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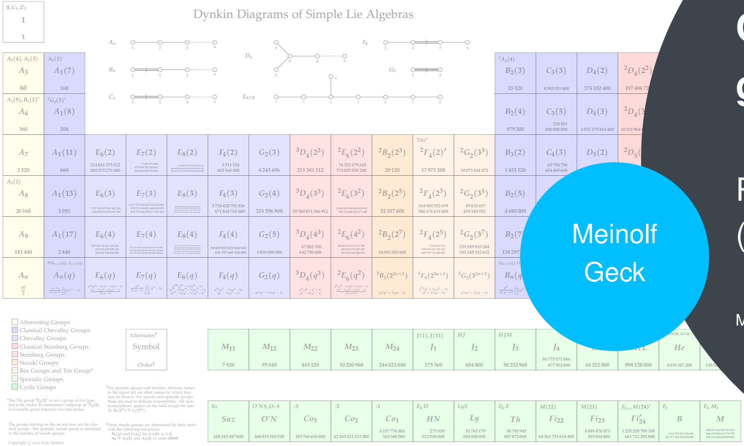

Computer algebra and groups of Lie type

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Representation Theory Days (in honor of George Lusztig)

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What can computers do in (pure) mathematics?

- Test working hypotheses in a large number of instances;
- discover new patterns by experiments previously unimaginable (by hand);
- complete the proof of theorems for which there is a reduction to the handling of a finite number of remaining cases;
- make data (that were difficult to obtain) easily/electronically accessible, thereby facilitating further work with or experiments on them.

CHEVIE **project: M. Geck, G. Hiss, F. Luebeck, G. Malle, J. Michel, G. Pfeiffer**

- Bring Lusztig's theories (characters of finite groups of Lie type, of Weyl groups and Hecke algebras, Kazhdan–Lusztig cells . . .) to the computer.
- Ongoing joint project since 1990s, see

  `http://www.math.rwth-aachen.de/~CHEVIE/`

  `https://github.com/jmichel7` (Jean Michel)

Aim of this talk: Show some applications of CHEVIE, relating to

- conjugacy classes and characters of Weyl groups;
- Kazhdan–Lusztig polynomials and cells;
- unipotent classes in algebraic groups;
- (generalised) Springer correspondence and Green functions.

My own first steps in this direction:

- ($\sim$ 1987) I "grew up" at the department of RWTH Aachen (Germany) where the computer algebra system GAP was developped.
- (Early 1990s) First GAP programs for working with finite Coxeter groups, for computing Kazhdan–Lusztig polynomials and cell representations ($\rightsquigarrow$ joint work with K. Lux on modular representations for Hecke algebra of type F_4).

Since about 2003: G. Lusztig is a quite regular user (even power user) of CHEVIE.

Example 1: Conjugacy classes and characters of Hecke algebras

W finite Coxeter group, S simple reflections, and $\ell\colon W \to \mathbb{Z}_{\geqslant 0}$ length function.

$\mathcal{H}$ Iwahori–Hecke algebra over $A = \mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ with standard basis $\{T_w \mid w \in W\}$;

$$\begin{aligned} T_w T_{w'} &= T_{ww'} && \text{if } \ell(ww') = \ell(w) + \ell(w'), \\ T_s^2 &= \mathbf{q}T_1 + (\mathbf{q} - 1)T_s && \text{for } s \in S. \end{aligned}$$

$\mathrm{cl}(W) :=$ set of conjugacy classes of W.

For $C \in \mathrm{cl}(W)$ let $d_C := \min\{\ell(w) \mid w \in C\}$ and $\boxed{C_{\min} := \{w \in C \mid \ell(w) = d_C\}}$.

For $w, w' \in W$ write $w \to w'$ if there is a sequence $w = w_1, w_2, \ldots, w_n = w'$ in W such that $\ell(w_1) \geqslant \ldots \geqslant \ell(w_n)$ and $w_{i+1} = s_i w_i s_i$ for some $s_i \in S$.

Theorem (G. and Pfeiffer 1993). Let $C \in \mathrm{cl}(W)$.

(a) For any $w \in C$, there exists some $w' \in C_{\min}$ such that $w \to w'$.

(b) If $w, w' \in C_{\min}$, then T_w and $T_{w'}$ are conjugate in $\mathcal{H}$.

Proof. Easy reduction to W irreducible; then case–by–case, using computer for W of exceptional type. Classification of $\mathrm{cl}(W)$ known by Carter (1972).

Note: to verify (a), one needs to look at *every* element of W. For (b), one needs explicit lists of $C_{\min}$. Example: $W = W(E_8)$; largest class $C \in \mathrm{cl}(W)$ with $|C| = 43{,}545{,}600$, $d_C = 7$, $|C_{\min}| = 64$. $\square$

X. He and S. Nie (Duke Math. J., 2012). Case/computer-free proof of theorem.

Our motivation for theorem: "q-deformation" of character table of W.

Let $K \supseteq A = \mathbb{Z}[\mathbf{q}, \mathbf{q}^{-1}]$ sufficiently large field and $\mathcal{H}_K := K \otimes_A \mathcal{H}$ (split semisimple).

Tits' Deformation Theorem: $\mathrm{Irr}(W) \leftrightarrow \mathrm{Irr}(\mathcal{H}_K)$, $\chi \leftrightarrow \chi_q$.

For $C \in \mathrm{cl}(W)$ fix $w_C \in C_{\min}$. Theorem $\Rightarrow$ $\chi_q(T_{w_C})$ independent of choice of w_C.

$W = \mathfrak{S}_3$	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$\mathcal{H}(\mathfrak{S}_3)$	$()$	(12)	(123)
$\chi_{q,1}$	1	$\mathbf{q}$	$\mathbf{q}^2$
$\chi_{q,2}$	1	-1	1
$\chi_{q,3}$	2	$\mathbf{q}-1$	$-\mathbf{q}$

Lusztig 1981:
$K = \mathbb{R}(\mathbf{q}^{1/2})$ sufficient,
all values in $\mathbb{R}[\mathbf{q}^{1/2}]$.

Example 2: Involutions and Kazhdan–Lusztig cells

W finite Coxeter group, S simple reflections, and $\mathbf{I} = \{w \in W \mid w^2 = 1\}$.

V vector space (over $\mathbb{C}$) with basis $\{a_w \mid w \in \mathbf{I}\}$. Linear action of W on V:

$$s.a_w = \begin{cases} -a_w & \text{if } sw = ws \text{ and } \ell(sw) < \ell(w), \\ a_{sws} & \text{otherwise.} \end{cases}$$

(Kottwitz 2000, above formulation from Lusztig–Vogan 2011.)

Kottwitz' Conjecture. Let $C \in \mathrm{cl}(\mathbf{I})$ and $V_C = \langle a_w \mid w \in C \rangle_{\mathbb{C}}$ (submodule of V).

Let $\Gamma \subseteq W$ be a Kazhdan–Lusztig left cell. Then

$$|C \cap \Gamma| = \dim \mathrm{Hom}_W(V_C, [\Gamma]_1), \quad \text{where} \quad [\Gamma]_1 = W\text{-module carried by } \Gamma.$$

Kottwitz, Casselman ("Verifying Kottwitz' conjecture by computer", 2000), Marberg (2013), Bonnafé and G. (2012–2015): known in all cases except for $W = W(E_8)$.

Big (computational) challenge: Find partition of $\mathbf{I}$ into left cells for type E_8.

For $\chi \in \mathrm{Irr}(W)$ let $D_\chi \in \mathbb{R}[\mathbf{q}]$ be the "generic degree" (Benson–Curtis 1972);

$$D_\chi = \frac{1}{f_\chi}\mathbf{q}^{a_\chi} + \text{higher powers of } \mathbf{q}, \quad \text{where } f_\chi \in \mathbb{R}_{>0},\ a_\chi \in \mathbb{Z}_{\geqslant 0}.$$

For $w \in W$, we have $\mathbf{q}^{(a_\chi - \ell(w))/2}\chi_{\mathbf{q}}(T_w) \in \mathbb{R}[\mathbf{q}^{1/2}]$. Lusztig's "leading coefficients":

$$c_{w,\chi} := \text{constant term of } (-1)^{\ell(w)}\mathbf{q}^{(a_\chi-\ell(w))/2}\chi_{\mathbf{q}}(T_w).$$

Theorem (Lusztig 1986). Let $w \in \mathbf{I}$ (involution).

- There exists some $\chi \in \mathrm{Irr}(W)$ such that $c_{w,\chi} \neq 0$.
- Every left cell Γ contains a unique $w \in \mathbf{I}$ such that $\sum_{\chi\in\mathrm{Irr}(W)} f_\chi^{-1} c_{w,\chi} \neq 0$; furthermore, $c_{w,\chi} =$ multiplicity of $\chi \in \mathrm{Irr}(W)$ in character of $[\Gamma]_1$.

Now W of type E_8; then $|\mathbf{I}| = 199952$ and number of left cells $= 101796$.

Y. Chen (2000): Let $w, w' \in \mathbf{I}$ and $\chi, \chi' \in \mathrm{Irr}(W)$ with $c_{w,\chi} \neq 0$ and $c_{w',\chi'} \neq 0$. Then w, w' belong to the same left cell if and only if $a_\chi = a_{\chi'}$ and w, w' have the same generalised τ-invariant (Vogan).

All this can be computed for the 199952 involutions (G.–Halls, Math. Comp. 2015).

Example 3: Bruhat cells and unipotent classes

Let $\mathbf{G}$ be a simple algebraic group over an algebraically closed field k,
$\mathbf{B} \subseteq \mathbf{G}$ a Borel subgroup, $\mathbf{T} \subseteq \mathbf{B}$ a maximal torus and $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$.

$\mathrm{cl}_{\mathrm{uni}}(\mathbf{G}) :=$ set of unipotent conjugacy classes of $\mathbf{G}$.
Consider intersections $\boxed{\mathcal{O} \cap \mathbf{B}\dot{w}\mathbf{B}}$ for $\mathcal{O} \in \mathrm{cl}_{\mathrm{uni}}(\mathbf{G})$ and $w \in W$.

Lemma (Lusztig). Let $w, w' \in C_{\min}$ for some $C \in \mathrm{cl}(W)$. Then

$$\mathcal{O} \cap \mathbf{B}\dot{w}\mathbf{B} \neq \varnothing \quad \Leftrightarrow \quad \mathcal{O} \cap \mathbf{B}\dot{w}'\mathbf{B} \neq \varnothing.$$

(Follows from theorem of G.–Pfeiffer.) For $\mathcal{O} \in \mathrm{cl}_{\mathrm{uni}}(\mathbf{G})$ and $C \in \mathrm{cl}(W)$ write

$C \dashv \mathcal{O}$ if $\mathcal{O} \cap \mathbf{B}\dot{w}\mathbf{B} \neq \varnothing$ for some/any $w \in C_{\min}$.

Theorem (Lusztig 2011, 2012 + Lusztig–Xue 2012).

There is a well-defined, surjective map $\Phi \colon \mathrm{cl}(W) \to \mathrm{cl}_{\mathrm{uni}}(\mathbf{G})$ such that
$C \dashv \Phi(C)$ and such that if $\mathcal{O}' \in \mathrm{cl}_{\mathrm{uni}}(\mathbf{G})$ and $C \dashv \mathcal{O}'$, then $\Phi(C) \subseteq \overline{\mathcal{O}}'$.

Thus, $\mathcal{O} = \Phi(C) \in \mathrm{cl}_{\mathrm{uni}}(\mathbf{G})$ is minimal such that $\mathcal{O} \cap \mathbf{B}\dot{w}\mathbf{B} \neq \varnothing$ for $w \in C_{\min}$

Examples: $\Phi(\{1_W\}) = \{1_{\mathbf{G}}\}$, $\Phi(\{\text{Coxeter elements}\}) =$ regular unipotent class.

Proof. Case–by–case, using very complicated computations for $\mathbf{G}$ of classical type. For $\mathbf{G}$ of exceptional type, work over $k = \overline{\mathbb{F}}_p$ and consider finite group of rational points $\mathbf{G}(q)$ where $q =$ large power of p. Then matrix of intersections

$$\Big(|(\mathcal{O} \cap \mathbf{B}\dot{w}_C\mathbf{B})(q)|\Big)_{\mathcal{O}\in\mathrm{cl}_{\mathrm{uni}}(\mathbf{G}),\, C\in\mathrm{cl}(W)}$$

can be expressed as the product of three matrices:

- q-deformation of character table of W (as considered above),
- non-abelian Fourier matrix from Lusztig's orange book (1984),
- the matrix of values of Green functions for $\mathbf{G}(q)$ ($\rightsquigarrow$ more in next sections).

All available in CHEVIE; can perform explicit computation. $\square$

Final remark: The above result is used, for example, in the proof of

Lusztig (Moscow Math. J. 2012): Cleanness of cuspidal character sheaves.

From now on: Character theory of finite groups of Lie type

Example: Let $\mathbf{G} = \mathrm{SL}_2(\overline{\mathbb{F}}_2)$, with Frobenius map $F_d(a_{ij}) = (a_{ij}^{2^d})$ for $d \geqslant 1$.
Let $\mathbf{G}^{F_d} := \{g \in \mathbf{G} \mid F_d(g) = g\} = \mathrm{SL}_2(\mathbb{F}_{2^d})$.

$\mathfrak{S}_3$ $(d{=}1)$	$()$	(12)	(123)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

$\mathfrak{A}_5$ $(d{=}2)$	$()$	$(12)(34)$	(123)	(12345)	(13524)
χ_1	1	1	1	1	1
χ_2	3	-1	0	$\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(1-\sqrt{5})$
χ_3	3	-1	0	$\frac{1}{2}(1-\sqrt{5})$	$\frac{1}{2}(1+\sqrt{5})$
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

Would like uniform ("generic") description of character table of $\mathbf{G}^{F_d}$ for all $d \geqslant 1$.

$q = 2^d$	C_1	C_2	$C_3(a)$	$C_4(b)$
χ_1	1	1	1	1
χ_2	q	0	1	-1
$\chi_3(n)$	$q+1$	1	$\zeta^{an} + \zeta^{-an}$	0
$\chi_4(m)$	$q-1$	-1	0	$-\xi^{bm} - \xi^{-bm}$
$\zeta = \exp(2\pi i/(q{-}1))$, $\xi = \exp(2\pi i/(q{+}1))$				

Unpacked table has size $(q+1) \times (q+1)$;
$\exists$ slightly more complicated table for q odd.

$\exists$ similar tables for other $\mathbf{G}$ of small dimension,
e.g., $\mathrm{Sp}_4(\mathbb{F}_q)$, $G_2(\mathbb{F}_q)$, ${}^3D_4(\mathbb{F}_q)$, ...

LATEST addition $F_4(2^d)$, $d \geqslant 1$, G. (2023).
ULTIMATE challenge $E_8(\mathbb{F}_q)$, any q.

General set-up: $\mathbf{G}$ connected reductive over $\overline{\mathbb{F}}_p$, with Frobenius map $F\colon \mathbf{G} \to \mathbf{G}$ corresponding to $\mathbb{F}_q$-rational structure (where $q = p^f$ for some $f \geqslant 1$).

$$\mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\} \quad \text{finite group of Lie type.}$$

Fix an F-stable Borel subgroup $\mathbf{B} \subseteq \mathbf{G}$ and F-stable maximal torus $\mathbf{T}_0 \subseteq \mathbf{B}$; let $W = N_{\mathbf{G}}(\mathbf{T}_0)/\mathbf{T}_0$ with induced automorphism $\sigma\colon W \to W$.

Deligne and Lusztig 1970s:

Let $w \in W$ and $\mathbf{T}_0[w] := \{t \in \mathbf{T}_0 \mid F(t) = \dot{w}^{-1} t \dot{w}\}$ (finite subgroup of $\mathbf{G}$)

$$\theta \in \mathrm{Irr}(\mathbf{T}_0[w]) \quad \rightsquigarrow \quad R_w^\theta \text{ virtual character of } G^F.$$

Lusztig 1984: Knowledge of all R_w^θ's $\rightsquigarrow$ "average value" character table of $\mathbf{G}^F$.

Let $\rho \in \mathrm{Irr}(\mathbf{G}^F)$ and $\mathcal{C}$ be an F-stable conjugacy class of $\mathbf{G}$. Then $\mathcal{C}^F$ splits into finitely many classes in $\mathbf{G}^F$, with representatives $g_1, \ldots, g_r \in \mathcal{C}^F$ say.

$$\rightsquigarrow \quad \text{"average value"} \quad \mathrm{AV}(\rho, \mathcal{C}) := \sum_{1 \leqslant i \leqslant r} [A_i : A_i^F]\, \rho(g_i),$$

where $A_i = C_{\mathbf{G}}(g_i)/C_{\mathbf{G}}^\circ(g_i)$ finite group (with induced action of F).

Example 4: Computation of Green functions

Let $\mathbf{G}_{\text{uni}}$ be the variety of unipotent elements of $\mathbf{G}$

$$\rightsquigarrow \quad \text{Green function} \quad Q_w \colon \mathbf{G}^F_{\text{uni}} \to \overline{\mathbb{Q}}_\ell, \quad u \mapsto R_w^\theta(u).$$

- Q_w has values in $\mathbb{Z}$, and does not depend on θ.
- Character formula: Get all values of R_w^θ from Q_w and inductive procedure.

Theorem (1976–2024). The Q_w are now known explicitly in all cases.

- Shoji (1982/83): $\mathbf{G}$ of type F_4 for $p > 2$, and $\mathbf{G}$ of classical type for $p > 2$.
- Beynon–Spaltenstein (1984): $\mathbf{G}$ of type E_6, E_7, E_8 and p "good".
- Malle (1990/93): F_4 and E_6 for $p = 2$; Porsch (1993): E_6 for $p = 3$.
- Shoji (2007): $\mathbf{G}$ of classical type and $p = 2$.
- G. (2020) $+$ Lübeck (2024): Last remaining cases E_7, E_8 for $p = 2, 3, 5$.

Ingredients: Springer correspondence; Lusztig's "Green functions and character sheaves" (1990, mild restriction on p, q) $+$ Shoji (1995, restrictions removed).

Generalised Springer correspondence (Lusztig 1984)

$\mathcal{N}_{\mathbf{G}} := \{(\mathcal{O}, \mathcal{E}) \mid \mathcal{O} \in \mathrm{cl_{uni}}(\mathbf{G}),\ \mathcal{E}$ irreducible $\mathbf{G}$-equivariant $\overline{\mathbb{Q}}_\ell$-local system on $\mathcal{O}\}$.

- Partition of $\mathcal{N}_{\mathbf{G}}$ into pieces called "unipotent blocks";
- collection of bijections $\{\mathcal{I} \leftrightarrow \mathrm{Irr}(\mathcal{W}_{\mathcal{I}}) \mid \mathcal{I}$ unipotent block of $\mathcal{N}_{\mathbf{G}}\}$, where $\mathcal{W}_{\mathcal{I}}$ is a certain finite Coxeter group associated with $\mathcal{I}$.

For Green functions, sufficient to consider $\mathcal{I}_0 =$ unipotent block with $(\{1\}, \overline{\mathbb{Q}}_\ell) \in \mathcal{I}_0$; $\mathcal{W}_{\mathcal{I}_0} = W$ (Weyl group of $\mathbf{G}$) and $\mathcal{I}_0 \leftrightarrow \mathrm{Irr}(W)$ "ordinary" Springer correspondence.

Also sufficient to consider $\mathbf{G}$ simple adjoint; further assume $F\colon \mathbf{G} \to \mathbf{G}$ split type. Then each unipotent class $\mathcal{O} \in \mathrm{cl_{uni}}(\mathbf{G})$ is F-stable and we can find $u_{\mathcal{O}} \in \mathcal{O}^F$ such that F acts trivially on $A(u_{\mathcal{O}}) = C_{\mathbf{G}}(u_{\mathcal{O}})/C^\circ_{\mathbf{G}}(u_{\mathcal{O}})$.

Lusztig–Shoji Algorithm. Set $Q_\chi := \frac{1}{|W|}\sum_{w \in W} \chi(w) Q_w$ for $\chi \in \mathrm{Irr}(W)$.

Let $\chi \leftrightarrow (\mathcal{O}, \mathcal{E}) \in \mathcal{I}_0$. Then $\{u \in \mathbf{G}^F_{\mathrm{uni}} \mid Q_\chi(u) \neq 0\} \subseteq \overline{\mathcal{O}}^F$ and

$$Q_\chi(u_{\mathcal{O}}) = \delta_\chi q^{\dim \mathcal{B}_{u_{\mathcal{O}}}} \dim \mathcal{E} \qquad \text{where} \qquad \delta_\chi \in \{\pm 1\}.$$

Once the signs $\delta_\chi = \pm 1$ (and the Springer correspondence) are known, all the values of the Green functions are determined by a purely combinatorial algorithm.

For $u \in \mathbf{G}^F_{\text{uni}}$ there are (explicitly computable) polynomials $P_{\chi,\chi',u} \in \mathbb{Z}[\mathbf{q}]$ such that

$$Q_\chi(u) = \sum_{\chi' \in \mathrm{Irr}(W)} \delta_{\chi'} P_{\chi,\chi',u}(q).$$

Remaining problem: Determine the signs $\delta_\chi = \pm 1$ for $\chi \in \mathrm{Irr}(W)$.

Idea: $R^{\theta=1}_{w=1}$ character of permutation representation of $\mathbf{G}^F$ on $\mathbf{G}^F/\mathbf{B}^F$. Hence

$$\begin{aligned} |(\mathbf{G}^F/\mathbf{B}^F)_u| &:= \text{number of cosets } \mathbf{G}^F/\mathbf{B}^F \text{ fixed by } u \\ &= Q_1(u) = \sum_{\chi \in \mathrm{Irr}(W)} \chi(1) Q_\chi(u) \\ &= \sum_{\chi' \in \mathrm{Irr}(W)} \underbrace{\Big(\sum_{\chi \in \mathrm{Irr}(W)} \chi(1) P_{\chi,\chi',u}(q) \Big)}_{\text{known, computable}} \delta_{\chi'}. \end{aligned}$$

Theorem (G. 2020). Let $r \geqslant 1$ and consider Green functions $Q^{(r)}_\chi$ for $\mathbf{G}^{F^r}$; have signs $\delta^{(r)}_\chi \in \{\pm 1\}$. Then $\delta^{(r)}_\chi = \delta^r_\chi$. So it is enough to consider $q = p$.

Example: $\mathbf{G}^F = E_8(\mathbb{F}_q)$ with $q = 2^f$

Let $\mathcal{O}$ = unipotent class denoted $E_8(b_6)$ with $u_{\mathcal{O}} = z_{77}$ (Mizuno 1980).

We have $A(u_{\mathcal{O}}) = C_G(u_{\mathcal{O}})/C_G^\circ(u_{\mathcal{O}}) \cong \mathfrak{S}_3$ and $|C_G(u_{\mathcal{O}})^F| = 6q^{28}$.

There are three $\chi \in \mathrm{Irr}(W)$ such that $\chi \leftrightarrow (\mathcal{O}, \mathcal{E})$, denoted $2240_{10}, 175_{12}, 840_{13}$.

(Springer correspondence known by Spaltenstein 1985.)

Want to determine $\delta_{2240_{10}} = \pm 1, \quad \delta_{175_{12}} = \pm 1, \quad \delta_{840_{13}} = \pm 1.$

$\mathcal{O}^F$ splits into 3 classes in $\mathbf{G}^F$, with representatives z_{77}, z_{78}, z_{79}. Run the CHEVIE function `ICCTable` to get the polynomials $P_{\chi,\chi',u} \in \mathbb{Z}[\mathbf{q}]$ for $u \in \{z_{77}, z_{78}, z_{79}\}$.

Theorem $\Rightarrow$ enough to consider $q = 2$:

$$|(\mathbf{G}^F/\mathbf{B}^F)_{z_{77}}| = 5{,}479{,}485\, \delta_{2240_{10}} + 358{,}400\, \delta_{175_{12}} + 1{,}233{,}920\, \delta_{840_{13}}$$

$$|(\mathbf{G}^F/\mathbf{B}^F)_{z_{78}}| = 5{,}479{,}485\, \delta_{2240_{10}} - 1{,}233{,}920\, \delta_{840_{13}}$$

$$|(\mathbf{G}^F/\mathbf{B}^F)_{z_{79}}| = 5{,}479{,}485\, \delta_{2240_{10}} + 179{,}200\, \delta_{175_{12}} + 1{,}233{,}920\, \delta_{840_{13}}$$

Left hand side is a non-negative integer $\Rightarrow$ $\delta_{2240_{10}} = 1$.

Claim. $\delta_{840_{13}} = -1$ and $\delta_{175_{12}} = 1$.

Idea of proof. Suppose $\delta_{840_{13}} = 1$; then above equation

$$|(\mathbf{G}^F/\mathbf{B}^F)_{z_{78}}| = 5{,}479{,}485\, \delta_{2240_{10}} - 1{,}233{,}920\, \delta_{840_{13}}$$

yields $|(\mathbf{G}^F/\mathbf{B}^F)_{z_{78}}| = 5{,}479{,}485 - 1{,}233{,}920 = 4{,}245{,}565$.

So, if we can find strictly more cosets fixed by z_{78}, then contradiction.

Explicitly count cosets fixed by z_{78}, using matrix realisation of $\mathbf{G}^F$ (G. 2020) or Steinberg presentation (Lübeck 2024).

Every double coset $\mathbf{B}^F \dot{w} \mathbf{B}^F$ contains precisely $q^{\ell(w)}$ cosets of $\mathbf{G}^F/\mathbf{B}^F$. Can write down explicit expressions for representatives of these cosets. Proceed along increasing value of $\ell(w)$ until sufficiently many cosets have been found. □

These computations require:

- Data base of (generalised) Springer correspondence in CHEVIE;
- implementation of Lusztig–Shoji algorithm (J. Michel's `ICCTable` function);
- programs for working inside $\mathbf{G}^F$ (with matrices, or words in generators $x_\alpha(t)$).

Example 5: Recovering geometry from algebra

Consider again generalised Springer correspondence.

Fix a unipotent block $\mathcal{I}$, a certain set of pairs $(\mathcal{O}, \mathcal{E})$ in $\mathcal{N}_{\mathbf{G}}$.

Using bijection $\mathcal{I} \leftrightarrow \mathrm{Irr}(\mathcal{W}_{\mathcal{I}})$, define equivalence relation $\sim$ on $\mathrm{Irr}(\mathcal{W}_{\mathcal{I}})$:

Let $\chi, \chi' \in \mathrm{Irr}(\mathcal{W}_{\mathcal{I}})$. $\chi \sim \chi' \stackrel{\text{def}}{\Longleftrightarrow} (\mathcal{O}, \mathcal{E}) \leftrightarrow \chi$ and $(\mathcal{O}, \mathcal{E}') \leftrightarrow \chi'$ (same $\mathcal{O}$).

Lusztig "Unipotent blocks and weighted affine Weyl groups" (2024) conjectures:
Partition of $\mathrm{Irr}(\mathcal{W}_{\mathcal{I}})$ defined by $\sim$ can be recovered in a purely algebraic way, using operations with characters of $\mathcal{W}_{\mathcal{I}}$ and inductive procedure.

Example: $\mathbf{G}$ of type E_7, simply connected and $p \neq 2$.

There is a unipotent bock $\mathcal{I}_1$ containing 25 pairs $(\mathcal{O}, \mathcal{E}) \in \mathcal{N}_{\mathbf{G}}$, with $\mathcal{W}_{\mathcal{I}}$ of type F_4.

Easy to program Lusztig's algebraic version of $\sim$ in CHEVIE.

G.–Hetz (2024): Found an inconsistency with Spaltenstein's tables (1985) for generalised Springer correspondence of $\mathbf{G}$. Using CHEVIE could correct table.

Adv. Studies Pure Math, vol. 6, AMS 1985

$\mathcal{O}$	dim $\mathcal{B}_u$		$\mathcal{I}_0$		$\mathcal{I}_1$		$\mathcal{I}_2$	
			-1	105_5				
D_6	4	Z_2	1	35_4	-1	$\chi_{3,1}$	-1	$(21, 0)$
$D_6(a_1)+A_1$	5	$Z^2_{(2,p-1)}$	1	189_5	ε'	$\chi_{1,2}$	—	
			ε	15_7 $(p\neq 2)$	ε''	$\chi_{8,3}$	—	
A_6	6	$Z_{(2,p)}$	1	105_6	—		—	
			-1	15_7 $(p=2)$	—		—	
$D_6(a_1)$	6	Z_2	1	210_6	-1	$\chi_{9,2}$	-1	$(0, 21)$
D_5+A_1	6	Z_2	1	168_6	-1	$\chi_{2,3}$	-1	$(1^2, 1)$
$D_6(a_2)+A_1$	7	$S_3\times Z_{(2,p-1)}$	1	315_7	-1	χ_{12}	—	
			θ	280_9	$-\theta$	$\chi_{8,2}$	—	
			ε	35_{13}	—		—	
D_5	7	$Z_{(2,p)}$	1	189_7	—		-1	$(1, 1^2)$
$(A_5+A_1)'$	8	Z_2	1	405_8	—		—	
			-1	189_{10}	—		—	
$D_6(a_2)$	8	$Z_{(2,p-1)}$	1	280_8	-1	χ_{16}	—	

$\mathcal{W}_{\mathcal{I}_0} : E_7$

$\mathcal{W}_{\mathcal{I}_1} : F_4$

$\mathcal{W}_{\mathcal{I}_2} : B_3$