N, S, E, W starting at the origin:

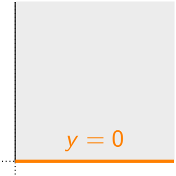
P	Q		w
U	U	$\mapsto$	E
U	D	$\mapsto$	N
D	U	$\mapsto$	S
D	D	$\mapsto$	W

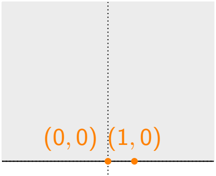


The bijection in terms of walks

Our bijection for paths gives bijections for *NSEW*-walks of length n:



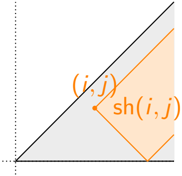




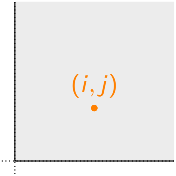
walks in first octant ending anywhere $\leftrightarrow$ walks in first quadrant ending on x-axis $\leftrightarrow$ walks in upper half-plane ending at $(0,0)$ or $(1,0)$

A generalization

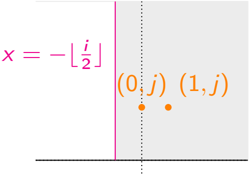
More generally, for every $i \geq j \geq 0$ with $i+j \equiv n \pmod 2$, we have bijections



walks in first octant ending in $\mathrm{sh}(i,j)$ $\leftrightarrow$

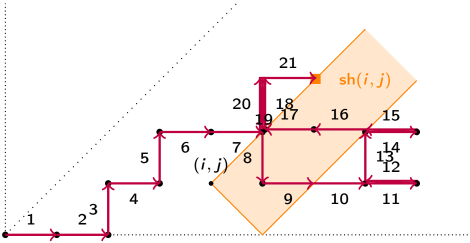


walks in first quadrant ending at (i,j) $\leftrightarrow$



walks in upper half-plane ending at $(0,j)$ or $(1,j)$ with leftmost point on $x=-\lfloor\frac{i}{2}\rfloor$

Example



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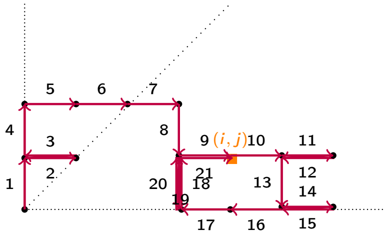


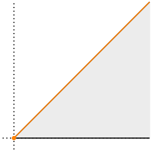
walks in first quadrant ending at (i,j)

Walks ending on the diagonal

Theorem (Bousquet-Mélou, Mishna '10)

The number of walks of length $2m$ in the first octant ending on the diagonal is the product $C_m C_{m+1}$ of Catalan numbers.

Proof uses kernel method and summation of hypergeometric seq.



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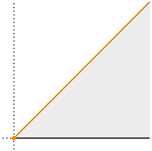
We now get a bijective proof by combining our bijection when $i = j = 0$



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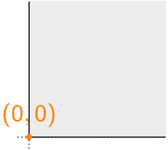

walks in first quadrant ending at $(0, 0)$

together with a bijection of Cori–Dulucq–Viennot '86 (or a more direct one of Bernardi '07).

Tidbit 2

Descents on 321-avoiding involutions

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$\pi \in \mathcal{S}_n$ is 321-**avoiding** if $\pi(1)\pi(2)\ldots\pi(n)$ has no decreasing subsequence of length 3.

π is an **involution** if $\pi^{-1} = \pi$.

$\mathcal{I}_n(321)$ = set of 321-avoiding involutions of length n

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Theorem (Simion-Schmidt '85)

$$|\mathcal{I}_n(321)| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

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i is a **descent** of π if $\pi(i) > \pi(i+1)$.

$\mathrm{Des}(\pi) =$ descent set of π

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$$\mathrm{maj}(\pi) = \sum_{i \in \mathrm{Des}(\pi)} i$$

Theorem (Barnabei-Bonetti-E.-Silimbani, Dahlberg-Sagan '14)

$$\sum_{\pi \in \mathcal{I}_n(321)} q^{\mathrm{maj}(\pi)} = \binom{n}{\lfloor \frac{n}{2} \rfloor}_q$$

where $\binom{n}{j}_q = \frac{(1-q^n)(1-q^{n-1})\dots(1-q^{n-j+1})}{(1-q^j)(1-q^{j-1})\dots(1-q)}$.

Richard Stanley again

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From:  Richard Stanley
Sent:  Wednesday, January 15, 2014
To:  Sergi Elizalde

Hi Sergi,
I like your paper (with various coauthors) on descent sets of
321-avoiding involutions.  Perhaps you would be interested to
know that the result is easy to prove nonbijectively and extends
(in principle) to k,k-1,...,2,1-avoiding involutions.  Namely, it
follows from Lemma 7.23.1 and Exercise 7.16(a) of EC2 that ...
```

Richard Stanley again

$$\sum_{\pi\in\mathcal{I}_n(321)} q^{\mathrm{maj}(\pi)} \overset{[\text{Lem. 7.23.1}]}{=\cdots=} \sum_{\substack{T\in\mathrm{SYT}_n\\ \le 2\text{ rows}}} q^{\mathrm{maj}(T)}$$

$$\overset{[\text{Prop. 7.19.11}]}{=} (1-q)(1-q^2)\cdots(1-q^n) \sum_{\substack{\lambda\vdash n\\ \le 2\text{ parts}}} s_\lambda(1,q,q^2,\dots)$$

$$\overset{[\text{Ex. 7.16a}]}{=} (1-q)\cdots(1-q^n)\, h_{\lfloor \frac{n}{2}\rfloor}(1,q,q^2,\dots)h_{\lceil \frac{n}{2}\rceil}(1,q,q^2,\dots)$$

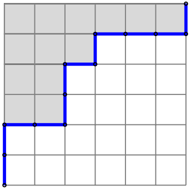
$$= \binom{n}{\lfloor \frac{n}{2}\rfloor}_q.$$

A bijective proof

Recall that $|\mathcal{G}_n| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

$\mathcal{G}_n$ is in bijection with the set Λ_n of partitions whose Young diagram fits inside a $\lfloor \frac{n}{2} \rfloor \times \lceil \frac{n}{2} \rceil$ box.

$$\binom{n}{\lfloor \frac{n}{2} \rfloor}_q = \sum_{\lambda \in \Lambda_n} q^{\mathrm{area}(\lambda)}$$



$\lambda = (6, 3, 2, 2)$
$\mathrm{area}(\lambda) = 13$

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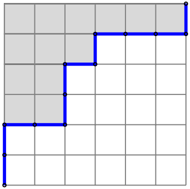

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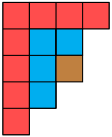
To give a bijective proof of

$$\sum_{\pi \in \mathcal{I}_n(321)} q^{\text{maj}(\pi)} = \binom{n}{\lfloor \frac{n}{2} \rfloor}_q$$

we need a bijection $\mathcal{I}_n(321) \to \Lambda_n$ that maps maj to area.

A refinement

For $\lambda \vdash m$, define its **hook decomposition** $\mathrm{HD}(\lambda)$ to be the set of hook lengths obtained by repeatedly peeling off the largest hook.



$$\lambda = (4,3,3,2,1)$$

$$\mathrm{HD}(\lambda) = \{1, 4, 8\}$$

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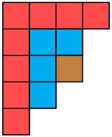

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Theorem (Barnabei–Bonetti–E.–Silimbani '14)

There is a bijection $\mathcal{I}_n(321) \to \Lambda_n$ *that maps* Des *to* HD *(and thus* maj *to area).*

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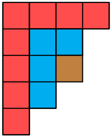

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Proof: Composition of bijections

$$\begin{array}{ccccccc} \mathcal{I}_n(321) & \longrightarrow & \mathcal{P}_n & \longrightarrow & \mathcal{G}_n & \longrightarrow & \Lambda_n \\ \text{Des} & \leftrightarrow & \text{Peak set} & \leftrightarrow & \text{Peak set} & \leftrightarrow & \text{HD} \end{array}$$

The bijections

$\mathcal{I}_n(321)$	$\longrightarrow$	$\mathcal{P}_n$	$\longrightarrow$	$\mathcal{G}_n$	$\longrightarrow$	Λ_n
Des	$\leftrightarrow$	Peak set	$\leftrightarrow$	Peak set	$\leftrightarrow$	HD

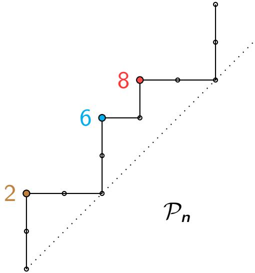
$3\ 4 \mid 1\ 2\ 7\ 9 \mid 5\ 10 \mid 6\ 8\ 11\ 12 \in \mathcal{I}_n(321)$

$\downarrow$ RSK

1	2	5	6	8	11	12
3	4	7	9	10		

Des $= \{2, 6, 8\}$

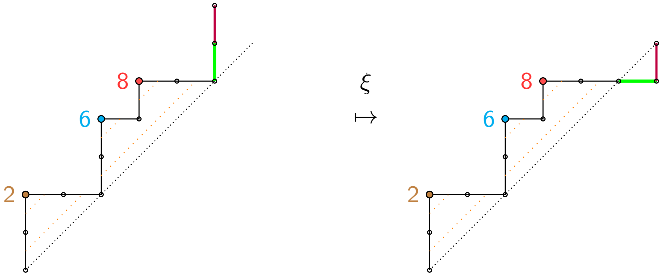
$\mapsto$



Peak set $= \{2, 6, 8\}$

The bijections

$\mathcal{I}_n(321)$	$\longrightarrow$	$\mathcal{P}_n$	$\longrightarrow$	$\mathcal{G}_n$	$\longrightarrow$	Λ_n
Des	$\leftrightarrow$	Peak set	$\leftrightarrow$	Peak set	$\leftrightarrow$	HD



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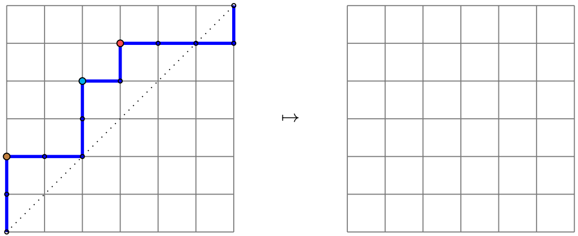

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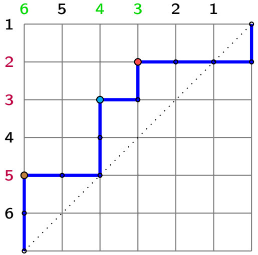
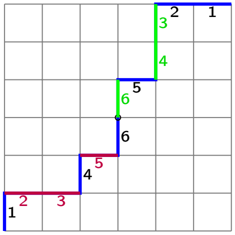
The bijections

$$\begin{array}{ccccccc} \mathcal{I}_n(321) & \longrightarrow & \mathcal{P}_n & \longrightarrow & \color{red}{\mathcal{G}_n} & \color{red}{\longrightarrow} & \color{red}{\Lambda_n} \\ \text{Des} & \leftrightarrow & \text{Peak set} & \leftrightarrow & \text{Peak set} & \leftrightarrow & \text{HD} \end{array}$$



Peak set $= \{2, 6, 8\}$

$\mapsto$



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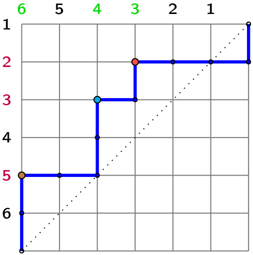

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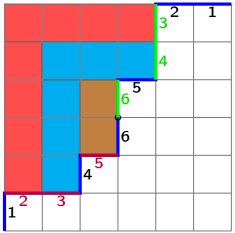

Peak set $= \{2, 6, 8\}$

HD $= \{2, 6, 8\}$

Conclusion

If you want to know all the material in EC1 and EC2

Conclusion

If you want to know all the material in EC1 and EC2 start learning it at an early age.

Happy 70th Birthday, Richard!