The FPP conjecture and computing the unitary dual

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Representation theory days
In honor of George Lusztig

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Outline

Slides at `http://www-math.mit.edu/~dav/paper.html`

Intro

This talk is about unitary reps of real groups. But. . .

1. The conference is to celebrate George's work
2. George doesn't talk about unitary representations
3. George doesn't talk about real groups
4. So what was I thinking?

I will tell you here what I omit in the rest of the talk.

Every time I say "we can compute," or "the `atlas` software can compute," what I mean is this:

The computation is completely inaccessible; but George found a straightforward way to do it.

Main example: character formulas for irr reps.

Beilinson, Bernstein, Kashiwara, and Brylinski related char formulas to symmetric-subgroup-equivariant perverse sheaves on flag varieties; and those George could write down in his sleep.

What's this about really?

$G(\mathbb{R})$ real reductive algebraic group.

$\widehat{G(\mathbb{R})}_u$ = (equiv classes of) irr unitary reps of $G(\mathbb{R})$.

I'll assume that studying this set (unitary dual) is the most world's best problem.

How can you approach it?

I'll start by saying what the answer looks like.

$G(\mathbb{R}) \rightsquigarrow$ {finite set of compact polyhedra U_j}.

Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

This just in...

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Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

The FPP conjecture (stated below) constrains the cpt polyhedra U_j and the cone-in-lattice factors C_j.

The FPP conjecture was recently proven by Dougal Davis and Lucas Mason-Brown.

The constraints make $\widehat{G(\mathbb{R})}_u$ computable (by the `atlas` software) for any particular value of $G(\mathbb{R})$.

Computing unitary dual of a series of *classical* groups is (thanks to Barbasch, Arthur...) a combinatorial problem for which one can hope for a complete and explicit answer.

(We don't yet have such an answer ☹.)

The *exceptional* groups are another matter.

Really computable?

Here is some information about the computations.

$G(\mathbb{R})$	time	memory	# unitary FPP faces
$SL(2,\mathbb{R})$	.010 sec	.4 mb	7
$SL(3,\mathbb{R})$	.020 sec	.4 mb	9
$SL(4,\mathbb{R})$	.241 sec	1.5 mb	47
$SL(5,\mathbb{R})$	.548 sec	1.8 mb	66
$SL(6,\mathbb{R})$	5.387 sec	5.5 mb	286
$SL(7,\mathbb{R})$	19.747 sec	18.2 mb	445
$Sp(4,\mathbb{R})$	.132 sec	1.0 mb	46
$Sp(6,\mathbb{R})$	2.180 sec	3.4 mb	319
$Sp(8,\mathbb{R})$	37.983 sec	10.2 mb	2043
$Sp(10,\mathbb{R})$	765.267 sec	70.5 mb	13768
$Sp(12,\mathbb{R})$	18841.898 sec	440.0 mb	88314
split G_2	.174 sec	.5 mb	60
split F_4	62.241 sec	22.8 mb	1864
split E_6	161.279 sec	116.1 mb	2217
quasisplit E_6	1892.922 sec	437.1 mb	19831
split E_7	15709588 sec	???	234381

Patterns

rank(K)	$G(\mathbb{R})$	time	memory	# unitary FPP faces
1	$SL(2,\mathbb{R})$	.010 sec	.4 mb	7
1	$SL(3,\mathbb{R})$	.020 sec	.4 mb	9
2	$SL(4,\mathbb{R})$	.241 sec	1.5 mb	47
2	$SL(5,\mathbb{R})$	.548 sec	1.8 mb	66
2	$Sp(4,\mathbb{R})$	.132 sec	1.0 mb	46
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6	quasisplit E_6	1892.922 sec	437.1 mb	19831
7	split E_7	15709588 sec	???	234381

E_7 done by Jeffrey Adams in 1000 `atlas` processes. Created overhead: sum of 1000 process times isn't comparable to single process times.

What we see and why we care

Very roughly: time, memory, and # unitary faces depend mostly on $\mathrm{rank}(K)$. Here are some approximations.

$\mathrm{rank}(K)$	time	memory	# unitary FPP faces
1	.015 sec	.4 mb	8
2	.25 sec	1.2 mb	55
3	6 sec	9 mb	330
4	85 sec	40 mb	2000
5	800 sec	70 mb	12000
6	8000 sec	450 mb	70000
R	$.01 \times 10^R$	$.02 \times 7^R$	2×5^R

Reason to make estimates: to guess how difficult it will be to calc FPP unitary faces in split E_8, and how complicated answer is.

First, expect several million FPP unitary faces.

If calculation is divided among many processors, need 150 gb for most processes; and perhaps 1 tb for a few of them.

To address the predicted weeks or months of CPU time, can consider separately each of 320,000 orbits of K on $\mathcal{B}$.

Steve Miller: many orbits take few secs; but some require day or two.

He is pursuing this work on hundreds of machines at Rutgers, and has completed about 280,000 orbits.

Immer mit dem einfachsten Beispiel. . .

This advice stayed on Michael Artin's board while he wrote *Algebra*.

$G(\mathbb{R}) \rightsquigarrow$ {finite set of compact polyhedra U_j}.

Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j U_j \times V_j \times C_j.$$

$$\widehat{SL(2,\mathbb{R})}_u \rightsquigarrow \left\{ \begin{array}{ll} (\text{point} \times \mathbb{R}^1 \times \{0\}) \longleftrightarrow \text{spherical unitary princ series} \\ (\text{point} \times \mathbb{R}^1 \times \{0\}) \longleftrightarrow \text{nonsph unitary princ series} \\ (\text{point} \times \mathbb{R}^0 \times \mathbb{N}) \longleftrightarrow \text{holomorphic discrete series} \\ (\text{point} \times \mathbb{R}^0 \times \mathbb{N}) \longleftrightarrow \text{antihol discrete series} \\ ([0,1] \times \mathbb{R}^0 \times \{0\}) \longleftrightarrow \text{complementary series} \end{array} \right\}$$

This is two lines, two half lattices, and one interval.

Picture for $SL(2,\mathbb{R})$ found by Valentine Bargmann in 1947.

For those with OCD or PhD: more words are needed to make this precise. Example: nonsph princ ser at 0 is sum of two irreps: $\text{nonsph}(\text{pt}, 0, 0) = \text{hol ds}(\text{pt}, 0, 0) + \text{antihol ds}(\text{pt}, 0, 0)$.

That answer has this form for any real reductive $G(\mathbb{R})$ comes from Harish-Chandra, Langlands, Knapp, Zuckerman $\approx$ 1985.

So what do we need to do?

$G(\mathbb{R}) \rightsquigarrow$ {finite set of compact polyhedra U_j}.

Each $U_j \rightsquigarrow$ (real vector space V_j, cone-in-a-lattice C_j)

$$\widehat{G(\mathbb{R})}_u = \coprod_j \; U_j \times V_j \times C_j.$$

Describe $\widehat{G(\mathbb{R})}_u \leftrightsquigarrow$ describe cpt polyhedra U_j.

Vec space V_j, cone-in-lattice C_j important but easy.

Main question today: what do cpt polyhed U_j look like?

Answer: U_j is finite union of product of simplices.

Goals for today:

1. say what kinds of simplices are allowed
2. recall work of Barbasch, (Barbasch and his friends) giving beautiful precise list of simplices in many cases
3. say how `atlas` software computes ugly precise list of simplices in all cases

Realistically: I'll mostly talk about (1).

Remind me about the Weyl group...

G cplx conn red alg group $\supset B$ Borel $\supset H$ max torus.

(X^* alg chars of H) $\supset$ (R roots) $\supset$ (Π simple roots).

(X_* alg cochars) $\supset$ ($R^\vee$ coroots) $\supset$ ($\Pi^\vee$ simple coroots).

Based root datum of G is $(X^*, \Pi, X_*, \Pi^\vee)$, $\mathfrak{h}^*_{\mathbb{R}} = X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

$\mathfrak{h}^*_{\mathbb{R}}$ is the real vector space where the classical root system lives.

Coroot hyperplanes: $E_{\alpha^\vee} = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) = 0\}$ ($\alpha^\vee$ in $R^\vee$).

Each coroot $\alpha^\vee$ defines simple reflection: $\mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$,

$$s_{\alpha^\vee}(\gamma) = \gamma - \gamma(\alpha^\vee) \cdot \alpha, \quad s_{\alpha^\vee}(\alpha) = -\alpha, \quad s_{\alpha^\vee} = \text{identity on } E_\alpha.$$

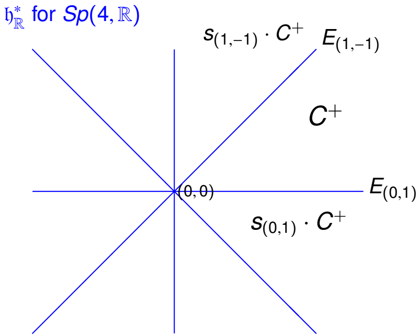
Weyl group of G is $W =$ group generated by all $s_{\alpha^\vee}$.

The open positive Weyl chamber is the open simplicial cone

$$C^+ = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) > 0 \quad (\alpha^\vee \in \Pi^\vee \ \text{simple})\}.$$

A Weyl chamber in $\mathfrak{h}^*_{\mathbb{R}}$ is a subset $w \cdot C^+$ (some $w \in W$).

What do Weyl chambers look like?



$\overline{C}^+$ is fundamental domain for W action on $\mathfrak{h}^*_{\mathbb{R}}$.

Action of W on Weyl chambers is simply transitive

dominant faces of $\overline{C}^+$ of codim $d \longleftrightarrow$ cardinality d subsets of $\Pi^\vee$

any face of $\mathfrak{h}^*_{\mathbb{R}}$ is in $W\cdot$(unique dom face)

And the affine Weyl group?

Standard terminology: what's below is the dual affine Weyl group.

Based root datum of G is $(X^*, \Pi, X_*, \Pi^\vee)$, $\mathfrak{h}^*_{\mathbb{R}} = X^* \otimes_{\mathbb{Z}} \mathbb{R}$.

Aff coroots are $R^{\vee,\text{aff}} = \{(\alpha^\vee, m) \mid \alpha^\vee \in R^\vee, m \in \mathbb{Z}\}$.

Pos aff coroots are $R^{\vee,\text{aff},+} = \{(\alpha^\vee, m) \mid m > 0 \text{ or } \alpha^\vee \in R^{\vee,+}, m = 0\}$.

Write $\alpha_0^\vee$ = lowest coroot (unique if G simple).

Simple aff coroots are $\Pi^{\vee,\text{aff}} = \{(\alpha^\vee, 0) \mid \alpha^\vee \in \Pi^\vee\} \cup \{(\alpha_0^\vee, 1)\}$.

Aff hyperplanes $E_{\alpha^\vee,m} = \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid \gamma(\alpha^\vee) + m = 0\}$.

aff coroot $\rightsquigarrow$ simple aff reflection: $\mathfrak{h}^*_{\mathbb{R}} \to \mathfrak{h}^*_{\mathbb{R}}$,

$$s_{\alpha^\vee,m}(\gamma) = \gamma - (\gamma(\alpha^\vee) + m)\cdot\alpha, \quad s_{\alpha^\vee,m} = \text{id on } E_{\alpha^\vee,m}.$$

Affine Weyl group of G is W^{aff} = group generated by all $s_{\alpha^\vee,m}$.

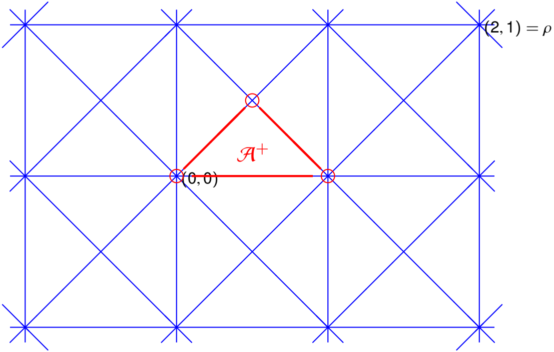
The open fundamental alcove is the open simplex

$$\begin{aligned}\mathcal{A}^+ &= \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} | \gamma(\alpha^\vee) + m > 0 \quad ((\alpha^\vee, m) \in \Pi^{\vee,\text{aff}} \text{ simple})\}\\ &= \{\gamma \in C^+ \mid \gamma(\alpha_0^\vee) < 1\}.\end{aligned}$$

An alcove in $\mathfrak{h}^*_{\mathbb{R}}$ is a subset $w \cdot \mathcal{A}^+$ (some $w \in W^{\text{aff}}$).

What do alcoves look like?

$\mathfrak{h}^*_{\mathbb{R}}$ for $Sp(4,\mathbb{R})$



$\overline{\mathcal{A}}^+$ is fundamental domain for W^{aff} action on $\mathfrak{h}^*_{\mathbb{R}}$.

Action of W^{aff} on alcoves is simply transitive

fund faces of $\overline{\mathcal{A}}^+$ of codim $d \longleftrightarrow$ order d subsets of $\Pi^{\vee,\text{aff}}$

any face of $\mathfrak{h}^*_{\mathbb{R}}$ is in $W^{\text{aff}}\cdot$(unique fundamental face)

What good are all these faces?

Langlands classif: irrs of real infl character indexed by

1. discrete parameter $(x, \lambda) \approx$ lowest K-type
2. continuous parameter $\gamma =$ infinitesimal character.

Here $x =$ KGB element: orbit of $K(\mathbb{C})$ on Borels in $G(\mathbb{C})$.

Finite # of x: 3 for $SL(2, R)$, 201 for $Sp(8, \mathbb{R})$, 320206 for split E_8.

Given x, set of allowed λ is finite # of cones in lattices

Given (x, λ), set of allowed γ is affine space $V_{\mathbb{R}}(x, \lambda) \subset \mathfrak{h}^*_{\mathbb{R}}$.

Therefore $V_{\mathbb{R}}(x, \lambda)$ is disjoint union of faces.

Theorem (Speh-V) Fix discrete parameter (x, λ).

1. If $\gamma_1, \gamma_2 \in$ same face of $V_{\mathbb{R}}(x, \lambda)$, then irr reps $J(x, \lambda, \gamma_1)$ and $J(x, \lambda, \gamma_2)$ are both unitary or both nonunitary.
2. Set of unitary γ is a compact polyhedron $U(x, \lambda) \subset V_{\mathbb{R}}(x, \lambda)$, a finite union of faces.

What does that say about the unitary dual?

Corollary Set $\widehat{G(\mathbb{R})}_{u,\text{real}}$ = unitary reps of real infl char. Then

$$\widehat{G(\mathbb{R})}_{u,\text{real}} = \bigcup_{x \in KGB} \bigcup_{\substack{\lambda \text{ allowed} \\ \text{for } x}} U(x, \lambda)$$

Claim in introduction:

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$$\widehat{G(\mathbb{R})}_u = \coprod_j \; U_j \times V_j \times C_j.$$

Polyhedra $U(x, \lambda)$ are the U_j in the introduction.

Extending Cor to all infl chars gives real vector spaces V_j.

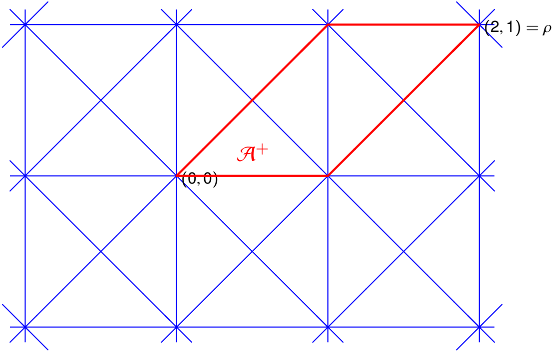
Given x, λ's are finite union of cones in lattices C_j.

To prove Claim, need to show $U(x, \lambda)$ is nearly independent of λ.

To describe unitary dual, need to compute all $U(x, \lambda)$.

What's the FPP…

FPP $\subset \mathfrak{h}^*_{\mathbb{R}}$ for $Sp(4,\mathbb{R})$



fundamental parallelepiped $= \{\gamma \in \mathfrak{h}^*_{\mathbb{R}} \mid 0 \leq \gamma(\alpha^\vee) \leq 1 \mid (\alpha \in \Pi)\}$

Union of $\#W/\#Z(G_{sc})$ alcoves.

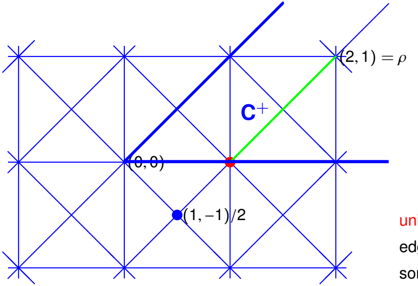
$G(\mathbb{R})$	# alcoves	# faces
$SL(2,\mathbb{R})$	1	3
$Sp(4,\mathbb{R})$	4	19
split E_8	696729600	24169704765

(Those numbers are 7×10^8 and 2.4×10^{10}.)

. . . and how does it help the unitary dual?

Real Langlands parameter (x, λ, γ) defines

1. Cartan involution $\theta = \theta(x)$ acting on $\mathfrak{h}^*_{\mathbb{R}}$
2. Cartan decomp $\mathfrak{h}^*_{\mathbb{R}} = \mathfrak{t}^*_{\mathbb{R}} + \mathfrak{a}^*_{\mathbb{R}}$ (± 1 eigenspaces)
3. differential of λ $d\lambda \in \mathfrak{t}^*_{\mathbb{R}}$
4. "A-parameter $\nu = \gamma(x, \lambda, \gamma) = \gamma - d\lambda$
5. Definition of param $\rightsquigarrow \gamma \in \overline{C^+}$ is dominant.



(x, λ) first disc series, Siegel par

$$\theta = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \; d\lambda = (1/2, -1/2)$$

$$\mathfrak{a}^*_{\mathbb{R}} = \{(t, t)\}$$

green line is allowed infl chars γ.

unitary part is some vertices $(2 + m_0, m_0)/2$, edges $\{(1 + t, t) \mid t \in (1 + m_1/2, 1 + (m_1 + 1)/2)\}$, some $m_0 \in \{-1, 0, 1\}$, $m_1 \in \{-1, 0\}$.

Define $U_{FPP}(x, \lambda) = \{\gamma \in FPP \mid J(x, \lambda, \gamma) \text{ is unitary}\}$.

$U_{FPP}(x, \lambda)$ is the single red point $[1, 0]$: only $m_0 = -1$ is unitary.

FPP conj (Davis, Mason-Brown theorem)

Suppose (x, λ, γ) is a real Langlands parameter of infinitesimal character γ.

FPP conjecture is distilled from work of Dan Barbasch.

Define $S(\gamma) = \{\alpha \in \Pi \mid \gamma(\alpha^\vee) \leq 1\}$, a set of simple roots,

$\mathfrak{q} = \mathfrak{q}(\gamma) = \mathfrak{l} + \mathfrak{u}$ parabolic with Levi generated by $S(\gamma)$.

1. γ belongs to the FPP if and only if $\mathfrak{q} = \mathfrak{g}$.
2. Conjecture If $J(x, \lambda, \nu)$ is unitary, then $\mathfrak{q}$ is θ-stable.
3. If $\mathfrak{q}$ is θ-stable, then $J(x, \lambda, \gamma)$ is good range cohomologically induced from $J(x_L, \lambda_L, \gamma_L)$ on L. Here $\lambda_L = \lambda - \rho(\mathfrak{u})$, $\gamma_L = \gamma - \rho(\mathfrak{u})$, $\gamma_L \in FPP(L)$.

$$U(x, \lambda) = \bigcup_{\theta\text{-stable } \mathfrak{q}} U_{FPP}(x_L, \lambda_L) + \rho(\mathfrak{u})$$

Conclusion: unitary dual is known if we compute (finitely many) $U_{FPP}(x, \lambda)$, the FPP infl characters for unitary reps in the series (x, λ).

Three cheers!

I chose this picture (from Bert Kostant's 80th birthday conference in Vancouver in 2008) for several reasons.

First: George is clearly the tallest person in the picture.

Second: the presence of my student Monica Nevins, who now works entirely on *p*-adic groups. George affects everyone he meets.

Third: our colleague Victor Guillemin is laughing, presumably at a joke from George.

Thank you George, for a memorable half century!