Affine Deligne-Lusztig varieties and affine Lusztig varieties

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Classical situtation

Let G be a connected reductive group over an algebraically closed field. Let B be a Borel subgroup of G and W be its Weyl group. We have the Bruhat decomposition $G = \sqcup_{w \in W} B\dot{w}B$.

The Deligne-Lusztig variety is defined by

$$X_w = \{gB \in G/B; g^{-1}Fr(g) \in B\dot{w}B\}.$$

It describes the interaction of Frobenius-twisted conjugation action with the Bruhat cell $B\dot{w}B$.

The Lusztig variety is defined by

$$Y_w(\gamma) = \{gB \in G/B; g^{-1}\gamma g \in B\dot{w}B\}.$$

It describes the interaction of ordinary conjugation action with the Bruhat cell $B\dot{w}B$.

Motivation

The finite group of Lie type G^F acts on X_w and acts on the cohomology of X_w. This leads to the Deligne-Lusztig representation of G^F:

$$\sum_i (-1)^i H_c^i(X_w, \mathcal{F}_\theta).$$

The Lusztig variety is a key ingredient in Lusztig's theory of character sheaves. We have the following diagram

$$B\backslash G/B \xleftarrow{q} \frac{G}{B} \xrightarrow{p} \frac{G}{G},$$

where $\frac{A}{H}$ denotes the quotient stack of A under the conjugation action of the group H. The Lusztig variety describes the intersection

$$q^{-1}(B\backslash B\dot{w}B/B) \cap p^{-1}(\{\gamma\}).$$

Connection

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[33] On the reflection representation of a finite Chevalley group, 1979

The work on this paper was done in the spring of 1977; the results were presented at an LMS Symposium on Representations of Lie Groups in Oxford (July 1977). I will explain the main result of this paper using concepts which were developed several years after the paper was written (theory of character sheaves). Let G be a connected reductive group over an algebraic closure of a finite field F_q with a fixed F_q-split rational structure and Frobenius map $F : G \to G$. For each w in the Weyl group one can consider (following [22]) the variety X_w of Borel subgroups B of G such that B, FB are in position w. Then $G(F_q)$ acts naturally on the l-adic cohomology $H^i_c(X_w)$. Replacing F by conjugation by an element $g \in G$ one can consider the variety $Y_{w,g}$ of Borel subgroups B of G such that B, gBg^{-1} are in position w. The union over g in G of these varieties maps naturaly to G and we can take the direct image K_w with compact support of the sheaf $\bar{Q}_l$ under this map. Then ${}^pH^iK_w$ are perverse sheaves on G. Now for any irreducible representation E of the Weyl group we denote by E_q the corresponding irreducible representation of $G(F_q)$ which appears in $H^0_c(X_1)$ (functions on the flag manifold of $G(F_q)$) and we denote by E_1 the simple perverse sheaf on G corresponding to E which appears in K_1 (a perverse sheaf on G up to shift, with W-action). The main result of this paper is that for any w we have

$$\sum_i (-1)^i (E_q : H^i_c(X_w)) = (-1)^{\dim G} \sum_i (-1)^i (E_1 : {}^pH^iK_w)$$

Figure: Lusztig, "Comments on my papers", arXiv:1707.09368

Affine analogy

F a non-arch. local field with valuation ring $\mathcal{O}_F$ and residue field $\mathbb{F}_q$.
$\breve{F}$ completion of F^{un} with valuation ring $\mathcal{O}_{\breve{F}}$ and residue field $k = \overline{\mathbb{F}}_q$.
σ Frobenius morphism of $\breve{F}$ over F.
$\mathbf{G}$ reductive group over F, σ Frobenius morphism on $\breve{G} := \mathbf{G}(\breve{F})$, $\breve{I}$ σ-stable Iwahori subgroup, $\tilde{W}$ Iwahori-Weyl group. We have

$$\breve{G} = \sqcup_{w \in \tilde{W}} \breve{I} \dot{w} \breve{I}.$$

The affine Deligne-Lusztig varieties in the affine flag variety $Fl = \breve{G}/\breve{I}$ is

$$X_w(b) = \{g\breve{I}; g^{-1} b \sigma(g) \in \breve{I} \dot{w} \breve{I}\} \subset Fl.$$

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Motivation and major problems

- Affine Deligne-Lusztig variety was introduced by Rapoport. It serve as a group-theoretic model for the reduction of Shimura varieties and shtukas with parahoric level structure and play a vital role in arithmetic geometry and the Langlands program.
- Affine Lusztig variety was first studied by Lusztig. It encode the orbital integrals of Iwahori–Hecke functions and serve as building blocks for the (conjectural) theory of affine character sheaves.

Below we list some major problems on the affine Deligne-Lusztig varieties:

- When is an affine Deligne-Lusztig variety nonempty?
- If it is nonempty, what is its dimension?
- What are the irreducible components?

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Example of G_2

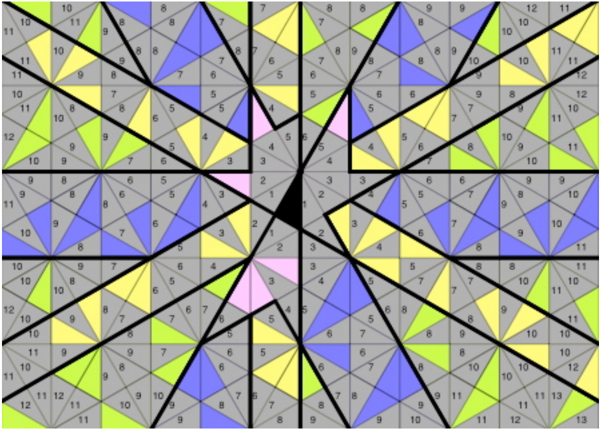


Figure: Görtz-Haines-Kottwitz-Reuman, "Dimensions of some affine Deligne-Lusztig varieties", Ann. Sci. École Norm. Sup. (2006).

State of art for ADLV

Problem	In affine Grassmannian
Nonemptiness	Mazur inequality [RR96], [Ko03], [Ga10]
Dimension	Rapoport conjecture [GHKR06], [Vi06], [Ha15], [Zhu17]
Irreducible component	Chen-Zhu conjecture [ZZ20], [Ni22], [HZZ24]

Problem	In affine flag
Nonemptiness	For basic *b*, Levi obstruction [GHKR10], [GHN15]
Dimension	Almost everywhere, dim=virtual dim [GHKR10], [H14], [H16], [H21]
Irreducible component	Known in a few cases Positive Coxeter type [HNY], [SSY]

Previously known results on ALV

Nonemptiness	In affine Grassmannian for split groups [KV12], [Chi19]
Dimension	In affine Grassmannian for split groups in equal char [Bo15], [Chi19]
Irreducible component	In affine Grassmannian for split groups for split γ in equal char [Chi19]

By global method using Hitchin fibration

Our motto

Affine Lusztig variety

↓ Affine Springer fiber

Affine Deligne-Lusztig varieties

From Affine Deligne-Lusztig varieties to affine Lusztig varieties

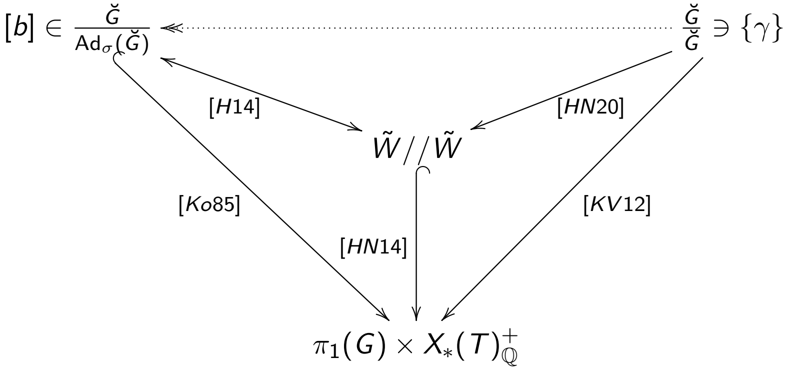
Nonemptiness ⇢ Nonemptiness

Dimension formula ⇢ Dimension formula

Enumeration of $Irr(X_w(b))$

$+$ ⇢ Enumeration of $Irr Y_w(\gamma)$

J_b-action on $Irr(X_w(b))$


$[b] \in \frac{\breve{G}}{\mathrm{Ad}_\sigma(\breve{G})}$
$\frac{\breve{G}}{\breve{G}} \ni \{\gamma\}$
[H14]
[HN20]
$\tilde{W}//\tilde{W}$
[Ko85]
[HN14]
[KV12]
$\pi_1(G) \times X_*(T)^+_{\mathbb{Q}}$

Main result, Part I

Theorem

Let γ be a regular semisimple element in $\breve{G}$ and $\{\gamma\} \mapsto [b]$. Then for any $w \in \tilde{W}$, we have

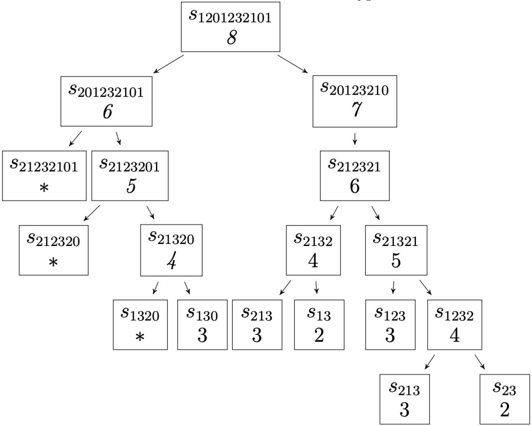
- $Y_w(\gamma) \neq \emptyset$ *if and only if* $X_w(b) \neq \emptyset$;
- $\dim Y_w(\gamma) = \dim X_w(b) + \dim Y_\gamma$.

Remark

1. *Here Y_γ is the affine Springer fiber associated with γ, the affine Springer fiber of γ in the affine flag of the Levi subgroup associated with γ.*
2. *If γ is bounded, then $Y_\gamma = Fl^\gamma$, the (ordinary) affine Springer fiber.*
3. *The dimension of affine Springer fiber in equal characteristic was conjectured by Kazhdan-Lusztig '88 and established by Bezrukavnikov '96. The mixed characteristic case was recently established by Chi '24.*

Reduction tree

FIGURE 1. Reduction tree for type $\tilde{A}_4$



Deligne-Lusztig reduction

Let $w \in \tilde{W}$ and $\mathcal{T}$ be a reduction tree of w. We denote by $\mathrm{RaP}(\mathcal{T})$ the set of reduction paths in $\mathcal{T}$.

Proposition

We have

$$X_w(b) = \bigsqcup_{\underline{p} \in RaP_{[b]}(\mathcal{T})} X_{\underline{p}}(b),$$

where $X_{\underline{p}}$ is J_b-equivariant universally homeomorphic to a fibration over $X_{end(\underline{p})}(b)$ with irreducible fibers of dimension $\ell(\underline{p})$.

$$Y_w(\gamma) = \bigsqcup_{\underline{p} \in RaP_{\{\gamma\}}(\mathcal{T})} Y_{\underline{p}}(\gamma),$$

where $Y_{\underline{p}}$ is $Z_{\breve{G}}(\gamma)$-equivariant isomorphic to a fibration over $Y_{end(\underline{p})}(\gamma)$ with irreducible fibers of dimension $\ell(\underline{p})$.

Main result, Part II

We denote by $Irr^{top}(-)$ the set of top-dimensional irreducible components. Then

Theorem

We have

$$\sharp J_b \backslash X_w(b) = \sharp\{\underline{p} \in RaP_{[b]}(\mathcal{T}) \text{ and of correct length}\},$$

$$\sharp Z_{\breve{G}}(\gamma) \backslash Y_w(\gamma) = \sum_{\underline{p} \in RaP_{\{\gamma\}}(\mathcal{T}) \text{ and of correct length}} n_{\underline{p},\gamma},$$

where $n_{\underline{p},\gamma}$ is the number of $Z_{\breve{G}}(\gamma)$-orbits on the irreducible components of affine Springer fibers associated with the pair $(\gamma, \underline{p})$.

Consequence

Theorem

For split groups, if γ has integral Newton vector and $\{\gamma\} \mapsto [b]$, then for "almost all" $w \in \tilde{W}$, we have

$$\sharp\big(Z_{\breve{G}}(\gamma)\backslash Irr^{top}(Y_w(\gamma))\big) = \sharp\big(J_b\backslash Irr^{top}(X_w(b))\big).$$

Here "almost all" means that w is in the antidominant chamber or has the regular translation part.

Theorem

For split groups, if γ has integral Newton vector λ, then

$$\sharp\big(Z_{\breve{G}}(\gamma)\backslash Irr^{top}(Y_\mu(\gamma))\big) = \dim V_\mu(\lambda).$$

In both results, we use Ngo '10 that the centralizer of γ acts transitively on the regular locus of affine Springer fibers in the affine Grassmannian.

Happy birthday, George!