“*Even more intriguing, if rather less plausible...*”

Louis J. Billera
Cornell University

Stanley@70, June 26, 2014

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Peter McMullen and Geoffrey Shephard end their 1971 London Mathematical Society Lecture Notes

Convex Polytopes and the Upper Bound Conjecture

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1. Where it came from
   - Upper Bounds for Polytopes
   - Upper Bounds for Spheres
   - Lower Bounds
2. The g-conjecture
   - Sufficiency
   - Necessity
3. Where it went (and is still going)
   - The polytope algebra
   - Nonsimplicial polytopes and the "toric" h-vector
   - Flag f-vectors and the **cd**-index
   - f-vectors of manifolds and other complexes
   - The equality case of the generalized lower bound conjecture
   - The g-conjecture for spheres

Where it came from: Upper bounds and cyclic polytopes

Upper Bound Theorem(McMullen 1970): If Q is an d-dimensional polytope with n vertices, then for any i,

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Note: It is sufficient to prove this for simplicial polytopes (every face a simplex).

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Earlier speculations by Grünbaum in 1970 and Klee in 1964.

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The h-vector and the f-vector of a polytope mutually determine

each other via the formulas (for $0 \leq i \leq d$):

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{i-j} f_{j-1}\,, \qquad f_{i-1} = \sum_{j=0}^{i} \binom{d-j}{i-j} h_j\,.$$

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$$A_\Delta = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

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$$h_0 = 1 \quad \text{and for each} \quad i \geq 1, \quad 0 \leq h_{i+1} \leq h_i^{\langle i \rangle}$$

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To prove UBC, McMullen showed for simplicial P with $f_0(P) = n$,

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Δ is a Cohen-Macaulay complex, *e.g.*, a sphere!

$h(\Delta)$ as a Hilbert function

Note: A_Δ CM means A_Δ is free module over the polynomial subring $K[\theta_1, \ldots, \theta_d]$ where $\theta_1, \ldots, \theta_d$ are generic forms in A_1

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i.e., $h_i = \dim_K B_i$

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5. Choose $t_1, t_2, \ldots, t_n$ defining $C(n, d+1)$ so that Δ is precisely the set of facets seen from some point $v \notin C(n, d+1)$. Then $\partial\Delta$ will be the boundary of a d-polytope.

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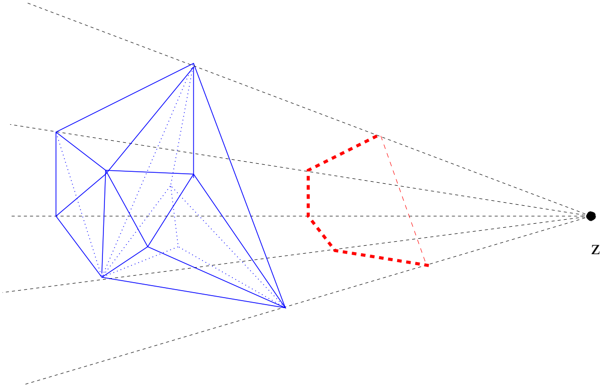

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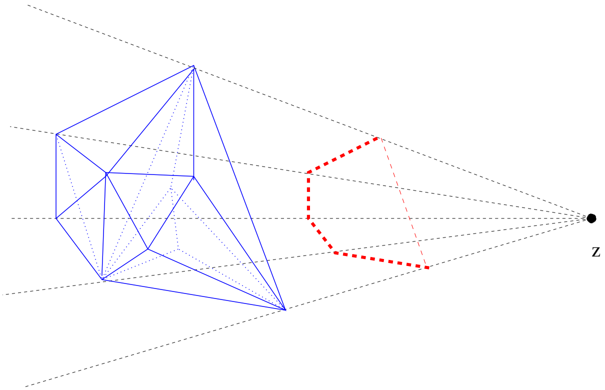

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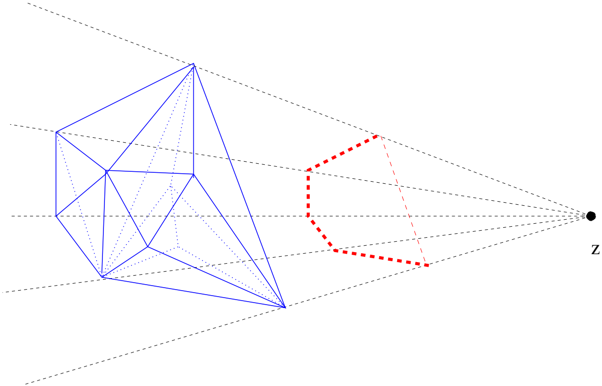

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Enter, toric varieties .....

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6. Thus $g(P)$ is an M-vector.

Where it went (and is going): Polytope algebra

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"*ridding the subject of this malignancy*".

The "toric" h-vector

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Karu (2004) showed toric g-vector nonnegative for all polytopes by an extension of the Hard Lefschetz Theorem to "combinatorial intersection homology" (piecewise polynomials on the fan but no toric variety).

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Stanley (and many others): f-vectors of simplicial posets .....

Equality case of the GLB conjecture

Murai & Nevo (2013) proved the equality case of the GLB using methods of commutative algebra. (See FPSAC 2014.)

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McMullen-Walkup (1971): *"Nevertheless, there are real differences as well as deep theoretical questions to be met with in extending results on simplicial polytopes to triangulated spheres (see Grünbaum [1970]). We have therefore satisfied ourselves with venturing the Generalized Lower-bound Conjecture for polytopes only."*

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