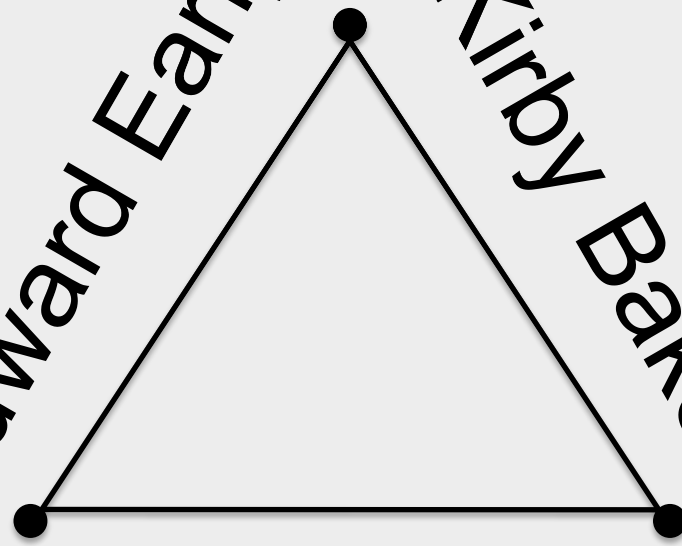
Edward Early
Kirby Baker



Character Polynomials

ST. EDWARD'S UNIVERSITY

Problem

- From Stanley's Positivity Problems in Algebraic Combinatorics
- Problem 12: Give a combinatorial interpretation of the row sums of the character table for S_n (combinatorial proof of non-negativity)

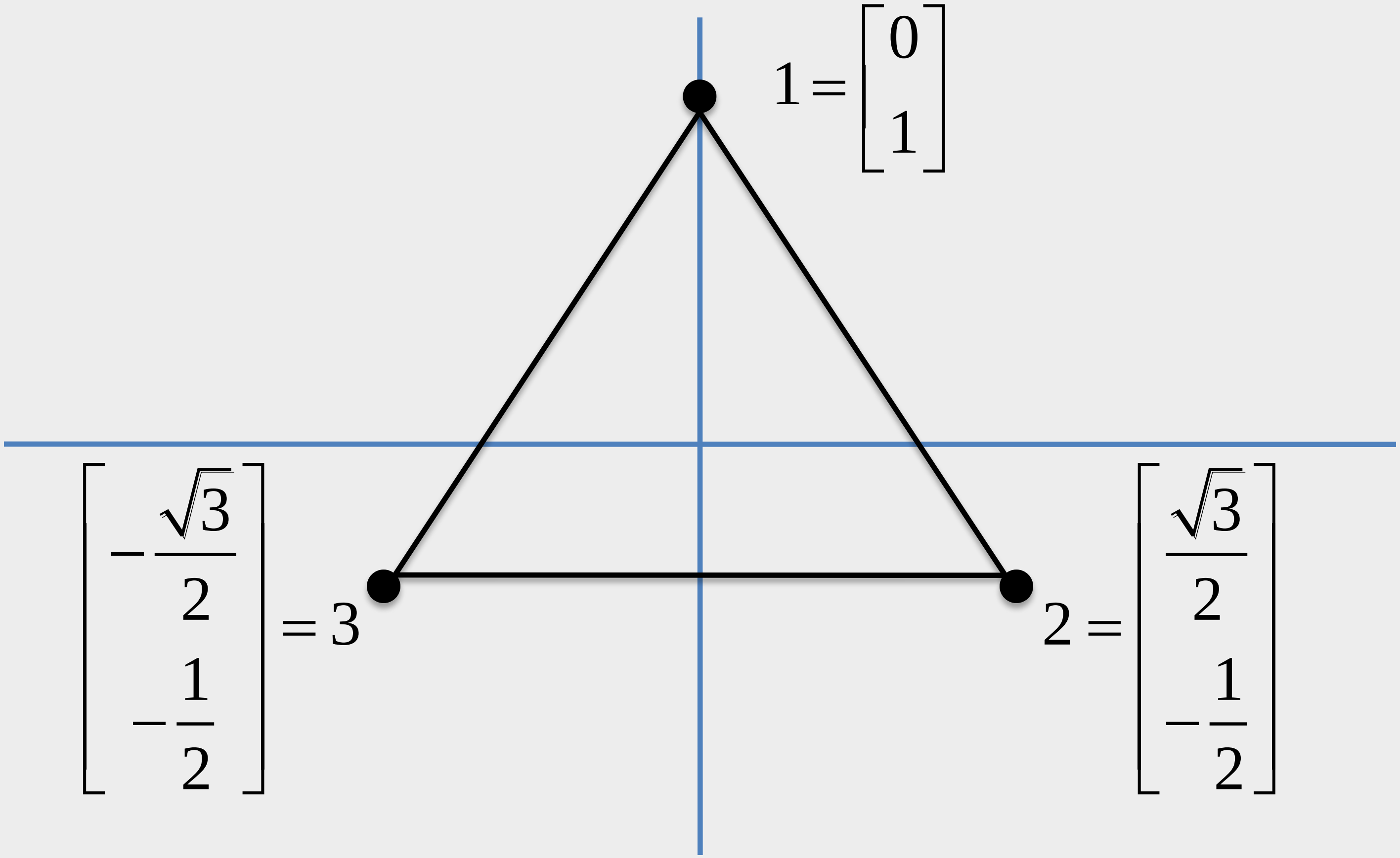
Symmetric Group

- S_n = permutations of n things
- Contains $n!$ elements
- S_3=permutations of $\{1,2,3\}$

  (123, 132, 213, 231, 312, 321)
- Permutations can be represented with $n \times n$ matrices
- Character: trace of a matrix representation
- Character Table: table of all irreducible characters of a group

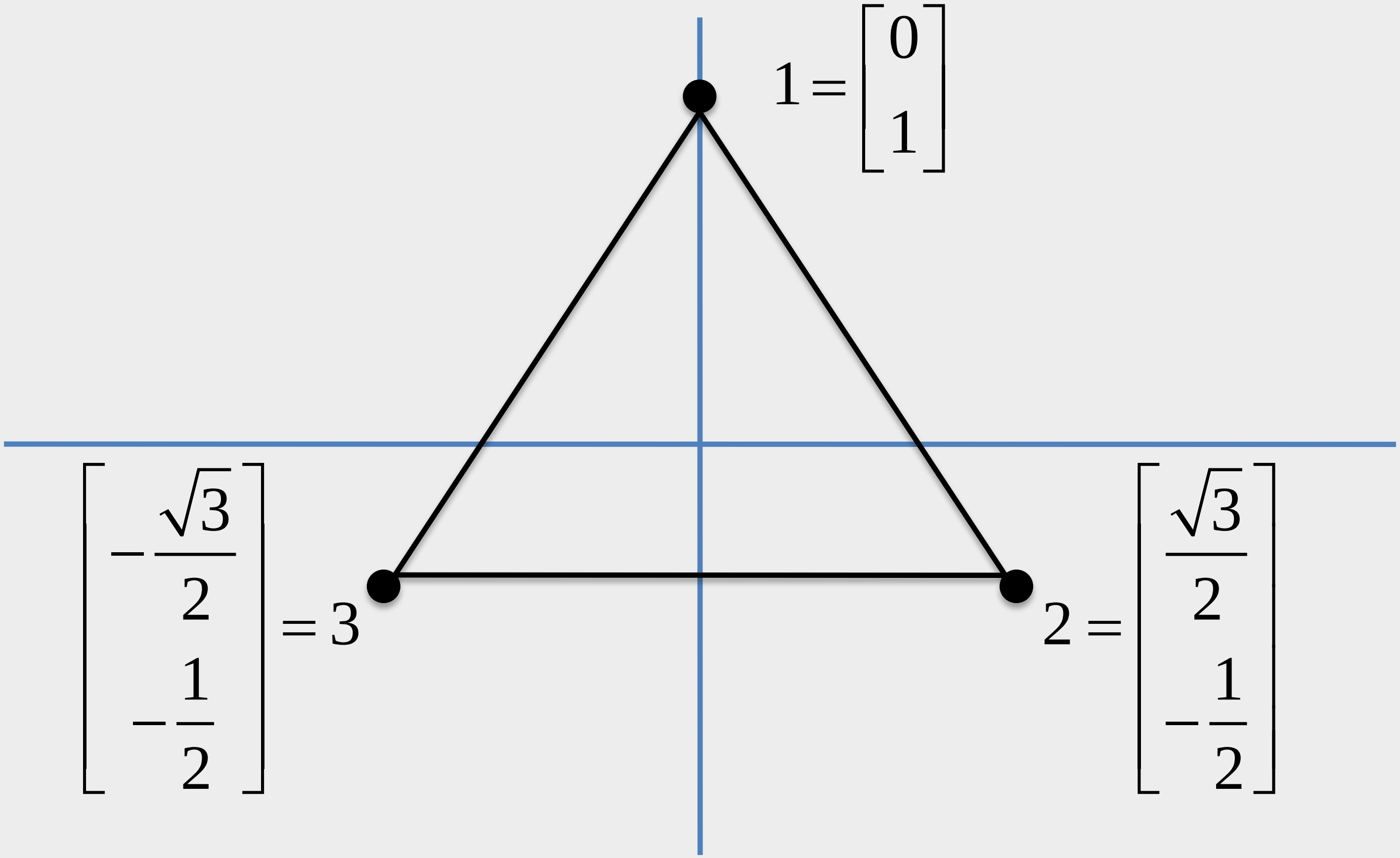
Representations of S_3

- vertices of an equilateral triangle



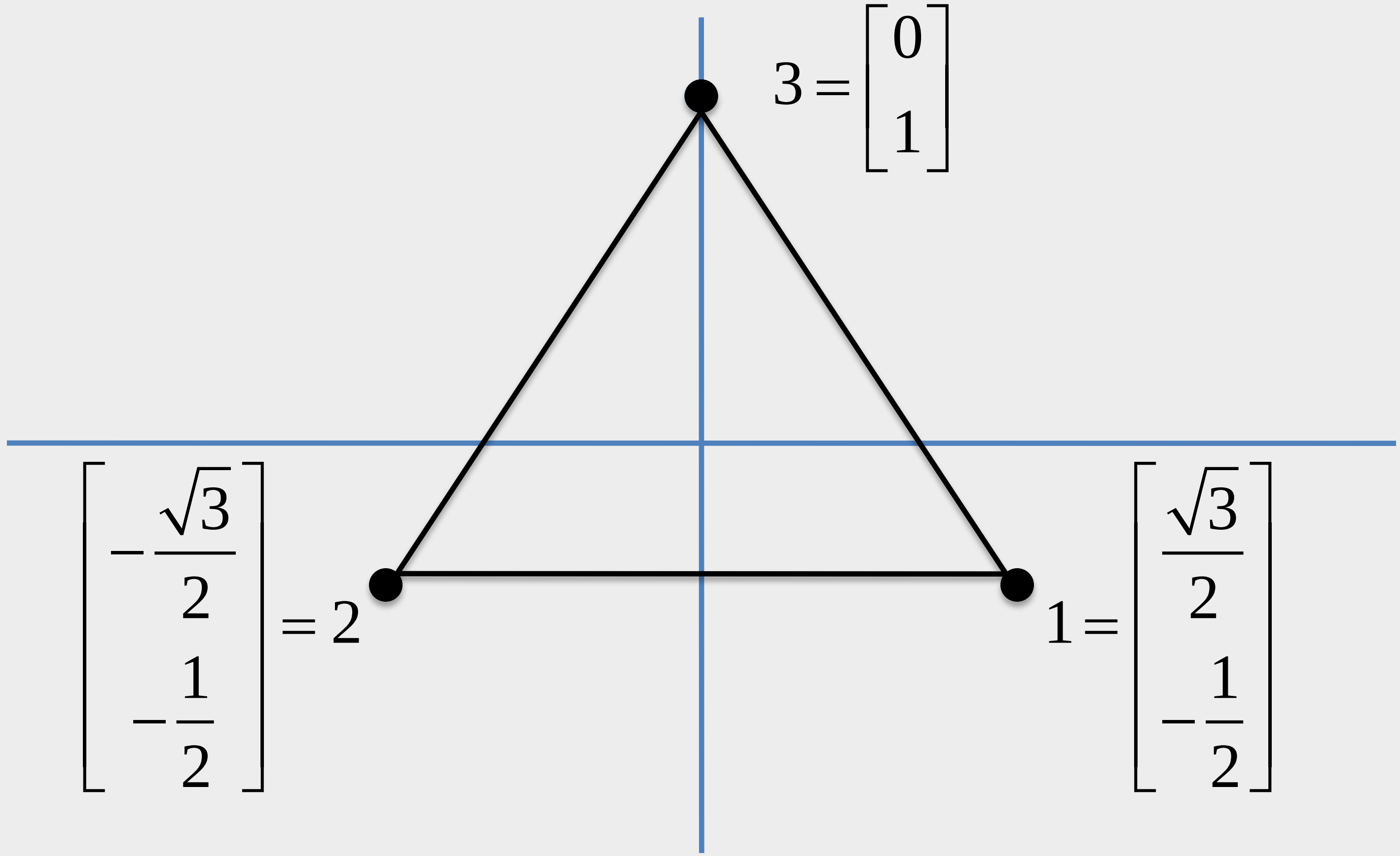
Representations of S_3

- vertices of an equilateral triangle
- pick a permutation: $123 \rightarrow 312$



Representations of S_3

- vertices of an equilateral triangle
- pick a permutation: $123 \rightarrow 312$



Representations of S_3

- $123 \rightarrow 312$ is $120°$ CW rotation

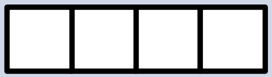
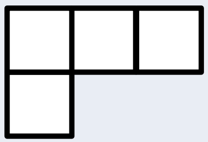
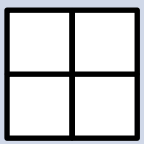
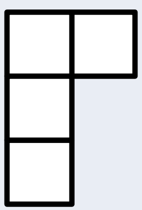
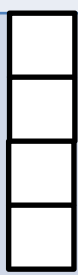
$$\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

- Character = Trace = $-\frac{1}{2}-\frac{1}{2}=-1$

Character Table for S_3

	1,1,1	**2,1**	**3**
3	1	1	1
2,1	2	0	-1
1,1,1	1	-1	1

Character Table for S_4

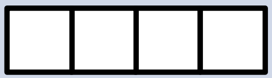
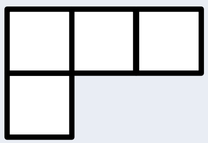
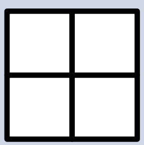
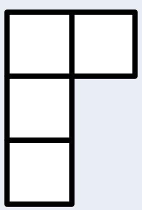
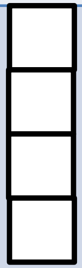
	1^4	$2,1^2$	2^2	$3,1$	4
	1	1	1	1	1
	3	1	-1	0	-1
	2	0	2	-1	0
	3	-1	-1	0	1
	1	-1	1	1	-1

Character Polynomials

- compute characters without matrices
- depend only on small parts of the cycle type

- connections to Murnaghan-Nakayama rule, Schur functions

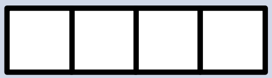
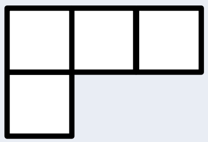
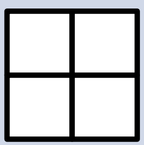
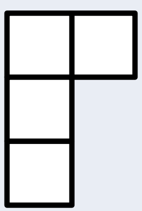
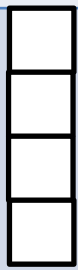
Character Table for S_4

	1^4	$2,1^2$	2^2	$3,1$	4	Sum
	1	1	1	1	1	5
	3	1	-1	0	-1	2
	2	0	2	-1	0	3
	3	-1	-1	0	1	2
	1	-1	1	1	-1	1

Character Polynomials

Partition	Polynomial
n	1
n-1,1	$a_1 - 1$
n-2,2	
n-2,1^2	
n-3,3	
n-3,2,1	
n-3,1^3	
n-4,1^4	

Character Table for S_4

	1^4	$2,1^2$	2^2	$3,1$	4
	1	1	1	1	1
	3	1	-1	0	-1
	2	0	2	-1	0
	3	-1	-1	0	1
	1	-1	1	1	-1

Character Polynomials

Partition	Polynomial
n	1
n-1,1	$a_1 - 1$
n-2,2	$a_2 - a_1 + \frac{a_1(a_1 - 1)}{2}$
n-2,1^2	$-a_2 - a_1 + 1 + \frac{a_1(a_1 - 1)}{2}$
n-3,3	
n-3,2,1	
n-3,1^3	
n-4,1^4	

Character Polynomials

Partition	Polynomial
n	1
n-1,1	$a_1 - 1$
n-2,2	$a_2 - a_1 + \frac{a_1(a_1-1)}{2}$
n-2,1^2	$-a_2 - a_1 + 1 + \frac{a_1(a_1-1)}{2}$
n-3,3	$a_3 + a_1 a_2 - a_2 - \frac{a_1(a_1-1)}{2} + \frac{a_1(a_1-1)(a_1-2)}{6}$
n-3,2,1	$-a_3 - a_1(a_1-1) + \frac{a_1(a_1-1)(a_1-2)}{3} + a_1$
n-3,1^3	$a_3 - a_1 a_2 + a_2 - \frac{a_1(a_1-1)}{2} + \frac{a_1(a_1-1)(a_1-2)}{6} + a_1 - 1$
n-4,1^4	

Character Polynomials

Partition	Polynomial
n	1
n-1,1	$a_1 - 1$
n-2,2	$a_2 - a_1 + \frac{a_1(a_1-1)}{2}$
n-2,1^2	$-a_2 - a_1 + 1 + \frac{a_1(a_1-1)}{2}$
n-3,3	$a_3 + a_1a_2 - a_2 - \frac{a_1(a_1-1)}{2} + \frac{a_1(a_1-1)(a_1-2)}{6}$
n-3,2,1	$-a_3 - a_1(a_1-1) + \frac{a_1(a_1-1)(a_1-2)}{3} + a_1$
n-3,1^3	$a_3 - a_1a_2 + a_2 - \frac{a_1(a_1-1)}{2} + \frac{a_1(a_1-1)(a_1-2)}{6} + a_1 - 1$
n-4,1^4	$-a_4 + a_1a_3 + \frac{a_2(a_2-1)}{2} - a_1\frac{a_2(a_2-1)}{2} - a_3 + a_1a_2 + \frac{a_1(a_1-1)(a_1-2)(a_1-3)}{24} - \frac{a_1(a_1-1)(a_1-2)}{6} - a_2 + \frac{a_1(a_1-1)}{2} + 1$

Generating Functions and Row Sums

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1} \frac{1}{1-x^i}$$

$$\prod_{i\geq 1} \frac{1}{1-x^i} = (1+x+x^2+x^3+x^4+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)(1+x^4+x^8+\cdots)+\cdots$$

Can get x^4 from:

1. $x^4 \cdot 1 \cdot 1 \cdot 1$ → $1,1,1,1$
2. $x \cdot 1 \cdot x^3 \cdot 1$ → $3,1$
3. $1 \cdot x^4 \cdot 1 \cdot 1$ → $2,2$
4. $x^2 \cdot x^2 \cdot 1 \cdot 1$ → $2,1,1$
5. $1 \cdot 1 \cdot 1 \cdot x^4$ → 4

- $p(4)=5$

Example: n-1,1

Character Polynomial: $a_1 - 1$

$$\frac{1}{1-ux} = 1 + ux + u^2x^2 + u^3x^3 + \cdots$$

$$\frac{\partial}{\partial u}\frac{1}{1-ux} = 0 + x + 2ux^2 + 3u^2x^3 + \cdots$$

$$\left.\frac{\partial}{\partial u}\frac{1}{1-ux}\right|_{u=1} = 0 + x + 2x^2 + 3x^3 + \cdots$$

counts number of 1s!

Example: n-1,1

$$\frac{\partial}{\partial u}(1-ux)^{-1} = x(1-ux)^{-2}$$

$$\left.\frac{\partial}{\partial u}(1-ux)^{-1}\right|_{u=1} = \frac{x}{(1-x)^2}$$

$$\sum_{n\geq 0} p(n)x^n = \prod_{i\geq 1}\frac{1}{1-x^i}$$

$$\left.\frac{\partial}{\partial u}\frac{1}{1-ux}\prod_{i\geq 2}\frac{1}{1-x^i}\right|_{u=1} = \frac{x}{1-x}\prod_{i\geq 1}\frac{1}{1-x^i}$$

Example: n-1,1

$$\frac{x}{1-x}\prod_{i\geq 1}\frac{1}{1-x^i} = \frac{x}{1-x}\sum_{n\geq 0} p(n)x^n$$

$$= (x + x^2 + x^3 + \cdots)\sum_{n\geq 0} p(n)x^n$$

$$\left[x^n\right] = \sum a_1$$

$$\left[x^n\right] = p(n-1) + p(n-2) + p(n-3) + \cdots$$

$$\text{Row Sum} = p(n-1) + p(n-2) + p(n-3) + \cdots - p(n)$$

Row	Row Sum
n	$p(n)$
n-1,1	$p(n-1)+p(n-2)+p(n-3)+p(n-4)+p(n-5)+\cdots-p(n)$
n-2,2	$p(n-2)+p(n-3)+3p(n-4)+3p(n-5)+5p(n-6)+\cdots-p(n-1)$
n-2,1^2	$p(n)+p(n-2)+p(n-3)+p(n-4)+3p(n-5)+3p(n-6)+\cdots-p(n-1)$
n-3,3	$p(n-3)+4p(n-5)+7p(n-6)+12p(n-7)+\cdots-2p(n-2)$
n-3,2,1	$p(n-1)+p(n-4)+5p(n-5)+10p(n-6)+\cdots-p(n-2)-2p(n-3)$
n-3,1^3	$p(n-1)+p(n-2)+p(n-4)+p(n-5)+6p(n-6)+\cdots-p(n)$

Growth of $p(n)$

- $p(n\text{-}1) \leq p(n) \leq p(n\text{-}1)+p(n\text{-}2)$

Row	Row Sum Positivity
n	$p(n)$
n-1,1	$p(n-1)+p(n-2)+p(n-3)+p(n-4)+p(n-5)+\cdots-p(n)$
n-2,2	$p(n-2)+p(n-3)+3p(n-4)+3p(n-5)+5p(n-6)+\cdots-p(n-1)$
n-2,1^2	$p(n)+p(n-2)+p(n-3)+p(n-4)+3p(n-5)+3p(n-6)+\cdots-p(n-1)$
n-3,3	$p(n-3)+4p(n-5)+7p(n-6)+12p(n-7)+\cdots-2p(n-2)$
n-3,2,1	$p(n-1)+p(n-4)+5p(n-5)+10p(n-6)+\cdots-p(n-2)-2p(n-3)$
n-3,1^3	$p(n-1)+p(n-2)+p(n-4)+p(n-5)+6p(n-6)+\cdots-p(n)$

Growth of $p(n)$

- $p(n\text{-}1) \leq p(n) \leq p(n\text{-}1)+p(n\text{-}2)$
- super-polynomial, sub-exponential

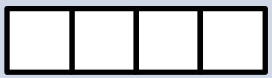
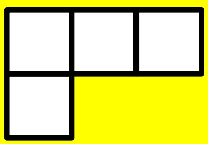
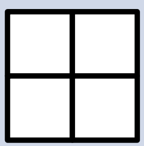
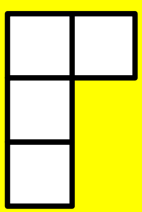
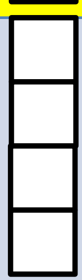
$$Q(x)\sum_{n\geq 0} p(n)x^n$$

- asymptotics good enough to show that finitely many subtracted terms guaranteed to cancel out for n sufficiently large

From the bottom up

- The sum of the last row is the number of self-conjugate partitions of n, call this $s(n)$.
- Conjugate row obtained by multiplying by bottom row

Character Table for S_4

	1^4	$2,1^2$	2^2	$3,1$	4
	1	1	1	1	1
	3	1	-1	0	-1
	2	0	2	-1	0
	3	-1	-1	0	1
	1	-1	1	1	-1

From the bottom up

- The sum of the last row is the number of self-conjugate partitions of n, call this $s(n)$.
- Conjugate row obtained by multiplying by bottom row

$$\sum_{n\geq 0} s(n)x^n = \prod_{i\geq 1} \frac{1}{1+(-1)^i x^i} \rightarrow Q(x)\sum_{n\geq 0} s(n)x^n$$

- For every row sum formula in terms of $p(n)$, the conjugate row has the same formula in terms of $s(n)$.
- $s(n\text{-}1) \leq s(n) \leq s(n\text{-}1)+s(n\text{-}2)$ for $n > 1$

Words of Wisdom

The worst thing you can do to a problem is to solve it completely… because then you have to find something else to work on.

– Dan Kleitman