Nodal sets of random spherical harmonics

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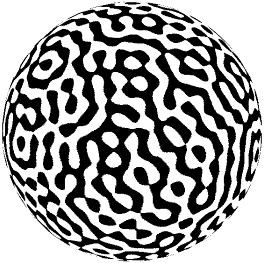

Representation Theory, Probability, and Symmetric Functions

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Random spherical harmonics:

$\mathcal{H}_n$ real Hilbert space of 2D spherical harmonics equipped with the $L^2(\mathbb{S}^2)$-norm, $\dim \mathcal{H}_n = 2n+1$, (Y_k) orthonormal basis in $\mathcal{H}_n$

(ξ_k) Gaussian IIDs, $\mathbb{E}|\xi_k|^2 = \frac{1}{2n+1}$

$f_n = \sum_{k=-n}^{n} \xi_k Y_k$ random spherical harmonic of degree n

The distribution of f_n

is independent of the choice of the ONB in $\mathcal{H}_n$

is invariant w.r.t. rotations of the sphere $\mathbb{S}^2$

$Z(f_n) = f^{-1}\{0\}$ the zero set of f_n

$N(f_n)$ the number of connected components of $Z(f_n)$ (large n)

Major difficulties to study the number of connected components:

Slow off-diagonal decay (and sign changes) of the covariance of the ensemble: $\mathbb{E}[f_n(x)f_n(y)] = P_n(\cos\Theta(x,y))$

P_n Legendre polynomial of degree n ($P_n(1) = 1$)

$\Theta(x,y)$ angle between $x, y \in \mathbb{S}^2$.

Scaled covariance: $P_n\left(cos\frac{z}{n}\right) \sim J_0(z)$, the zeroth Bessel function, $J_0(z) \sim \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{\pi}{4}\right)$

It is natural to think of f_n as defined on the sphere $n\mathbb{S}^2$ of radius n and of area $\simeq n^2$. In this scale the covariance decays as $\text{dist}^{-1/2}$.

Another difficulty: "non-locality" (contrary to the length or the Euler characteristics).

Bogomolny and Schmit percolation model

In 2001, Bogomolny and Schmit proposed a remarkable random loop model for description of the topology of the zero set $Z(f_n)$.

Their model completely ignores slow decaying correlations and is very far from being rigorous.

Attempts to digest their work stimulated much of the progress recently achieved in this area.

LLN + Exponential concentration:

THEOREM 1 (F.Nazarov, M.S., Amer. J. Math., 2009) There exists $\nu > 0$ s.t.

$$\mathbb{P}\Big[\big|N(f_n) - \nu n^2\big| > \varepsilon n^2\Big] < Ce^{-c(\varepsilon)n}$$

with $c(\varepsilon) \gtrsim \varepsilon^{15}$.

The proof gives

$$\nu = \lim_{n\to\infty} \mathbb{E}\Big[\frac{1}{\mathrm{area}(G_n)}\Big],$$

where G_n is a nodal domain of f_n on $n\mathbb{S}^2$ that contains a marked point x.

Later, we have shown that the Law of Large Numbers with a positive limit (but without the exponential concentration) holds for rather general classes of smooth Gaussian fields on $\mathbb{R}^d$ and of smooth Gaussian ensembles on manifolds (J. Math. Phys., Analysis, Geometry, 2016).

Related results and extensions:

- "derandomization" on the torus: Bourgain, Buckley – Wigman, Ingremeau;
- other topological observables: Gayet – Welschinger, Lerario – Lundberg (upper and lower bounds for mean values), Sarnak – Wigman, Canzani – Sarnak (the Law of Large Numbers);
- fields and ensembles with positive correlations: Malevich (1972, sic!), Beffara – Gayet, Beliaev – Muirhead – Wigman, Rivera – Vanneuville;
- excursion sets: Swerling (1963, sic!), Beliaev – McAuley – Muirhead.

By no means is this list complete.

Size of fluctuations of $N(f_n)$:

$N(f_n)$ the number of connected components of the zero set $Z(f_n)$.

Trivial bounds: $1 \lesssim \mathrm{Var}[N(f_n)] \lesssim n^4$.

The Bogomolny and Schmit prediction says that $\mathrm{Var}[N(f_n)]$ grows as n^2, that is, as $\mathbb{E}[N(f_n)]$.

The exponential concentration from Theorem 1

$$\mathbb{P}\left[\left|N(f_n) - \nu n^2\right| > \varepsilon n^2\right] < Ce^{-c\varepsilon^{15} n}$$

yields the upper bound: $\mathrm{Var}[N(f_n)] \lesssim n^{4-\frac{2}{15}}$.

Recent advance:

THEOREM 2 (work in progress with Fedya Nazarov)

$$\mathrm{Var}[N(f_n)] \gtrsim n^{\sigma}$$

with some $\sigma > 0$

REMARK This lower bound holds for *any* non-degenerated isotropic smooth Gaussian fields on $n\,\mathbb{S}^2$ with decay of correlations $\gtrsim \mathrm{dist}^{-c}$ with some $c > 0$.

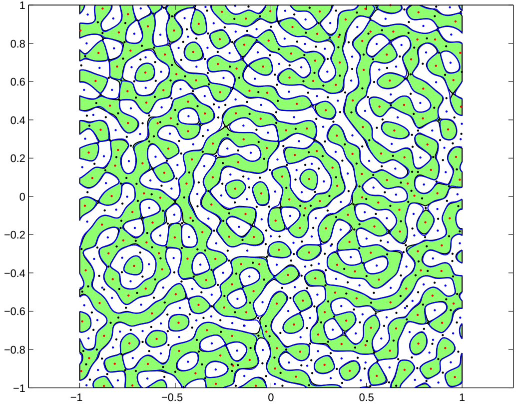
We will discuss main ideas from the proof of the lower bound. They can be viewed as the first (though, modest) step towards justification of the Bogomolny-Schmit heuristcs.

Saddle points with small critical values:

Heuristically, the fluctuations in the topology of $Z(f_n)$ are caused by saddle points of f_n with small critical values that yield so called "avoided crossings" of the zero set $Z(f_n)$.

I.e., switches in the topology of the zero set of f_n are caused by a point process that has a low intensity but strong long range dependence, as illustrated on the following simulation produced by Dima Belyaev.

Instead of random spherical harmonics Belyaev simulated so called random plane waves (RPWs) but one may safely ignore the difference (the RPW is a scaling limit of the our random spherical harmonics on $n\mathbb{S}^2$ as $n \to \infty$).



Blue lines are zero lines of a RPW F_0, blue and red points are maxima and minima of F_0, and black points are saddle points of F_0. Black lines are zero lines of the sum $F_0 + \frac{1}{10}F_1$, where F_1 is another RPW, equidistributed with F_0 and independent of F_0, green domains are connected components of the set where this sum is positive.

Step 1: Low level critical points

$f = f_n$ random spherical harmonic of degree n on $n\mathbb{S}^2$, $\mathbb{E}|f|^2 = 1$

$\mathrm{Cr}(\alpha) = \{z \in n\mathbb{S}^2 \colon \nabla f(z) = 0, |f(z)| \leqslant \alpha\}$, $0 < \alpha \ll 1$

"With high probability" (w.h.p.) means except of an event of probability $O(n^{-c})$ with some $c > 0$.

LEMMA 1 Let $n^{-2+\varepsilon} \leqslant \alpha \leqslant n^{-2+2\varepsilon}$. Then, w.h.p., the set $\mathrm{Cr}(\alpha)$ is relatively large: $|\,\mathrm{Cr}(\alpha)| \gtrsim n^{c\varepsilon}$, and the points in this set are $n^{1-C\varepsilon}$-separated.

Step 2: Introducing a small perturbation

$f_\alpha = \sqrt{1-\alpha^2} f + \alpha g$, g is an independent copy of f. The random function f_α has the same distribution as f.

We condition on f.

LEMMA 2 Let $\alpha = n^{-2+\varepsilon}$ and $\alpha' = \alpha n^{\varepsilon} = n^{-2+2\varepsilon}$. Then, w.h.p., topology of $Z(f_\alpha)$ is determined by the collection of signs of $f_\alpha(z)$ at $z \in \mathrm{Cr}(\alpha')$.

This lemma allows us "to localize" the problem. Its proof needs a caricature of a quantitative Morse theory.

Step 3: Reduction to independent percolation

Recall: $\alpha = n^{-2+\varepsilon}$, $\alpha' = n^{-2+2\varepsilon}$, $f_\alpha = \sqrt{1-\alpha^2} f + \alpha g$, g is an independent copy of f

We replace g by its independent copy g_z (some linear algebra with estimates).

This step needs a good separation between the points of $\mathrm{Cr}(\alpha')$ provided Lemma 1.

Define a collection of independent random functions $\widetilde{f_\alpha} = \sqrt{1-\alpha^2} f + \alpha g_z$, $z \in \mathrm{Cr}(\alpha')$.

LEMMA 3 W.h.p., $\mathrm{sgn}(f_\alpha(z)) = \mathrm{sgn}(\widetilde{f_\alpha}(z))$, $z \in \mathrm{Cr}(\alpha')$.

This reduces the problem to the independent random loop model on a graph with the degree of each vertex either 4 (saddle points of f) or 0 (maxima and minima of f).

Discard the latter case and assume that the degree of each vertex is 4.

Step 4: Independent percolation on some graphs

$G = G(V, E)$ a graph embedded in $n\mathbb{S}^2$

The degree of each vertex $v \in V$ is 4.

In each vertex v, we independently replace the edges crossing by one of two possible avoided crossing configuration, $p(v), 1 - p(v)$ are the corresponding probabilities.

Γ random configuration of loops,

$N(\Gamma)$ the number of loops in Γ.

LEMMA 4 : For any $p_0 > 0$,

$$\mathrm{Var}[N(\Gamma)] \geqslant c(p_0)\,|\{v \in V \colon p_0 \leqslant p(v) \leqslant 1 - p_0\}|$$

This completes the proof of the lower bound for fluctuations of $N(f_n)$ but not this talk.

Happy Birthday To Grisha!