Bijective Enumerations on Humps and Peaks in (k, a)-paths and (n, m)-Dyck paths

Rosena Ruoxia Du

East China Normal University

Stanley@70, MIT, June 27, 2014

Combinatorics, Special Functions, and Physics, August 2004
Nankai University, Tianjin.

Similing Richard, August 2004, Nankai University, Tianjin.

3-Noncrossing Matchings

William Y. C. Chen[1], Eva Y. P. Deng[2] and Rosena R. X. Du[3]

Center for Combinatorics, LPMC
Nankai University, Tianjin 300071, P. R. China

[1]chen@nankai.edu.cn, [2]dengyp@eyou.com, [3]du@nankai.edu.cn

Dedicated to Richard P. Stanley on the Occasion of His Sixtieth Birthday

July 18, 2004

Abstract. The problem of enumerating 3-noncrossing matchings (123123-avoiding matchings) was recently raised by Klazar as the first unsolved case of a more general problem of enumerating k-noncrossing partitions. In this paper, we show that all the following three classes of matchings with forbidden patterns are in one-to-one correspondence with pairs of Dyck paths with the same origin and the same destination: 3-noncrossing matchings (123123-avoiding matchings), 3-nonnesting matchings (123321-avoiding matchings), non-double-nesting matchings (123312-avoiding matchings). The enumeration of 3-nonnesting matchings is solved by Gouyou-Beauchamps.

7.24. **a.** [1] Let $U : \Lambda \to \Lambda$ and $D : \Lambda \to \Lambda$ be linear transformations defined by $U(f) = p_1 f$ and

$$D(f) = \frac{\partial}{\partial p_1} f,$$

where $\partial/\partial p_1$ is applied to f written as a polynomial in the p_i's. Show that $DU - UD = I$, the identity operator.

b. [1] Show that $DU^k = kU^{k-1} + U^k D$.

c. [2] Deduce from (a) and (b) that if $\ell \in \mathbb{N}$ then

$$(U + D)^\ell = \sum_{\substack{i+j \le \ell \\ r := (\ell - i - j)/2 \in \mathbb{N}}} \frac{\ell!}{2^r r! \, i! \, j!} U^i D^j.$$

d. [2+] An *oscillating tableau* (or *up–down tableau*) of shape λ and length ℓ is a sequence $\emptyset = \lambda^0, \lambda^1, \ldots, \lambda^\ell = \lambda$ of partitions such that for all $1 \le i \le \ell - 1$, the diagram of λ^i is obtained from that of λ^{i-1} by either adding one square or removing one square. (If we add a square each time, then $\ell = |\lambda|$ and we have an SYT of shape λ.) Clearly if such an oscillating tableau exists, then $\ell = |\lambda| + 2r$ for some $r \in \mathbb{N}$. Deduce from (c) that the number $\tilde{f}^\lambda_\ell$ of oscillating tableaux of shape λ and length $\ell = |\lambda| + 2r$ is given by

$$\tilde{f}^\lambda_\ell = \frac{\ell! \, f^\lambda}{(\ell - 2r)! \, r! \, 2^r}.$$

e. [3−] Give a bijective proof of (d).

TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 359, Number 4, April 2007, Pages 1555–1575
S 0002-9947(06)04210-3
Article electronically published on September 19, 2006

CROSSINGS AND NESTINGS OF MATCHINGS AND PARTITIONS

WILLIAM Y. C. CHEN, EVA Y. P. DENG, ROSENA R. X. DU, RICHARD P. STANLEY, AND CATHERINE H. YAN

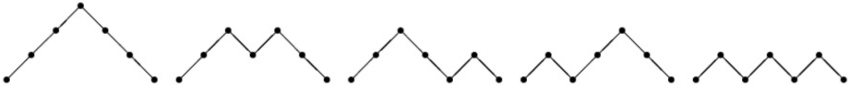
ABSTRACT. We present results on the enumeration of crossings and nestings for matchings and set partitions. Using a bijection between partitions and vacillating tableaux, we show that if we fix the sets of minimal block elements and maximal block elements, the crossing number and the nesting number of partitions have a symmetric joint distribution. It follows that the crossing numbers and the nesting numbers are distributed symmetrically over all partitions of $[n]$, as well as over all matchings on $[2n]$. As a corollary, the number of k-noncrossing partitions is equal to the number of k-nonnesting partitions. The same is also true for matchings. An application is given to the enumeration of matchings with no k-crossing (or with no k-nesting).


FRANKLIN

Total number of peaks in Dyck paths

Exercise 6.19, EC2, Page 221.

(i) Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



- The number of all Dyck paths of order n is the Catalan number C_n.

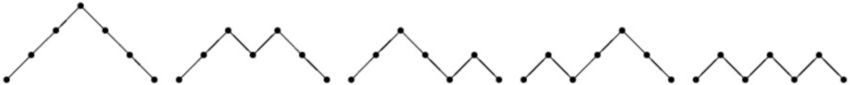
$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

- peak: an up step followed by a down step.
- Question: How many peaks are there in all Dyck paths of order n?

Total number of peaks in Dyck paths

Exercise 6.19, EC2, Page 221.

(i) Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



- The number of all Dyck paths of order n is the Catalan number C_n.

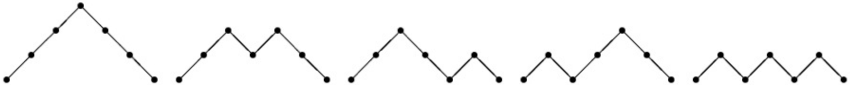
$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

- peak: an up step followed by a down step.
- Question: How many peaks are there in all Dyck paths of order n?

Total number of peaks in Dyck paths

Exercise 6.19, EC2, Page 221.

(i) Dyck paths from $(0,0)$ to $(2n,0)$, i.e., lattice paths with steps $(1,1)$ and $(1,-1)$, never falling below the x-axis



- The number of all Dyck paths of order n is the Catalan number C_n.

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

- peak: an up step followed by a down step.
- Question: How many peaks are there in all Dyck paths of order n?

Total number of peaks in Dyck paths

- (EC2, Exercise 6.36): The number of Dyck paths of order n with k peaks is the Narayana number.

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

- Summing over k, we get the number of peaks in all Dyck paths of order n:

$$\sum_{k=0}^{n} kN(n,k) = \binom{2n-1}{n}.$$

- Is there a simple explaination without summation?
- Yes! Note that $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, and $\binom{2n}{n}$ is the number of all super Dyck paths, or free Dyck paths. (Dyck paths allowed to go bellow the x-axis.) We can give a simple bijective proof.
- Similar relations hold for more generalized lattice paths.

Total number of peaks in Dyck paths

- (EC2, Exercise 6.36): The number of Dyck paths of order n with k peaks is the Narayana number.

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

- Summing over k, we get the number of peaks in all Dyck paths of order n:

$$\sum_{k=0}^{n} kN(n,k) = \binom{2n-1}{n}.$$

- Is there a simple explaination without summation?
- Yes! Note that $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, and $\binom{2n}{n}$ is the number of all super Dyck paths, or free Dyck paths. (Dyck paths allowed to go bellow the x-axis.) We can give a simple bijective proof.
- Similar relations hold for more generalized lattice paths.

Total number of peaks in Dyck paths

- (EC2, Exercise 6.36): The number of Dyck paths of order n with k peaks is the Narayana number.

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

- Summing over k, we get the number of peaks in all Dyck paths of order n:

$$\sum_{k=0}^{n} kN(n,k) = \binom{2n-1}{n}.$$

- Is there a simple explaination without summation?
- Yes! Note that $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, and $\binom{2n}{n}$ is the number of all super Dyck paths, or free Dyck paths. (Dyck paths allowed to go bellow the x-axis.) We can give a simple bijective proof.
- Similar relations hold for more generalized lattice paths.

Total number of peaks in Dyck paths

- (EC2, Exercise 6.36): The number of Dyck paths of order n with k peaks is the Narayana number.

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

- Summing over k, we get the number of peaks in all Dyck paths of order n:

$$\sum_{k=0}^{n} kN(n,k) = \binom{2n-1}{n}.$$

- Is there a simple explaination without summation?
- Yes! Note that $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, and $\binom{2n}{n}$ is the number of all super Dyck paths, or free Dyck paths. (Dyck paths allowed to go bellow the x-axis.) We can give a simple bijective proof.
- Similar relations hold for more generalized lattice paths.

Total number of peaks in Dyck paths

- (EC2, Exercise 6.36): The number of Dyck paths of order n with k peaks is the Narayana number.

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}.$$

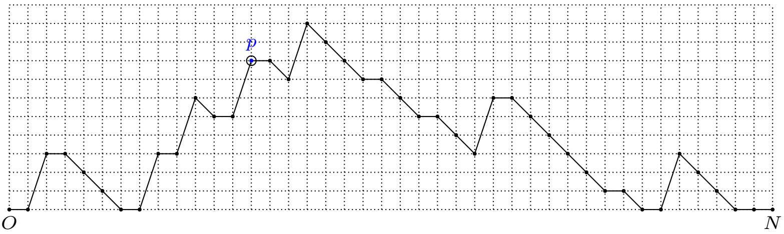
- Summing over k, we get the number of peaks in all Dyck paths of order n:

$$\sum_{k=0}^{n} kN(n,k) = \binom{2n-1}{n}.$$

- Is there a simple explaination without summation?
- Yes! Note that $\binom{2n-1}{n} = \frac{1}{2}\binom{2n}{n}$, and $\binom{2n}{n}$ is the number of all super Dyck paths, or free Dyck paths. (Dyck paths allowed to go bellow the x-axis.) We can give a simple bijective proof.
- Similar relations hold for more generalized lattice paths.

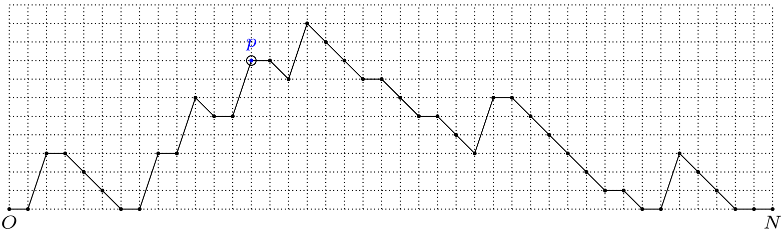
(k, a)-paths

- A (k, a)-path of order n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using up steps $(1, k)$, down steps $(1, -1)$ and horizontal steps $(a, 0)$ and never goes below the x-axis.
- $\mathcal{P}_n(k, a)$: the set of all (k, a)-paths of order n.
- $\mathcal{P}_n(1, \infty)$: Dyck paths; $\mathcal{P}_n(1, 1)$: Motzkin paths; $\mathcal{P}_n(1, 2)$: the set of Schröder paths; $\mathcal{P}_n(k, \infty)$: k-ary paths.
- *peak*: an up step followed by a down step.
- *hump*: an up step followed by zero or more horizontal steps followed by a down step.



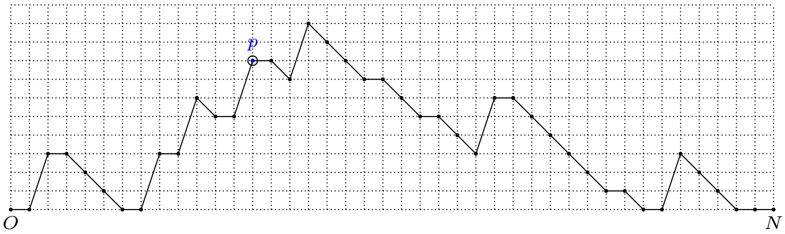
(k, a)-paths

- A (k, a)-path of order n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using up steps $(1, k)$, down steps $(1, -1)$ and horizontal steps $(a, 0)$ and never goes below the x-axis.
- $\mathcal{P}_n(k, a)$: the set of all (k, a)-paths of order n.
- $\mathcal{P}_n(1, \infty)$: Dyck paths; $\mathcal{P}_n(1, 1)$: Motzkin paths; $\mathcal{P}_n(1, 2)$: the set of Schröder paths; $\mathcal{P}_n(k, \infty)$: k-ary paths.
- *peak*: an up step followed by a down step.
- *hump*: an up step followed by zero or more horizontal steps followed by a down step.



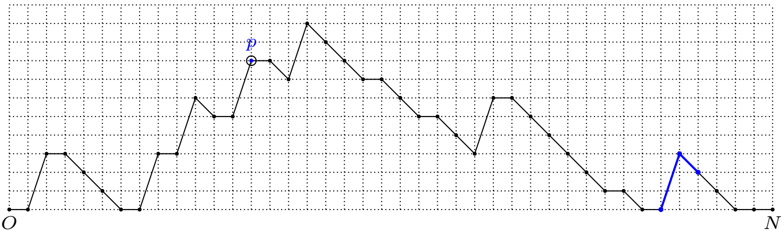
(k, a)-paths

- A (k, a)-path of order n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using up steps $(1, k)$, down steps $(1, -1)$ and horizontal steps $(a, 0)$ and never goes below the x-axis.
- $\mathcal{P}_n(k, a)$: the set of all (k, a)-paths of order n.
- $\mathcal{P}_n(1, \infty)$: Dyck paths; $\mathcal{P}_n(1, 1)$: Motzkin paths; $\mathcal{P}_n(1, 2)$: the set of Schröder paths; $\mathcal{P}_n(k, \infty)$: k-ary paths.
- *peak*: an up step followed by a down step.
- *hump*: an up step followed by zero or more horizontal steps followed by a down step.



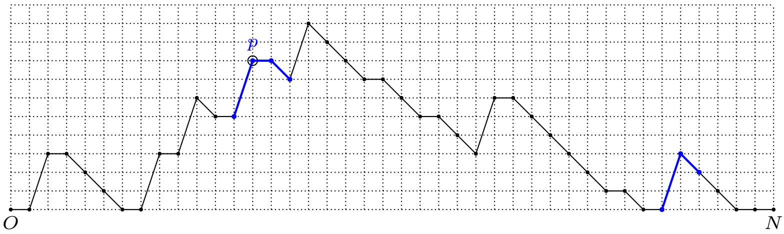
(k, a)-paths

- A (k, a)-path of order n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using up steps $(1, k)$, down steps $(1, -1)$ and horizontal steps $(a, 0)$ and never goes below the x-axis.
- $\mathcal{P}_n(k, a)$: the set of all (k, a)-paths of order n.
- $\mathcal{P}_n(1, \infty)$: Dyck paths; $\mathcal{P}_n(1, 1)$: Motzkin paths; $\mathcal{P}_n(1, 2)$: the set of Schröder paths; $\mathcal{P}_n(k, \infty)$: k-ary paths.
- *peak*: an up step followed by a down step.
- *hump*: an up step followed by zero or more horizontal steps followed by a down step.



(k, a)-paths

- A (k, a)-path of order n is a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using up steps $(1, k)$, down steps $(1, -1)$ and horizontal steps $(a, 0)$ and never goes below the x-axis.
- $\mathcal{P}_n(k, a)$: the set of all (k, a)-paths of order n.
- $\mathcal{P}_n(1, \infty)$: Dyck paths; $\mathcal{P}_n(1, 1)$: Motzkin paths; $\mathcal{P}_n(1, 2)$: the set of Schröder paths; $\mathcal{P}_n(k, \infty)$: k-ary paths.
- *peak*: an up step followed by a down step.
- *hump*: an up step followed by zero or more horizontal steps followed by a down step.



Humps and peaks of (k, a)-paths and super (k, a)-paths

- In 2008 Regev noticed the curious relation between the number of peaks in all Dyck paths and free Dyck paths. He also counted the number of humps in all Motzkin paths and found that similar relations holds.
- In 2013, using generating function methods, Mansour and Shattuck generalized Regev's results to (k, a)-paths and proved the following euqations:

$$(k+1)\sum_{P\in\mathcal{P}_n(k,a)}\#\text{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n}, \tag{1.1}$$

$$(k+1)\sum_{P\in\mathcal{P}_n(k,a)}\#\text{Peaks}(P) = |\mathcal{SP}_n(k,a)| - |\mathcal{SP}_{n-a}(k,a)|, \tag{1.2}$$

where $\delta_{a|n} = 1$ if a divides n or 0 otherwise.

Humps and peaks of (k, a)-paths and super (k, a)-paths

- In 2008 Regev noticed the curious relation between the number of peaks in all Dyck paths and free Dyck paths. He also counted the number of humps in all Motzkin paths and found that similar relations holds.
- In 2013, using generating function methods, Mansour and Shattuck generalized Regev's results to (k, a)-paths and proved the following euqations:

$$(k+1) \sum_{P \in \mathcal{P}_n(k,a)} \#\mathrm{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n}, \qquad (1.1)$$

$$(k+1) \sum_{P \in \mathcal{P}_n(k,a)} \#\mathrm{Peaks}(P) = |\mathcal{SP}_n(k,a)| - |\mathcal{SP}_{n-a}(k,a)|, \qquad (1.2)$$

where $\delta_{a|n} = 1$ if a divides n or 0 otherwise.

What does $\frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right)$ count?

$$(k+1) \sum_{P \in \mathcal{P}_n(k,a)} \#\text{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n},$$

Is equivalent to the following:

$$\sum_{P \in \mathcal{P}_n(k,a)} \#Humps(P) = \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right). \tag{2.1}$$

The following lemma explains what the right hand side of (2.1) counts.

Lemma 2.1

There is a 1-to-$(k+1)$ correspondence between $\mathcal{SP}_n^U(k,a)$ and $\mathcal{SP}_n^0(k,a)$, and we have

$$|\mathcal{SP}_n^U(k,a)| = \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right). \tag{2.2}$$

$\mathcal{SP}_n^0(k,a)$: the set of super (k,a)-paths of order n with at least one up step;
$\mathcal{SP}_n^U(k,a)$: the set of super (k,a)-paths or order n whose first non-horizontal step is U.

What does $\frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right)$ count?

$$(k+1)\sum_{P\in\mathcal{P}_n(k,a)} \#\mathrm{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n},$$

Is equivalent to the following:

$$\sum_{P\in\mathcal{P}_n(k,a)} \#Humps(P) = \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right). \quad (2.1)$$

The following lemma explains what the right hand side of (2.1) counts.

Lemma 2.1

There is a 1-to-$(k+1)$ correspondence between $\mathcal{SP}_n^U(k,a)$ and $\mathcal{SP}_n^0(k,a)$, and we have

$$|\mathcal{SP}_n^U(k,a)| = \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right). \quad (2.2)$$

$\mathcal{SP}_n^0(k,a)$: the set of super (k,a)-paths of order n with at least one up step;
$\mathcal{SP}_n^U(k,a)$: the set of super (k,a)-paths or order n whose first non-horizontal step is U.

What does $\frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right)$ count?

$$(k+1)\sum_{P\in\mathcal{P}_n(k,a)} \#\mathrm{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n},$$

Is equivalent to the following:

$$\sum_{P\in\mathcal{P}_n(k,a)} \#Humps(P) = \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right). \tag{2.1}$$

The following lemma explains what the right hand side of (2.1) counts.

Lemma 2.1

There is a 1-to-$(k+1)$ correspondence between $\mathcal{SP}_n^U(k,a)$ and $\mathcal{SP}_n^0(k,a)$, and we have

$$|\mathcal{SP}_n^U(k,a)| = \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right). \tag{2.2}$$

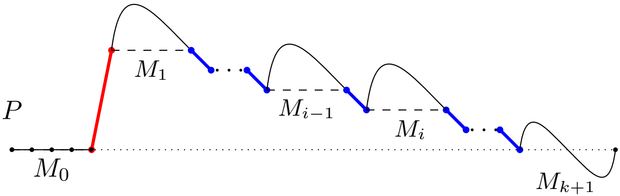
$\mathcal{SP}_n^0(k,a)$: the set of super (k,a)-paths of order n with at least one up step;
$\mathcal{SP}_n^U(k,a)$: the set of super (k,a)-paths or order n whose first non-horizontal step is U.

Proof of Lemma 2.1: $\phi : \mathcal{SP}_n^U(k,a) \longrightarrow \mathcal{SP}_n^0(k,a)$,

Given $P \in \mathcal{SP}_n^U(k,a)$, we can uniquely decompose it into the following form:

$$P = M_0\ U\ M_1\ D\ M_2\ D\ \cdots\ D\ M_k\ D\ M_{k+1},$$

in which $M_0, M_1, \ldots, M_k$ are (k,a)-paths, M_{k+1} is a super (k,a)-path, and M_0 consists of only horizontal steps.



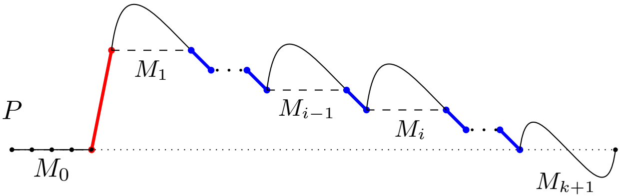
$\psi(P) = \{P_i = M_0\ D\ \overline{M}_1\ D\ \cdots\ D\ \overline{M}_{i-1}\ U\ M_i\ D\ \cdots\ D\ M_{k+1} : 1 \leqslant i \leqslant k+1\}$.
Here $\overline{M}_i$ means the supper (k,a)-path obtained from M_i by reading the steps in reversing order, i.e., if $M_i = HUUDHD$, then $\overline{M}_i = DHDUUH$.

Proof of Lemma 2.1: $\phi : \mathcal{SP}_n^U(k,a) \longrightarrow \mathcal{SP}_n^0(k,a)$,

Given $P \in \mathcal{SP}_n^U(k,a)$, we can uniquely decompose it into the following form:

$$P = M_0 \ U \ M_1 \ D \ M_2 \ D \ \cdots \ D \ M_k \ D \ M_{k+1},$$

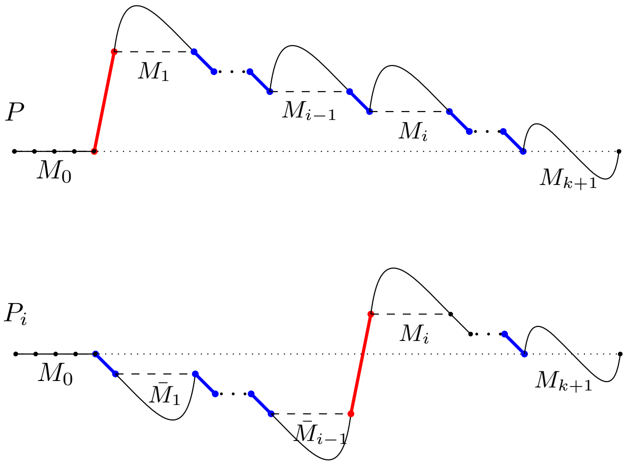
in which $M_0, M_1, \ldots, M_k$ are (k,a)-paths, M_{k+1} is a super (k,a)-path, and M_0 consists of only horizontal steps.



$\psi(P) = \{P_i = M_0 \ D \ \overline{M}_1 \ D \ \cdots \ D \ \overline{M}_{i-1} \ U \ M_i \ D \ \cdots \ D \ M_{k+1} : 1 \leqslant i \leqslant k+1\}$. Here $\overline{M}_i$ means the supper (k,a)-path obtained from M_i by reading the steps in reversing order, i.e., if $M_i = HUUDHD$, then $\overline{M}_i = DHDUUH$.

Proof of Lemma 2.1: $\phi : \mathcal{SP}_n^U(k,a) \longrightarrow \mathcal{SP}_n^0(k,a)$,

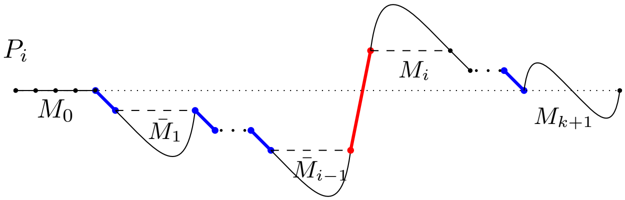
$$\psi(P) = \{P_i = M_0\, D\, \overline{M}_1\, D\, \cdots\, D\, \overline{M}_{i-1}\, U\, M_i\, D\, \cdots\, D\, M_{k+1} : 1 \leqslant i \leqslant k+1\}.$$



Proof of Lemma 2.1: $\phi : \mathcal{SP}_n^U(k,a) \longrightarrow \mathcal{SP}_n^0(k,a)$,

Key point to recover P from $\psi(P)$: find the left-most up step U in P_i whose right end point has positive y-coordinate, then decompose P_i into the following form:

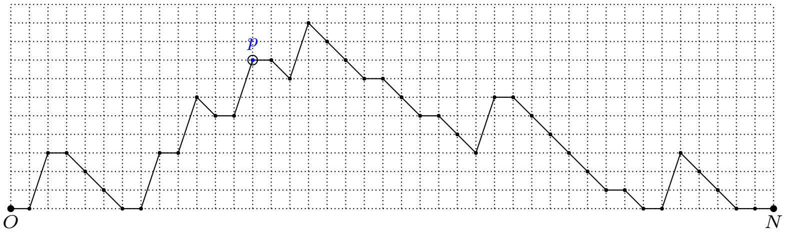
$$P_i = M_0\ D\ M_1\ D\ \cdots\ D\ M_{i-1}\ U\ M_i\ D\ \cdots\ D\ M_{k+1}, 1 \leqslant i \leqslant k+1.$$



$\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.

Theorem 2.2

Let $\mathcal{LP}_n(k,a)$ denote the set of pairs (L,p), where $L \in \mathcal{P}_n(k,a)$, and p is a specified hump in L. Then there is a bijection $\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.

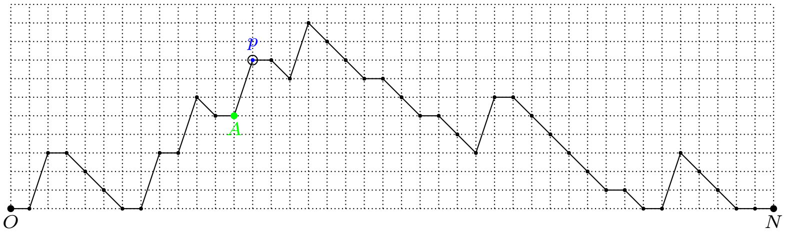


- A: the leftmost point in L_{Op} that is followed by an up step, and there is no down step in L_{Ap};
- B: the leftmost point in L such that $x_B > x_P$ and $y_B = y_A$;
- C: the rightmost point in L_{OA} such that $y_C = 0$.

$\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.

Theorem 2.2

Let $\mathcal{LP}_n(k,a)$ denote the set of pairs (L,p), where $L \in \mathcal{P}_n(k,a)$, and p is a specified hump in L. Then there is a bijection $\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.

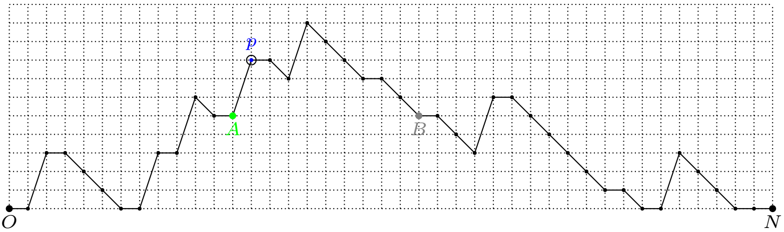


- A: the leftmost point in L_{Op} that is followed by an up step, and there is no down step in L_{Ap};
- B: the leftmost point in L such that $x_B > x_P$ and $y_B = y_A$;
- C: the rightmost point in L_{OA} such that $y_C = 0$.

$\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.

Theorem 2.2

Let $\mathcal{LP}_n(k,a)$ denote the set of pairs (L,p), where $L \in \mathcal{P}_n(k,a)$, and p is a specified hump in L. Then there is a bijection $\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.

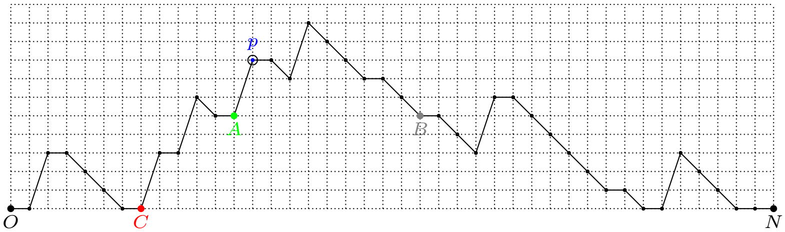


- A: the leftmost point in L_{Op} that is followed by an up step, and there is no down step in L_{Ap};
- B: the leftmost point in L such that $x_B > x_P$ and $y_B = y_A$;
- C: the rightmost point in L_{OA} such that $y_C = 0$.

$\Phi : \mathcal{LP}_n(k, a) \to \mathcal{SP}_n^U(k, a)$.

Theorem 2.2

Let $\mathcal{LP}_n(k, a)$ denote the set of pairs (L, p), where $L \in \mathcal{P}_n(k, a)$, and p is a specified hump in L. Then there is a bijection $\Phi : \mathcal{LP}_n(k, a) \to \mathcal{SP}_n^U(k, a)$.

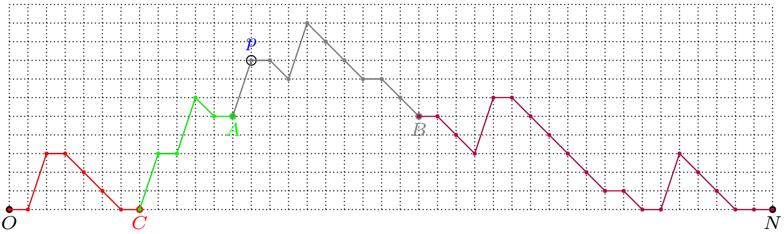


- A: the leftmost point in L_{Op} that is followed by an up step, and there is no down step in L_{Ap};
- B: the leftmost point in L such that $x_B > x_P$ and $y_B = y_A$;
- C: the rightmost point in L_{OA} such that $y_C = 0$.

$\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.

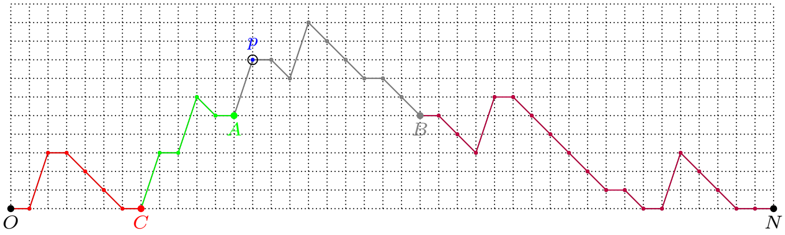
Theorem 2.2

Let $\mathcal{LP}_n(k,a)$ denote the set of pairs (L,p), where $L \in \mathcal{P}_n(k,a)$, and p is a specified hump in L. Then there is a bijection $\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.

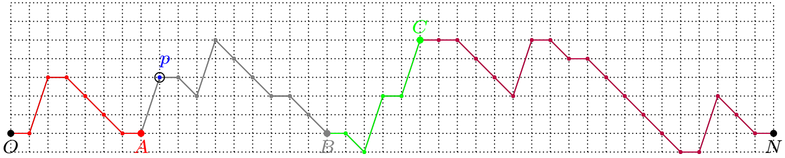


- A: the leftmost point in L_{Op} that is followed by an up step, and there is no down step in L_{Ap};
- B: the leftmost point in L such that $x_B > x_P$ and $y_B = y_A$;
- C: the rightmost point in L_{OA} such that $y_C = 0$.

$\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.



Define $\Phi(L,p) = L_{OC}L_{AB}\overline{L}_{CA}\overline{L}_{BN} \triangleq SL$.



$\Phi^{-1} : \mathcal{SP}_n^U(k, a) \to \mathcal{LP}_n(k, a)$.

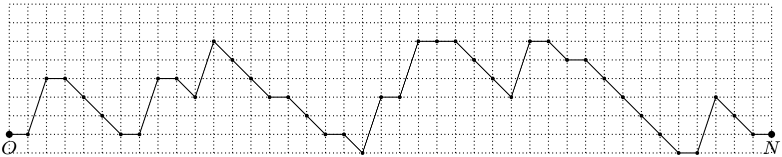


Figure : A super $(3, 1)$-path $\Phi(L, p) \in \mathcal{SP}_{41}^U(3, 1)$.

- B: the point on the x-axis that following a down step, and the next down step is the first down step that goes below the x-axis;
- A: the rightmost point that $y_A = 0$, $x_A < x_B$ and A is followed by an up step;
- C: the leftmost point such that $x_C \geq x_B$ and $\forall G$, $x_G \geq x_B$ implies that $y_G \leq y_C$;
- p: the leftmost hump in L_{AB}.

$\Phi^{-1} : \mathcal{SP}_n^U(k,a) \to \mathcal{LP}_n(k,a)$.

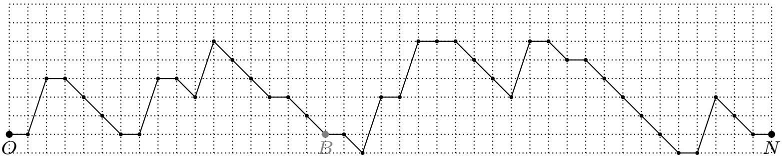


Figure : A super $(3,1)$-path $\Phi(L,p) \in \mathcal{SP}_{41}^U(3,1)$.

- B: the point on the x-axis that following a down step, and the next down step is the first down step that goes below the x-axis;
- A: the rightmost point that $y_A = 0$, $x_A < x_B$ and A is followed by an up step;
- C: the leftmost point such that $x_C \geq x_B$ and $\forall G$, $x_G \geq x_B$ implies that $y_G \leq y_C$;
- p: the leftmost hump in L_{AB}.

$\Phi^{-1} : \mathcal{SP}_n^U(k,a) \to \mathcal{LP}_n(k,a)$.

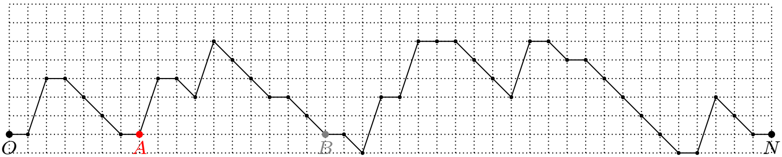


Figure : A super $(3,1)$-path $\Phi(L,p) \in \mathcal{SP}_{41}^U(3,1)$.

- B: the point on the x-axis that following a down step, and the next down step is the first down step that goes below the x-axis;
- A: the rightmost point that $y_A = 0$, $x_A < x_B$ and A is followed by an up step;
- C: the leftmost point such that $x_C \geq x_B$ and $\forall G$, $x_G \geq x_B$ implies that $y_G \leq y_C$;
- p: the leftmost hump in L_{AB}.

$\Phi^{-1} : \mathcal{SP}_n^U(k,a) \to \mathcal{LP}_n(k,a)$.

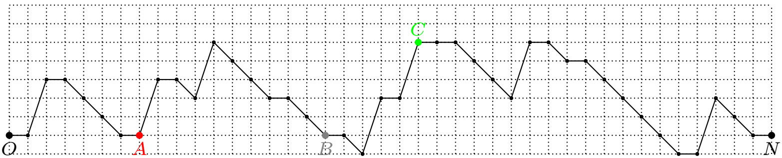


Figure : A super $(3,1)$-path $\Phi(L,p) \in \mathcal{SP}_{41}^U(3,1)$.

- B: the point on the x-axis that following a down step, and the next down step is the first down step that goes below the x-axis;
- A: the rightmost point that $y_A = 0$, $x_A < x_B$ and A is followed by an up step;
- C: the leftmost point such that $x_C \geq x_B$ and $\forall G$, $x_G \geq x_B$ implies that $y_G \leq y_C$;
- p: the leftmost hump in L_{AB}.

$\Phi^{-1} : \mathcal{SP}_n^U(k,a) \to \mathcal{LP}_n(k,a)$.

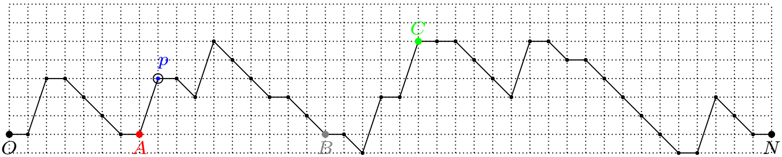


Figure : A super $(3,1)$-path $\Phi(L,p) \in \mathcal{SP}_{41}^U(3,1)$.

- B: the point on the x-axis that following a down step, and the next down step is the first down step that goes below the x-axis;
- A: the rightmost point that $y_A = 0$, $x_A < x_B$ and A is followed by an up step;
- C: the leftmost point such that $x_C \geq x_B$ and $\forall G$, $x_G \geq x_B$ implies that $y_G \leq y_C$;
- p: the leftmost hump in L_{AB}.

$\Phi^{-1} : \mathcal{SP}_n^U(k,a) \to \mathcal{LP}_n(k,a)$.

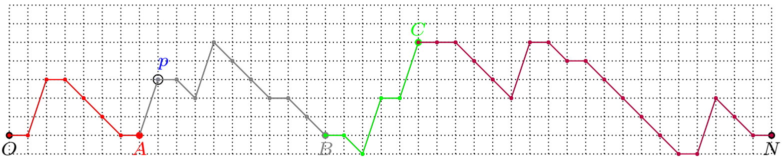
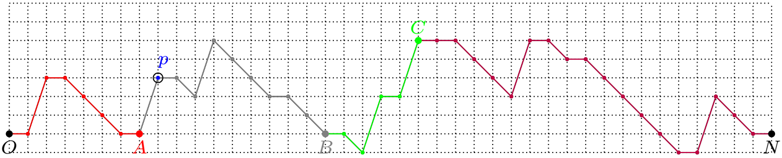
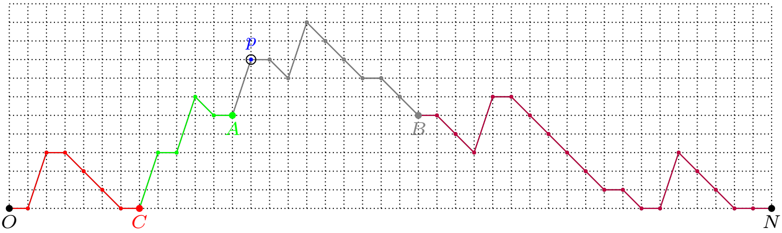


Figure : A super $(3,1)$-path $\Phi(L,p) \in \mathcal{SP}_{41}^U(3,1)$.

- B: the point on the x-axis that following a down step, and the next down step is the first down step that goes below the x-axis;
- A: the rightmost point that $y_A = 0$, $x_A < x_B$ and A is followed by an up step;
- C: the leftmost point such that $x_C \geq x_B$ and $\forall G$, $x_G \geq x_B$ implies that $y_G \leq y_C$;
- p: the leftmost hump in L_{AB}.

$\Phi : \mathcal{LP}_n(k,a) \to \mathcal{SP}_n^U(k,a)$.





Proof of Equation (1.1) and (1.2)

$$(k+1) \sum_{P \in \mathcal{P}_n(k,a)} \#\mathrm{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n},$$

follows immediately from Lemma 2.1 and Theorem 2.2.

For Equation (1.2), we see that given $(L, p) \in \mathcal{LP}_n(k, a)$, if the specified hump p in L is a not peak, then in the resulting super (k, a)-path SL, the leftmost hump in L_{AB} is not a peak. From Lemma 2.1 we know that there are

$$\frac{1}{k+1}\left(|\mathcal{SP}_{n-a}(k,a)| - \delta_{a|(n-a)}\right)$$

such paths in $\mathcal{SP}^U_n(k, a)$. Therefore we have

$$\begin{aligned} &\sum_{P \in \mathcal{P}_n(k,a)} \#\mathrm{Peaks}(P) \\ = \quad & \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right) - \frac{1}{k+1}\left(|\mathcal{SP}_{n-a}(k,a)| - \delta_{a|(n-a)}\right) \\ = \quad & \frac{1}{k+1}(|\mathcal{SP}_n(k,a)| - |\mathcal{SP}_{n-a}(k,a)|). \end{aligned}$$

Proof of Equation (1.1) and (1.2)

$$(k+1)\sum_{P\in\mathcal{P}_n(k,a)}\#\text{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n},$$

follows immediately from Lemma 2.1 and Theorem 2.2.

For Equation (1.2), we see that given $(L,p)\in\mathcal{LP}_n(k,a)$, if the specified hump p in L is a not peak, then in the resulting super (k,a)-path SL, the leftmost hump in L_{AB} is not a peak. From Lemma 2.1 we know that there are

$$\frac{1}{k+1}\left(|\mathcal{SP}_{n-a}(k,a)| - \delta_{a|(n-a)}\right)$$

such paths in $\mathcal{SP}_n^U(k,a)$. Therefore we have

$$\begin{aligned}&\sum_{P\in\mathcal{P}_n(k,a)}\#\text{Peaks}(P)\\ =\;& \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right) - \frac{1}{k+1}\left(|\mathcal{SP}_{n-a}(k,a)| - \delta_{a|(n-a)}\right)\\ =\;& \frac{1}{k+1}(|\mathcal{SP}_n(k,a)| - |\mathcal{SP}_{n-a}(k,a)|).\end{aligned}$$

Proof of Equation (1.1) and (1.2)

$$(k+1)\sum_{P\in\mathcal{P}_n(k,a)} \#\text{Humps}(P) = |\mathcal{SP}_n(k,a)| - \delta_{a|n},$$

follows immediately from Lemma 2.1 and Theorem 2.2.

For Equation (1.2), we see that given $(L,p) \in \mathcal{LP}_n(k,a)$, if the specified hump p in L is a not peak, then in the resulting super (k,a)-path SL, the leftmost hump in L_{AB} is not a peak. From Lemma 2.1 we know that there are

$$\frac{1}{k+1}\left(|\mathcal{SP}_{n-a}(k,a)| - \delta_{a|(n-a)}\right)$$

such paths in $\mathcal{SP}^U_n(k,a)$. Therefore we have

$$\begin{aligned}
&\sum_{P\in\mathcal{P}_n(k,a)} \#\text{Peaks}(P) \\
=\;& \frac{1}{k+1}\left(|\mathcal{SP}_n(k,a)| - \delta_{a|n}\right) - \frac{1}{k+1}\left(|\mathcal{SP}_{n-a}(k,a)| - \delta_{a|(n-a)}\right) \\
=\;& \frac{1}{k+1}(|\mathcal{SP}_n(k,a)| - |\mathcal{SP}_{n-a}(k,a)|).
\end{aligned}$$

Remark 1

Yan also give a bijection which proofs (1.1) and (1.2), but her bijection is different from our bijection Φ.

Remark 2

Note that when defining the bijection Φ, the parameters k and a do not really matter. Let S be a set of positive integers, we define an (S, a)-path of order n to be a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0,0)$ to $(n, 0)$ using up steps $U = (1, k), k \in S$, down steps $D = (1, -1)$ and horizontal steps $H = (a, 0)$ and never goes below the x-axis.

Therefore, our bijection Φ proves the following stronger result for (S, a)-paths:

Corollary 2.3

The total number of humps in all (S, a)-paths of order n equals the total number of supper (S, a)-paths of order n whose first non-horziontal step is an up step.

Remark 1

Yan also give a bijection which proofs (1.1) and (1.2), but her bijection is different from our bijection Φ.

Remark 2

Note that when defining the bijection Φ, the parameters k and a do not really matter. Let S be a set of positive integers, we define an (S, a)-path of order n to be a lattice path in $\mathbb{Z} \times \mathbb{Z}$ from $(0, 0)$ to $(n, 0)$ using up steps $U = (1, k), k \in S$, down steps $D = (1, -1)$ and horizontal steps $H = (a, 0)$ and never goes below the x-axis.

Therefore, our bijection Φ proves the following stronger result for (S, a)-paths:

Corollary 2.3

The total number of humps in all (S, a)-paths of order n equals the total number of supper (S, a)-paths of order n whose first non-horzional step is an up step.

Remark 1

Yan also give a bijection which proofs (1.1) and (1.2), but her bijection is different from our bijection Φ.

Remark 2

Note that when defining the bijection Φ*, the parameters* k *and* a *do not really matter. Let* S *be a set of positive integers, we define an* (S, a)*-path of order* n *to be a lattice path in* $\mathbb{Z} \times \mathbb{Z}$ *from* $(0, 0)$ *to* $(n, 0)$ *using up steps* $U = (1, k), k \in S$*, down steps* $D = (1, -1)$ *and horizontal steps* $H = (a, 0)$ *and never goes below the* x*-axis.*

Therefore, our bijection Φ proves the following stronger result for (S, a)-paths:

Corollary 2.3

The total number of humps in all (S, a)*-paths of order* n *equals the total number of supper* (S, a)*-paths of order* n *whose first non-horzional step is an up step.*

(n, m)-Dyck paths

We also extend our study to the relation between peaks of (n, m)-Dyck paths and free (n, m)-paths. (n, m)-Dyck paths are related to simultaneous core partitions, and are studied by many authors: Bizley (1954), Fukukama(2013), and Armstrong, Rhoades and Williams (2013).

An *(n, m)-Dyck path* is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0, 0)$ to (n, m), using up steps $(0, 1)$ and down steps $(1, 0)$ and never goes below the diagonal line.

Example 1

There are 7 $(3, 5)$-Dyck paths:

$\mathcal{D}(n, m)$: the set of (n, m)-Dyck paths.

$\mathcal{F}(n, m)$: the set of free (n, m)-paths (allowed to go below the diagonal).

(n, m)-Dyck paths

We also extend our study to the relation between peaks of (n, m)-Dyck paths and free (n, m)-paths. (n, m)-Dyck paths are related to simultaneous core partitions, and are studied by many authors: Bizley (1954), Fukukama(2013), and Armstrong, Rhoades and Williams (2013).

An *(n, m)-Dyck path* is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0, 0)$ to (n, m), using up steps $(0, 1)$ and down steps $(1, 0)$ and never goes below the diagonal line.

Example 1

There are 7 $(3, 5)$-Dyck paths:

$\mathcal{D}(n, m)$: the set of (n, m)-Dyck paths.

$\mathcal{F}(n, m)$: the set of free (n, m)-paths (allowed to go below the diagonal).

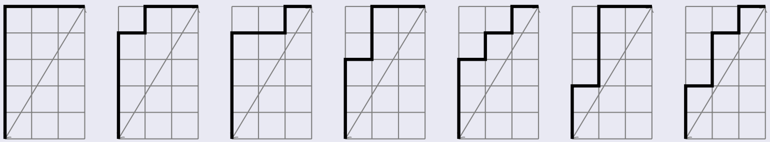
(n, m)-Dyck paths

We also extend our study to the relation between peaks of (n, m)-Dyck paths and free (n, m)-paths. (n, m)-Dyck paths are related to simultaneous core partitions, and are studied by many authors: Bizley (1954), Fukukama(2013), and Armstrong, Rhoades and Williams (2013).

An *(n, m)-Dyck path* is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0, 0)$ to (n, m), using up steps $(0, 1)$ and down steps $(1, 0)$ and never goes below the diagonal line.

Example 1

There are 7 $(3, 5)$-Dyck paths:



$\mathcal{D}(n, m)$: the set of (n, m)-Dyck paths.

$\mathcal{F}(n, m)$: the set of free (n, m)-paths (allowed to go below the diagonal).

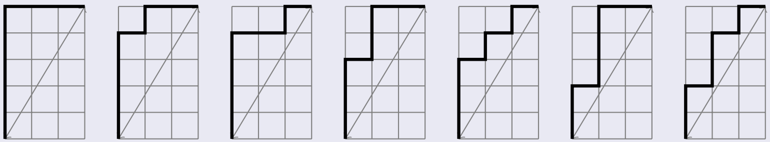
(n, m)-Dyck paths

We also extend our study to the relation between peaks of (n, m)-Dyck paths and free (n, m)-paths. (n, m)-Dyck paths are related to simultaneous core partitions, and are studied by many authors: Bizley (1954), Fukukama(2013), and Armstrong, Rhoades and Williams (2013).

An *(n, m)-Dyck path* is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0, 0)$ to (n, m), using up steps $(0, 1)$ and down steps $(1, 0)$ and never goes below the diagonal line.

Example 1

There are 7 $(3, 5)$-Dyck paths:



$\mathcal{D}(n, m)$: the set of (n, m)-Dyck paths.

$\mathcal{F}(n, m)$: the set of free (n, m)-paths (allowed to go below the diagonal).

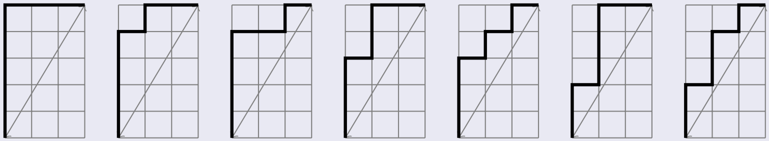
(n, m)-Dyck paths

We also extend our study to the relation between peaks of (n, m)-Dyck paths and free (n, m)-paths. (n, m)-Dyck paths are related to simultaneous core partitions, and are studied by many authors: Bizley (1954), Fukukama(2013), and Armstrong, Rhoades and Williams (2013).

An *(n, m)-Dyck path* is a lattice path in $\mathbb{Z} \times \mathbb{Z}$, from $(0, 0)$ to (n, m), using up steps $(0, 1)$ and down steps $(1, 0)$ and never goes below the diagonal line.

Example 1

There are 7 $(3, 5)$-Dyck paths:



$\mathcal{D}(n, m)$: the set of (n, m)-Dyck paths.

$\mathcal{F}(n, m)$: the set of free (n, m)-paths (allowed to go below the diagonal).

Some Known Properties of the equivalence class of P when $gcd(n, m) = 1$.

$P = u_1 u_2 \cdots u_{m+n}$ and $Q = v_1 v_2 \cdots v_{n+m}$ are equivalent if and only if there is some $i, 1 \leq i \leq n+m$ such that $u_{i+1} \cdots u_{n+m} u_1 \cdots u_i = v_1 v_2 \cdots v_{n+m}$.
$[P]$: the equivalence class of P.

Lemma 3.1

For any free path P from $(0,0)$ to (n, m), if $gcd(n, m) = 1$, then

1) $|[P]| = n + m$;
2) *There is a unique (n, m)-Dyck path in $[P]$.*

Figure : A free path P from $(0,0)$ to $(2,3)$, and the 5 different free paths in $[P]$.

Some Known Properties of the equivalence class of P when $gcd(n, m) = 1$.

$P = u_1 u_2 \cdots u_{m+n}$ and $Q = v_1 v_2 \cdots v_{n+m}$ are equivalent if and only if there is some $i, 1 \leq i \leq n+m$ such that $u_{i+1} \cdots u_{n+m} u_1 \cdots u_i = v_1 v_2 \cdots v_{n+m}$.
$[P]$: the equivalence class of P.

Lemma 3.1

For any free path P from $(0, 0)$ to (n, m), if $gcd(n, m) = 1$, then

1) $|[P]| = n + m$;
2) *There is a unique (n, m)-Dyck path in $[P]$.*

Figure : A free path P from $(0, 0)$ to $(2, 3)$, and the 5 different free paths in $[P]$.

Some Known Properties of the equivalence class of P when $gcd(n,m)=1$.

$P = u_1u_2\cdots u_{m+n}$ and $Q = v_1v_2\cdots v_{n+m}$ are equivalent if and only if there is some $i, 1 \le i \le n+m$ such that $u_{i+1}\cdots u_{n+m}u_1\cdots u_i = v_1v_2\cdots v_{n+m}$.
$[P]$: the equivalence class of P.

Lemma 3.1

For any free path P from $(0,0)$ to (n,m), if $gcd(n,m)=1$, then

1) $|[P]| = n+m$;
2) *There is a unique (n,m)-Dyck path in $[P]$.*

Figure : A free path P from $(0,0)$ to $(2,3)$, and the 5 different free paths in $[P]$.

Some Known Properties of the equivalence class of P when $gcd(n, m) = 1$.

$P = u_1u_2 \cdots u_{m+n}$ and $Q = v_1v_2 \cdots v_{n+m}$ are equivalent if and only if there is some $i, 1 \le i \le n + m$ such that $u_{i+1} \cdots u_{n+m}u_1 \cdots u_i = v_1v_2 \cdots v_{n+m}$.
$[P]$: the equivalence class of P.

Lemma 3.1

For any free path P from $(0, 0)$ to (n, m), if $gcd(n, m) = 1$, then

1) $|[P]| = n + m$;
2) *There is a unique (n, m)-Dyck path in $[P]$.*

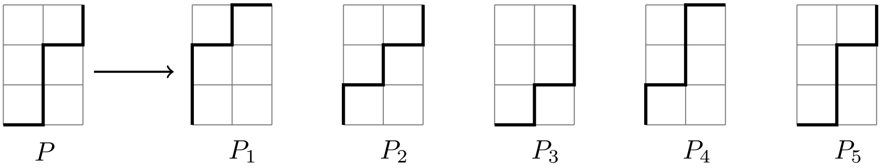


Figure : A free path P from $(0, 0)$ to $(2, 3)$, and the 5 different free paths in $[P]$.

Number of (n, m)-Dyck paths

Corollary 3.2

If $gcd(n, m) = 1$, then the number of (n, m)-Dyck paths is

$$D(n, m) = \frac{1}{n+m}\binom{n+m}{n}. \tag{3.1}$$

It is also proved that when $gcd(n, m) = d$, the number of (n, m)-Dyck paths is

$$\sum_{a}\prod_{i=1}^{d}\left(\frac{1}{a_i!}D^{a_i}(\frac{i}{d}m, \frac{i}{d}n)\right). \tag{3.2}$$

where the sum $\sum_a$ is taken over all sequences of non-negative integers $a = (a_1, a_2, \cdots)$ such that $\sum_{i=1}^{\infty} ia_i = d$.

Number of (n, m)-Dyck paths

Corollary 3.2

If $gcd(n, m) = 1$, *then the number of* (n, m)*-Dyck paths is*

$$D(n, m) = \frac{1}{n+m}\binom{n+m}{n}. \tag{3.1}$$

It is also proved that when $gcd(n, m) = d$, the number of (n, m)-Dyck paths is

$$\sum_{a}\prod_{i=1}^{d}\left(\frac{1}{a_i!}D^{a_i}(\frac{i}{d}m, \frac{i}{d}n)\right). \tag{3.2}$$

where the sum $\sum_a$ is taken over all sequences of non-negative integers $a = (a_1, a_2, \cdots)$ such that $\sum_{i=1}^{\infty} ia_i = d$.

Theorem 3.3: $\hat{\Phi} : \mathcal{PD}(n,m;j) \to \mathcal{F}^{UD}(n,m;j)$

$\mathcal{PD}(n,m;j) = \{(P,p) | P \in \mathcal{D}(n,m;j),\ p \text{ is a peak of } P\}$,
$\mathcal{F}^{UD}(n,m;j)$: the set of free paths in $\mathcal{F}^{UD}(n,m;j)$ that start with an up step and end with a down step.

Theorem 3.3

Then there is a bijection $\hat{\Phi} : \mathcal{PD}(n,m;j) \to \mathcal{F}^{UD}(n,m;j)$ when $gcd(n,m) = 1$.

Theorem 3.3: $\hat{\Phi} : \mathcal{PD}(n,m;j) \to \mathcal{F}^{UD}(n,m;j)$

$\mathcal{PD}(n,m;j) = \{(P,p) | P \in \mathcal{D}(n,m;j),\ p$ is a peak of $P\}$,
$\mathcal{F}^{UD}(n,m;j)$: the set of free paths in $\mathcal{F}^{UD}(n,m;j)$ that start with an up step and end with a down step.

Theorem 3.3

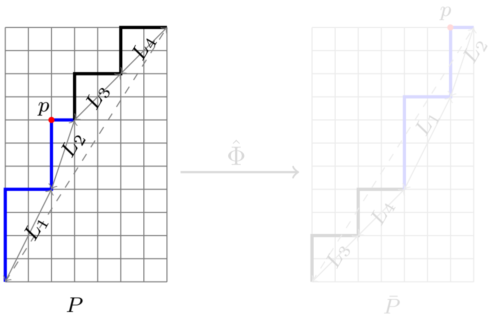
Then there is a bijection $\hat{\Phi} : \mathcal{PD}(n,m;j) \to \mathcal{F}^{UD}(n,m;j)$ when $gcd(n,m) = 1$.

Theorem 3.3: $\hat{\Phi} : \mathcal{PD}(n,m;j) \to \mathcal{F}^{UD}(n,m;j)$

$\mathcal{PD}(n,m;j) = \{(P,p) | P \in \mathcal{D}(n,m;j),\ p$ is a peak of $P\}$,
$\mathcal{F}^{UD}(n,m;j)$: the set of free paths in $\mathcal{F}^{UD}(n,m;j)$ that start with an up step and end with a down step.

Theorem 3.3

Then there is a bijection $\hat{\Phi} : \mathcal{PD}(n,m;j) \to \mathcal{F}^{UD}(n,m;j)$ *when* $gcd(n,m) = 1$.

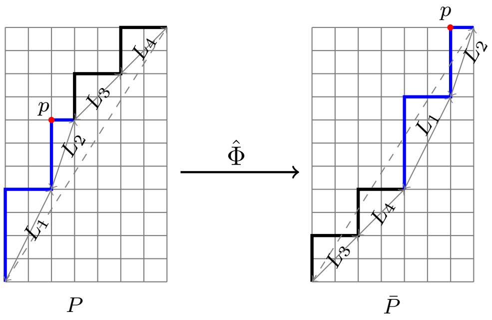


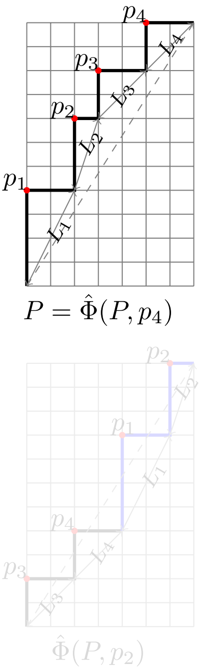
Theorem 3.3: $\hat{\Phi} : \mathcal{PD}(n, m; j) \to \mathcal{F}^{UD}(n, m; j)$

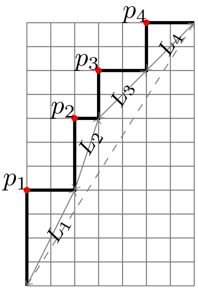
$\mathcal{PD}(n, m; j) = \{(P, p) | P \in \mathcal{D}(n, m; j),\ p$ is a peak of $P\}$,
$\mathcal{F}^{UD}(n, m; j)$: the set of free paths in $\mathcal{F}^{UD}(n, m; j)$ that start with an up step and end with a down step.

Theorem 3.3

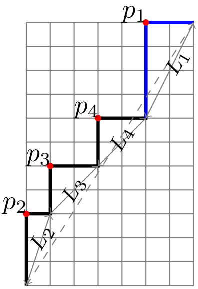
Then there is a bijection $\hat{\Phi} : \mathcal{PD}(n, m; j) \to \mathcal{F}^{UD}(n, m; j)$ *when* $gcd(n, m) = 1$.




p_4
p_3
p_2
p_1
L_1
L_2
L_3
L_4
$P = \hat{\Phi}(P, p_4)$
$\hat{\Phi}(P, p_1)$
$\hat{\Phi}(P, p_2)$
$\hat{\Phi}(P, p_3)$



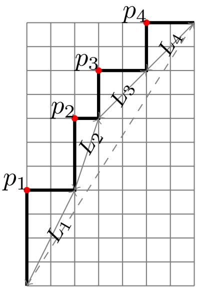
$P = \hat{\Phi}(P, p_4)$



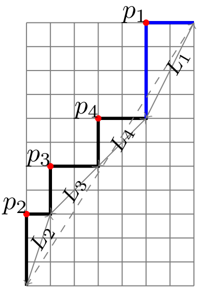
$\hat{\Phi}(P, p_1)$

$\hat{\Phi}(P, p_2)$

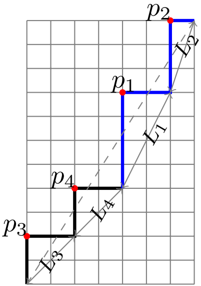
$\hat{\Phi}(P, p_3)$



$P = \hat{\Phi}(P, p_4)$

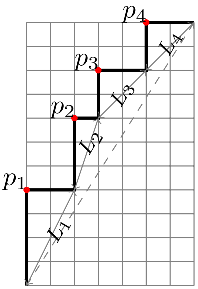


$\hat{\Phi}(P, p_1)$

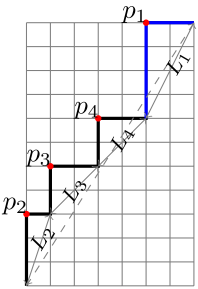


$\hat{\Phi}(P, p_2)$

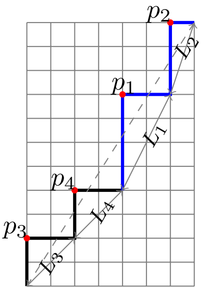
$\hat{\Phi}(P, p_3)$



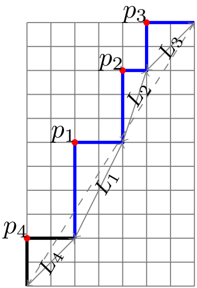
$P = \hat{\Phi}(P, p_4)$



$\hat{\Phi}(P, p_1)$



$\hat{\Phi}(P, p_2)$



$\hat{\Phi}(P, p_3)$

Enumerating (n, m)-Dyck paths with a given number of peaks

Lemma 3.4

The number of free paths from $(0, 0)$ to (n, m) with j peaks is

$$|\mathcal{F}(n, m; j)| = \binom{n}{j}\binom{m}{j}; \tag{3.3}$$

and

$$|\mathcal{F}^{UD}(n, m; j)| = \binom{n-1}{j-1}\binom{m-1}{j-1}; \tag{3.4}$$

Theorem 3.5

When $gcd(n, m) = 1$, the number of (n, m)-Dyck paths with exactly j peaks is:

$$D(n, m; j) = \frac{1}{j}\binom{n-1}{j-1}\binom{m-1}{j-1}. \tag{3.5}$$

(3.5) is also given by Armstrong, Rhoades and Williams, in which the authors call it "rational Narayana number".

Enumerating (n, m)-Dyck paths with a given number of peaks

Lemma 3.4

The number of free paths from $(0,0)$ *to* (n,m) *with* j *peaks is*

$$|\mathcal{F}(n,m;j)| = \binom{n}{j}\binom{m}{j}; \tag{3.3}$$

and

$$|\mathcal{F}^{UD}(n,m;j)| = \binom{n-1}{j-1}\binom{m-1}{j-1}; \tag{3.4}$$

Theorem 3.5

When $gcd(n,m) = 1$, *the number of* (n,m)*-Dyck paths with exactly* j *peaks is:*

$$D(n,m;j) = \frac{1}{j}\binom{n-1}{j-1}\binom{m-1}{j-1}. \tag{3.5}$$

(3.5) is also given by Armstrong, Rhoades and Williams, in which the authors call it "rational Narayana number".

Enumerating (n, m)-Dyck paths with a given number of peaks

Lemma 3.4

The number of free paths from $(0,0)$ to (n,m) with j peaks is

$$|\mathcal{F}(n,m;j)| = \binom{n}{j}\binom{m}{j}; \tag{3.3}$$

and

$$|\mathcal{F}^{UD}(n,m;j)| = \binom{n-1}{j-1}\binom{m-1}{j-1}; \tag{3.4}$$

Theorem 3.5

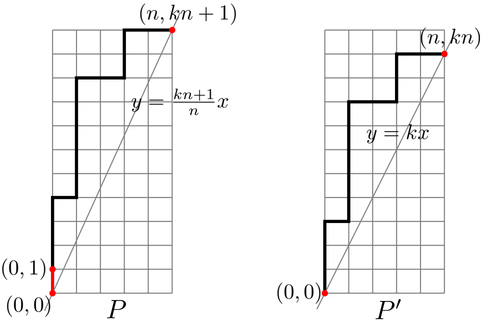
When $gcd(n,m) = 1$, the number of (n,m)-Dyck paths with exactly j peaks is:

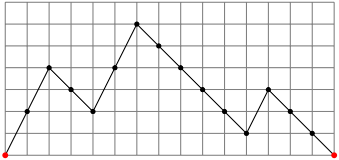
$$D(n,m;j) = \frac{1}{j}\binom{n-1}{j-1}\binom{m-1}{j-1}. \tag{3.5}$$

(3.5) is also given by Armstrong, Rhoades and Williams, in which the authors call it "rational Narayana number".

Lemma 4.1

There is one-to-one correspondences between the following sets: k-ary paths of order n; (n, kn)-Dyck paths, and $(n, kn+1)$ paths.





k-ary paths of order of n with a given number of peaks

Corollary 4.2

- *The number of k-ary paths of order n is:*

$$\frac{1}{kn+1}\binom{(k+1)n}{n}, \tag{4.1}$$

- *The number of k-ary paths of order n with exactly j peaks is:*

$$D(n,k;j) = \frac{1}{j}\binom{n-1}{j-1}\binom{kn}{j-1}. \tag{4.2}$$

Note that when $k = 1$. Equation (4.1) and (4.2) coincide with the well-known result that Dyck path of order n is counted by the n-th Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$, and the number of Dyck path of order n with exactly j peaks is the Narayana number $N(n;j) = \frac{1}{j}\binom{n-1}{j-1}\binom{n}{j-1}$.

k-ary paths of order of n with a given number of peaks

Corollary 4.2

- *The number of k-ary paths of order n is:*

$$\frac{1}{kn+1}\binom{(k+1)n}{n}, \tag{4.1}$$

- *The number of k-ary paths of order n with exactly j peaks is:*

$$D(n,k;j) = \frac{1}{j}\binom{n-1}{j-1}\binom{kn}{j-1}. \tag{4.2}$$

Note that when $k = 1$. Equation (4.1) and (4.2) coincide with the well-known result that Dyck path of order n is counted by the n-th Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$, and the number of Dyck path of order n with exactly j peaks is the Narayana number $N(n;j) = \frac{1}{j}\binom{n-1}{j-1}\binom{n}{j-1}$.

k-ary paths of order of n with a given number of peaks

Corollary 4.2

- *The number of k-ary paths of order n is:*

$$\frac{1}{kn+1}\binom{(k+1)n}{n}, \tag{4.1}$$

- *The number of k-ary paths of order n with exactly j peaks is:*

$$D(n,k;j) = \frac{1}{j}\binom{n-1}{j-1}\binom{kn}{j-1}. \tag{4.2}$$

Note that when $k = 1$. Equation (4.1) and (4.2) coincide with the well-known result that Dyck path of order n is counted by the n-th Catalan number $C(n) = \frac{1}{n+1}\binom{2n}{n}$, and the number of Dyck path of order n with exactly j peaks is the Narayana number $N(n;j) = \frac{1}{j}\binom{n-1}{j-1}\binom{n}{j-1}$.

Motzkin paths and Standard Young Tableaux

f^λ : the number of SYT of shape λ;

$\mathcal{H}(k, l; n) = \{\lambda = (\lambda_1, \lambda_2, \cdots) | \lambda \vdash n, \lambda_{k+1} \leq l\}$;

$S(k, l; n) = \sum_{\lambda \in \mathcal{H}(k,l;n)} f^\lambda$: number of SYT in a (k, l)-hook.

Figure : Standard Young Tableaux in S(2,1;4)

Motzkin paths and Standard Young Tableaux

f^λ : the number of SYT of shape λ;

$\mathcal{H}(k,l;n) = \{\lambda = (\lambda_1, \lambda_2, \cdots)|\lambda \vdash n, \lambda_{k+1} \leq l\}$;

$S(k,l;n) = \sum_{\lambda \in \mathcal{H}(k,l;n)} f^\lambda$: number of SYT in a (k,l)-hook.

1	2	3	4

1	2	3
4		

1	2	4
3		

1	3	4
2		

1	2
3	4

1	3
2	4

1	2
3	
4	

1	3
2	
4	

1	4
2	
3	

1
2
3
4

Figure : Standard Young Tableaux in S(2,1;4)

Motzkin paths and Standard Young Tableaux

f^λ : the number of SYT of shape λ;

$\mathcal{H}(k,l;n) = \{\lambda = (\lambda_1, \lambda_2, \cdots)|\lambda \vdash n, \lambda_{k+1} \le l\}$;

$S(k,l;n) = \sum_{\lambda \in \mathcal{H}(k,l;n)} f^\lambda$: number of SYT in a (k,l)-hook.

Figure : Standard Young Tableaux in S(2,1;4)

Motzkin paths and Standard Young Tableaux

f^λ : the number of SYT of shape λ;

$\mathcal{H}(k,l;n) = \{\lambda = (\lambda_1, \lambda_2, \cdots)|\lambda \vdash n, \lambda_{k+1} \leq l\}$;

$S(k,l;n) = \sum_{\lambda \in \mathcal{H}(k,l;n)} f^\lambda$: number of SYT in a (k,l)-hook.

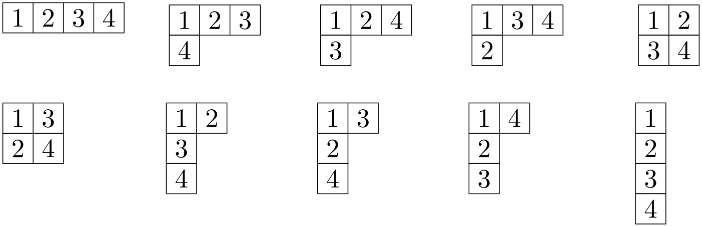


Figure : Standard Young Tableaux in S(2,1;4)

In *Asymptotic values for degrees associated with strips of Young tableau*, Adv. Math., 41, (1981), 115-136, A. Regev proved that $S(3,0;n)$ equals the n-th Motzkin number.

In *Skew-standard Tableaux with Three Rows*, arXiv: 1002. 4060 v3 [math.CO] 12 May 2010, Sen-peng Eu gave a bijection between Motzkin paths of order n and SYT of order n with at most three rows.

In *Probabilities in the (k,l) hook*, Israel J. Math. 169 61-88, 2009, A. Regev computed $S(2,1;n)$.

In *Humps for Dyck and for Motzkin paths*, arXiv: 1002. 4504v1 [math. CO] 24 Feb 2010, A. Regev noticed that the total number of humps in all Motzkin paths of order n is equal to $S(2,1;n) - 1$.

In *Asymptotic values for degrees associated with strips of Young tableau*, Adv. Math., 41, (1981), 115-136, A. Regev proved that $S(3, 0; n)$ equals the n-th Motzkin number.

In *Skew-standard Tableaux with Three Rows*, arXiv: 1002. 4060 v3 [math.CO] 12 May 2010, Sen-peng Eu gave a bijection between Motzkin paths of order n and SYT of order n with at most three rows.

In *Probabilities in the (k, l) hook*, Israel J. Math. 169 61-88, 2009, A. Regev computed $S(2, 1; n)$.

In *Humps for Dyck and for Motzkin paths*, arXiv: 1002. 4504v1 [math. CO] 24 Feb 2010, A. Regev noticed that the total number of humps in all Motzkin paths of order n is equal to $S(2, 1; n) - 1$.

In *Asymptotic values for degrees associated with strips of Young tableau*, Adv. Math., 41, (1981), 115-136, A. Regev proved that $S(3,0;n)$ equals the n-th Motzkin number.

In *Skew-standard Tableaux with Three Rows*, arXiv: 1002. 4060 v3 [math.CO] 12 May 2010, Sen-peng Eu gave a bijection between Motzkin paths of order n and SYT of order n with at most three rows.

In *Probabilities in the (k,l) hook*, Israel J. Math. 169 61-88, 2009, A. Regev computed $S(2,1;n)$.

In *Humps for Dyck and for Motzkin paths*, arXiv: 1002. 4504v1 [math. CO] 24 Feb 2010, A. Regev noticed that the total number of humps in all Motzkin paths of order n is equal to $S(2,1;n) - 1$.

In *Asymptotic values for degrees associated with strips of Young tableau*, Adv. Math., 41, (1981), 115-136, A. Regev proved that $S(3, 0; n)$ equals the n-th Motzkin number.

In *Skew-standard Tableaux with Three Rows*, arXiv: 1002. 4060 v3 [math.CO] 12 May 2010, Sen-peng Eu gave a bijection between Motzkin paths of order n and SYT of order n with at most three rows.

In *Probabilities in the (k, l) hook*, Israel J. Math. 169 61-88, 2009, A. Regev computed $S(2, 1; n)$.

In *Humps for Dyck and for Motzkin paths*, arXiv: 1002. 4504v1 [math. CO] 24 Feb 2010, A. Regev noticed that the total number of humps in all Motzkin paths of order n is equal to $S(2, 1; n) - 1$.

Is there a bijective proof?

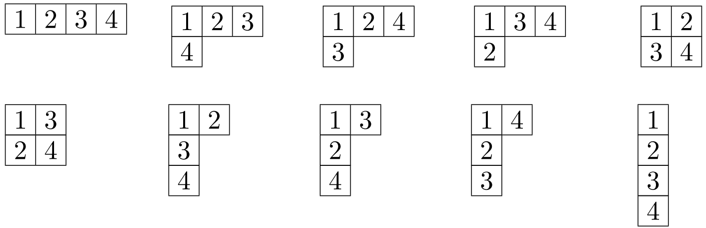


Figure : Standard Young Tableaux in S(2,1;4)

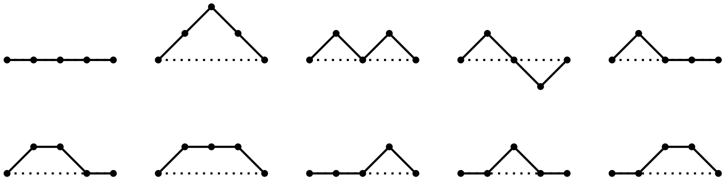


Figure : Super Motzkin paths of order 4 whose first non-horizontal step is an up step

תודה

Dankie Gracias

Спасибо شكراً

Merci Takk

Köszönjük Terima kasih

Grazie Dziękujemy Dėkojame

Ďakujeme Vielen Dank Paldies

Kiitos Täname teid 谢谢

Thank You

Tak

感謝您 Obrigado Teşekkür Ederiz

감사합니다

Σας Ευχαριστούμ

ขอบคุณ

Bedankt Děkujeme vám

ありがとうございます

Tack