Knot Categorification from Geometry, via String Theory

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Based on joint work (to appear) with Andrei Okounkov

I will describe

two geometric approaches to the

knot categorification problem.

These approaches should be viewed as emerging from string theory.

However, this gives string theory too much credit.

To extract string theory predictions,

and put the pieces together,

required the work of Andrei and his many

collaborators over the years,

Bezrukavnikov, Maulik, Nekrasov, Smirnov…

One of approaches we will derive
turns out to have
the same flavor as that in the work of
Kamnitzer and Cautis.

The other is a relative of the approach of
Seidel and Smith.

The relation between the

two approaches

is a variant of

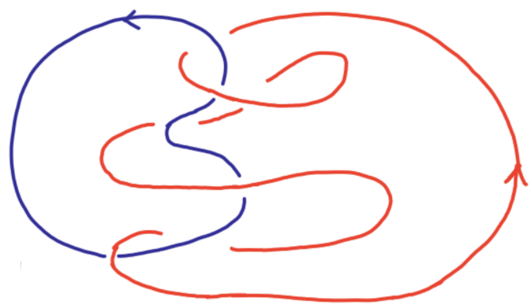
two dimensional mirror symmetry.

There is a third approach
with the same string theory origin,
due to Witten.

What emerges from string theory
is unified framework
for knot categorification.

To begin with, it is useful to recall
some well known aspects of knot invariants.

To get a quantum invariant of a link K



one starts with a Lie algebra,

$$^L\mathfrak{g}$$

and a coloring
of its strands by representations of $^L\mathfrak{g}$.

The link invariant,

in addition to the choice of a group

$$^{L}\mathfrak{g}$$

and its representations,

depends on one parameter

$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

Witten showed in his famous ’89 paper,

that the knot invariant

comes from

Chern-Simons theory with gauge group based on the Lie algebra

$$^L\mathfrak{g}$$

and (effective) Chern-Simons level

$$\kappa$$

In the same paper, he showed that
underlying Chern-Simons theory is a
two-dimensional conformal field theory
corresponding to

$$\widehat{{}^L\mathfrak{g}}_\kappa \,,$$

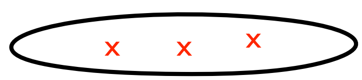
the affine Lie algebra of ${}^L\mathfrak{g}$, at level κ .

Start with the space conformal blocks of

$$\widehat{{}^L\mathfrak{g}}_\kappa$$

on a Riemann surface $\mathcal{A}$ with punctures

$\mathcal{A}$



To eventually get invariants of knots in $\mathbb{R}^3$ or S^3

we want to take

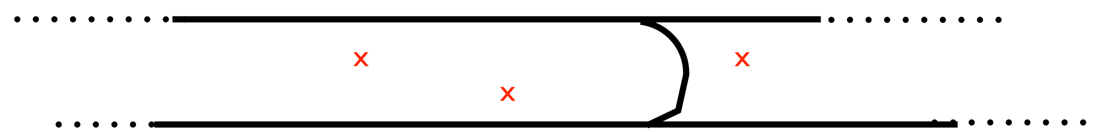
$$\mathcal{A}$$

to be a complex plane with punctures,

$\mathcal{A}$

or equivalently, a punctured infinite cylinder.

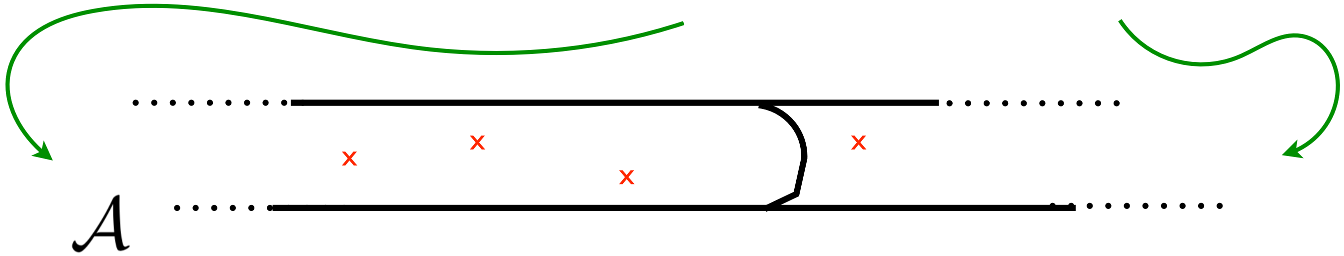
The corresponding $\widehat{{}^L\mathfrak{g}_\kappa}$ conformal blocks



are correlators of chiral vertex operators on $\mathcal{A}$

$$\Psi(a_1, \ldots, a_n) = \langle \mu| \; \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n) \; |\mu'\rangle$$

Punctures at $x = 0$ and ∞ ,



are labeled by a pair of highest weight vectors

$$|\mu\rangle \quad \text{and} \quad |\mu'\rangle$$

of Verma module representations of the algebra.

A chiral vertex operator

$$\Phi_{{}^L\rho_I}(a_I)$$

associated to a finite dimensional representation

$${}^L\rho_I$$

of ${}^L\mathfrak{g}$ adds a puncture at a finite point on $\mathcal{A}$

$$x = a_I$$

It acts as an intertwiner

$$\Phi_{{}^L\rho_I}(a_I): \quad {}^L\rho_{\mu_I} \; \rightarrow \; {}^L\rho_{\mu_{I+1}} \otimes {}^L\rho_I(a_I)$$

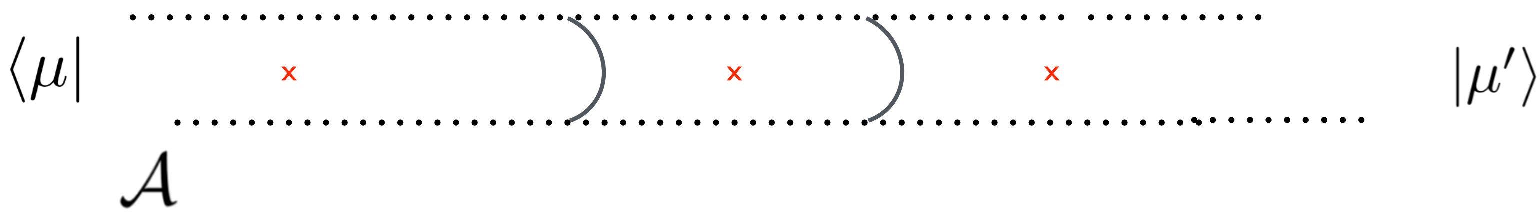
between a pair of Verma module representations,

$\langle \mu_I |$

$| \mu_{I+1} \rangle$

${}^L\rho_I(a_I)$

Sewing the chiral vertex operators



we get different conformal blocks,

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \; \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n) \; | \mu' \rangle$$

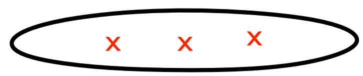
depending on choices of intermediate Verma module representations.

(which the notation hides).

The space conformal blocks of

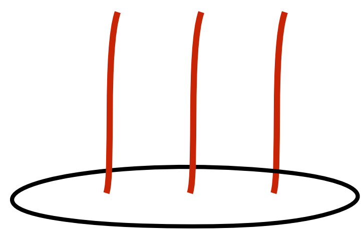
$$\widehat{{}^L\mathfrak{g}_\kappa}$$

on the Riemann surface $\mathcal{A}$ with punctures

$\mathcal{A}$ 

is the Hilbert space of Chern-Simons theory on

$$\mathcal{A} \times \mathrm{time}$$

$\mathcal{A}$ 

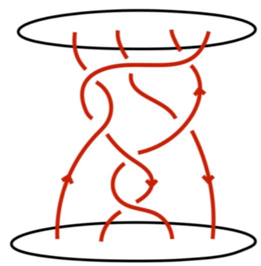
where the strands get colored

by representations of the corresponding punctures.

The Chern-Simons path integral on

$$\mathcal{A} \times \text{interval}$$

in the presence of a braid



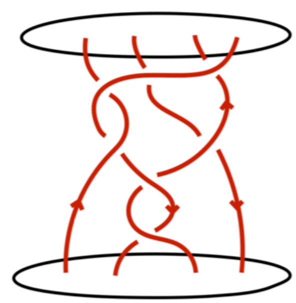
$$\mathfrak{B} = \mathfrak{B}(B)$$

gives the corresponding

quantum braid invariant.

The braid invariant

$$\mathfrak{B} = \mathfrak{B}(B)$$



is a matrix that transports

the space of conformal blocks,

along the braid B

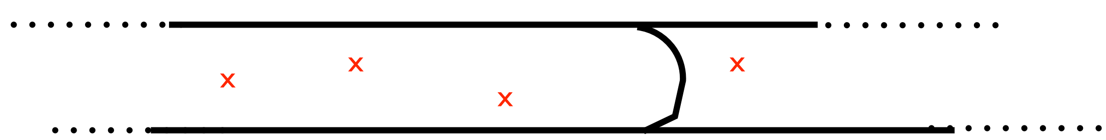
To describe the transport,

instead of characterizing $\widehat{{}^L\mathfrak{g}}_\kappa$ conformal blocks

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \, \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n) \, | \mu' \rangle$$

in terms of vertex operators and sewing,

$\mathcal{A}$



it is better to describe them as solutions to a differential equation.

The equation solved by

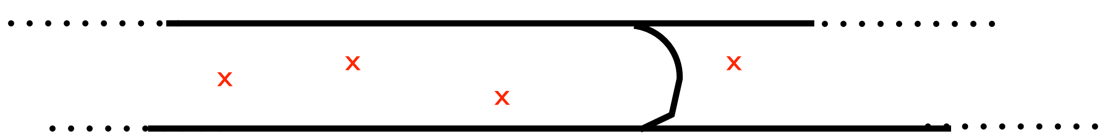
conformal blocks of $\widehat{{}^L\mathfrak{g}}_\kappa$ on $\mathcal{A}$

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \; \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n) \; | \mu' \rangle$$

is the equation discovered by Knizhnik and Zamolodchikov in '84:

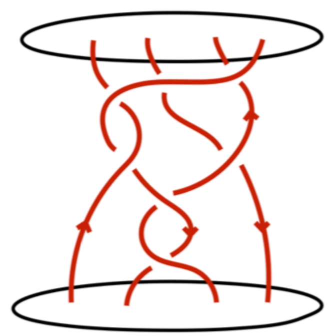
$$\kappa \, a_I \frac{\partial}{\partial a_I} \Psi = \left(\sum_{J \neq I} r_{IJ}(a_I/a_J) + r_{I0} + r_{I\infty} \right) \Psi$$

It is regular as long as the punctures are distinct.



The quantum braid invariant

$$\mathfrak{B}(B)$$



is the monodromy matrix of the Knizhnik-Zamolodchikov equation, along the path in the parameter space corresponding to the braid B .

The monodromy problem of the $\widehat{{}^L\mathfrak{g}}_\kappa$ Knizhnik-Zamolodchikov equation

$$\kappa\, a_I \frac{\partial}{\partial a_I}\,\Psi = \sum_{J \neq I} r_{IJ}(a_I/a_J)\,\Psi$$

was solved by Drinfeld and Kohno in '89.

They showed that its monodromy matrices are given in terms of the R-matrices of the quantum group

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

corresponding to ${}^L\mathfrak{g}$

Action by monodromies

turns the space of conformal blocks into a module for the

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

quantum group in representation,

$${}^L\rho = \otimes_I {}^L\rho_I$$

The representation ${}^L\rho$ is viewed here as a representation of $U_{\mathfrak{q}}({}^L\mathfrak{g})$ and not of ${}^L\mathfrak{g}$, but we will denote by the same letter.

The monodromy action

is irreducible only in the subspace of

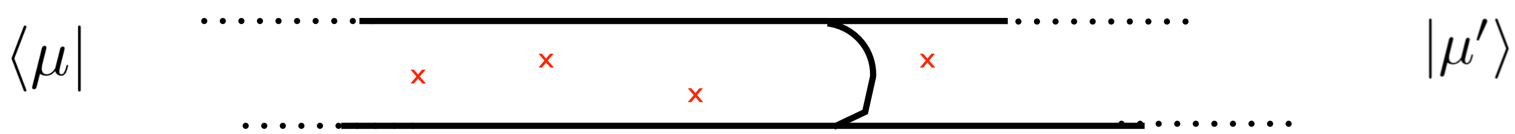
$$ {}^L\rho = \otimes_I {}^L\rho_I $$

of fixed weight

$$ \nu $$

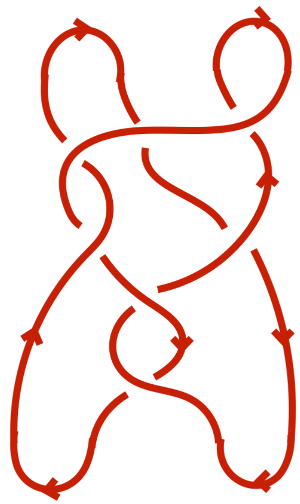
which in our setting equals to

$$ \nu = \mu - \mu' $$



This perspective leads to
quantum invariants of not only braids
but knots and links as well.

Any link K can be represented as a



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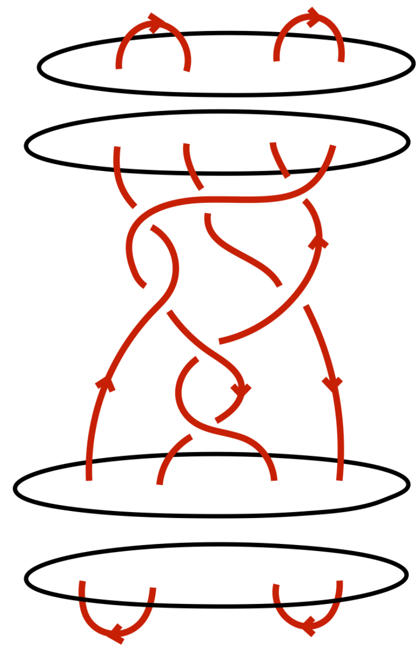
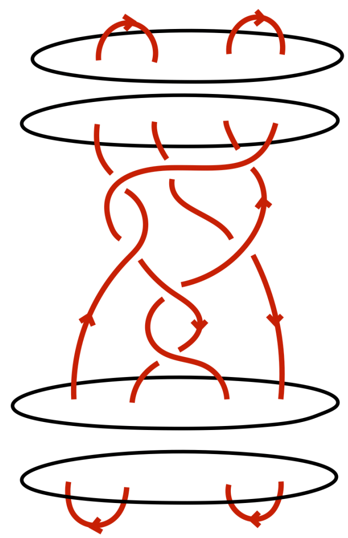


a closure of some braid B

The corresponding quantum link invariant is the matrix element

$$\left(\Psi_{\mathcal{L}_{out}}\right| \mathfrak{B} \left|\Psi_{\mathcal{L}_{in}}\right)$$



$\mathfrak{B}(B)$

of the braiding matrix,

taken between a pair of conformal blocks

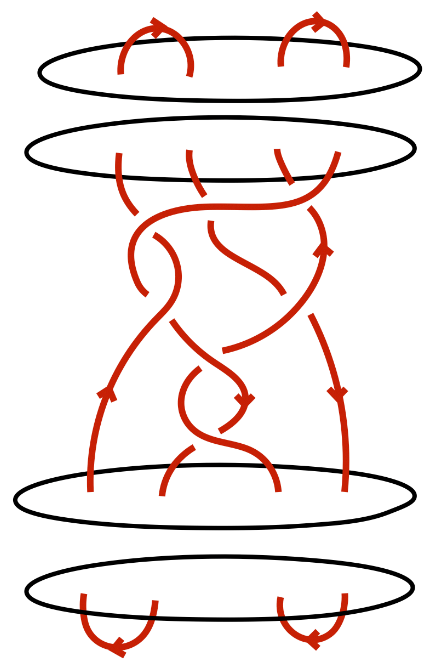
$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

The pair of conformal blocks

$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

that pick out the matrix element

$$\left(\Psi_{\mathcal{L}_{out}}\right| \mathfrak{B} \left|\Psi_{\mathcal{L}_{in}}\right)$$



correspond to the top and the bottom of the picture.

The conformal blocks

we need are specific solutions to KZ equations

$$\Psi_{\mathcal{L}_{in}}, \qquad \Psi_{\mathcal{L}_{out}}$$

which describe pairwise fusing vertex operators

into copies of trivial representation.

Necessarily they correspond to subspace of

$${}^L\rho = \otimes_I {}^L\rho_I$$

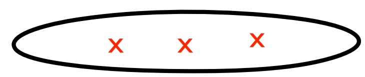
of weight

$$\nu = 0$$

To categorify quantum knot invariants,

one would like to associate

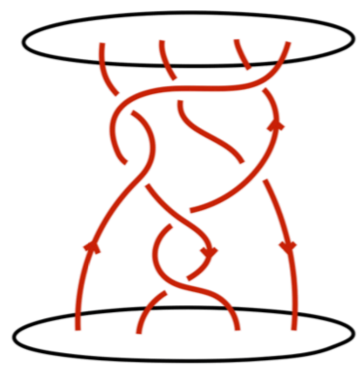
to the space conformal blocks one obtains at a fixed time slice

$\mathcal{A}$ 

a bi-graded category,

and to each conformal block an object of the category.

To braids,



one would like to associate

functors between the categories

corresponding to the

top and the bottom.

Moreover,

we would like to do that in the way that

recovers the quantum knot invariants upon

de-categorification.

One typically proceeds by coming up with a category,
and then one has to work to prove
that de-categorification gives
the quantum knot invariants one aimed to categorify.

In the two of the approaches
we are about to describe,
the second step is automatic.

The starting point for us is

a geometric realization of conformal blocks,

coming from supersymmetric quantum field theory

and

string theory.

We will eventually find not
one but two such interpretations.

To explain how they come about,
and to find a relation between them,
it is useful to ask a slightly different question first.

Namely, we will first ask for a

geometric interpretation of

q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

the quantum affine algebra that is a q-deformation of

$$\widehat{{}^L\mathfrak{g}}$$

the affine Lie algebra.

The q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

are q-deformations of conformal blocks of $\widehat{{}^L\mathfrak{g}}$

which I. Frenkel and Reshetikhin

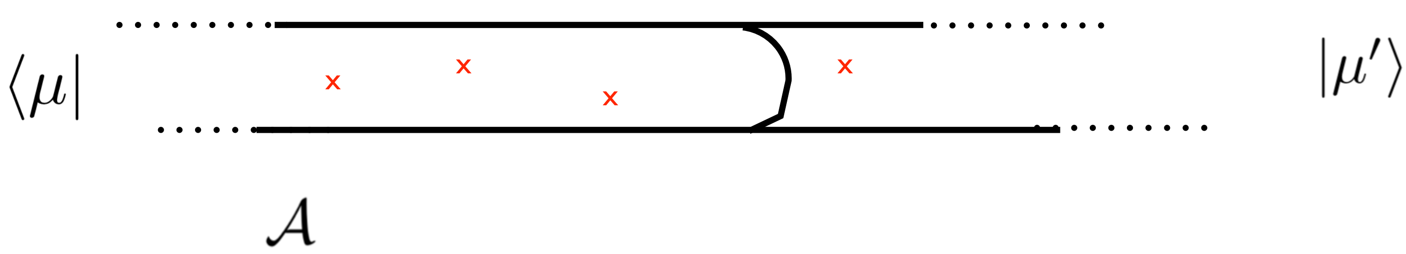
discovered in the '80's.

They are defined as correlation functions

$$\Psi(a_1, \dots, a_n) = \langle\mu|\ \Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\ |\mu'\rangle$$

of chiral vertex operators,

like in the conformal case, except all the operators are q-deformed.



Just like conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

may be defined as solutions of the Knizhnik-Zamolodchikov equation,

the q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

are solutions of the quantum Knizhnik-Zamolodchikov equation.

The quantum Knizhnik-Zamolodchikov (qKZ) equation

is a regular difference equation

$$\Psi(a_1, \dots pa_I, \dots a_n) = R_{II-1}(pa_I/a_{I-1}) \cdots R_{I1}(pa_I/a_{I-1})(\hbar^\rho)_I$$

$$\times R_{In}(a_I/a_n) \dots R_{II+1}(a_I/a_{I+1}) \Psi(a_1, \dots a_I, \dots a_n)$$

which reduces to the Knizhnik-Zamolodchikov equation

$$\kappa\, a_I \frac{\partial}{\partial a_I} \Psi = \left(\sum_{J \neq I} r_{IJ}(a_I/a_J) + r_{I0} + r_{I\infty} \right) \Psi$$

in the conformal limit.

It turns out that q-conformal blocks of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

have a geometric realization,

coming from

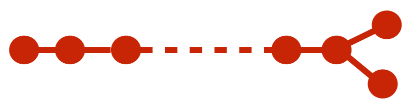
supersymmetric gauge theory.

Let ${}^L\mathfrak{g}$ be a simply laced Lie algebra so in particular

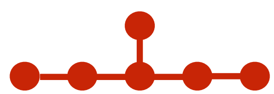
$$ {}^L\mathfrak{g} = \mathfrak{g} $$

and of the following types:

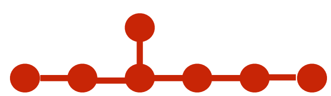
$\mathfrak{g} = A_n$



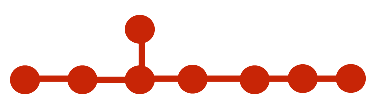
$\mathfrak{g} = D_n$



$\mathfrak{g} = E_6$



$\mathfrak{g} = E_7$



$\mathfrak{g} = E_8$

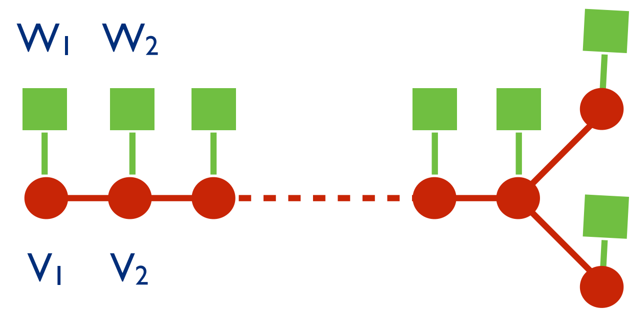
The gauge theory we need

is a

three dimensional quiver gauge theory

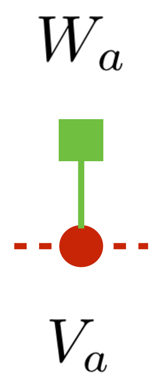
with quiver $\mathcal{Q}$

$\mathfrak{g} = D_n$



based on the Dynkin diagram of $\mathfrak{g}$

The ranks of the vector spaces



are determined by the representation

$$^{L}\rho = \otimes_I \, ^{L}\rho_I$$

and the fixed weight ν in that representation.

The corresponding quiver variety

$$X = T^* \mathrm{Rep}\, \mathcal{Q} /\!/\!/\!/ G_{\mathcal{Q}}$$

entered representation theory previously,

in the work of Nakajima,

who showed that

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

acts on its classical K-theory.

With Davesh Maulik and Andrey Smirnov,

Andrei

developed

quantum K-theory

of

Nakajima quiver varieties such as

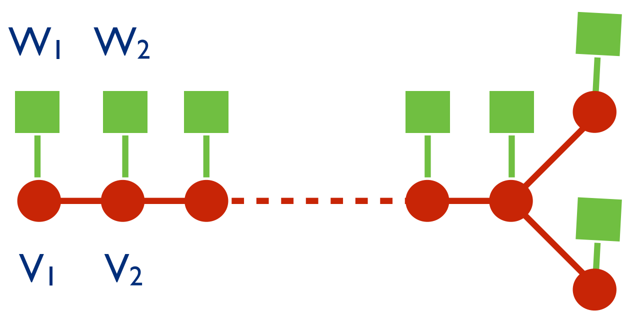
$$X = T^*\mathrm{Rep}\, \mathcal{Q} /\!\!/\!\!/\!\!/ G_{\mathcal{Q}}$$

They showed that,

quantum K-theory

of the quiver variety corresponding to

$\mathcal{Q}$



provides solutions to the

qKZ equation of

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

we started with, as its most basic objects.

One gets solutions of the qKZ equation as

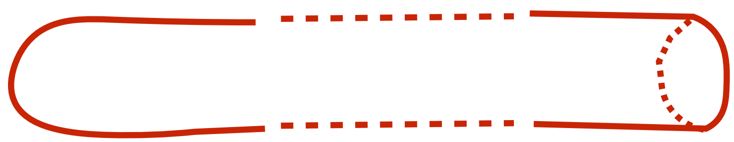
generating functions of

equivariant, K-theoretic counts

of quasi-maps

$$\mathrm{D} \dashrightarrow X$$

D



of all degrees.

These generating functions go under the name

K-theoretic $\text{Vertex}^{K}(X)$ functions.

One works equivariantly with respect to:

$$\mathsf{T} = \mathsf{A} \times \mathbb{C}^\times_\hbar$$

A is the maximal torus of rotations of X
that preserve the symplectic form, and $\mathbb{C}^\times_\hbar$ scales it,

and with respect to

$$\mathbb{C}^\times_p$$

which rotates the domain curve.

Physically,

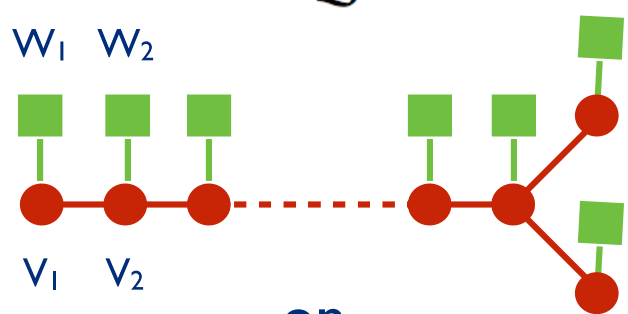
K-theoretic $\mathsf{Vertex}^K(X)$ functions.

are supersymmetric partition functions

of the three dimensional quiver gauge theory

with quiver

$\mathcal{Q}$

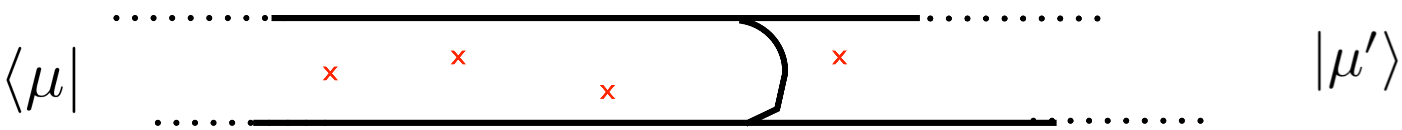


on

$\mathrm{D} \times S^1$

All the ingredients in the q-conformal block

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \; \Phi_{L_{\rho_1}}(a_1) \cdots \Phi_{L_{\rho_n}}(a_n) \; | \mu' \rangle$$

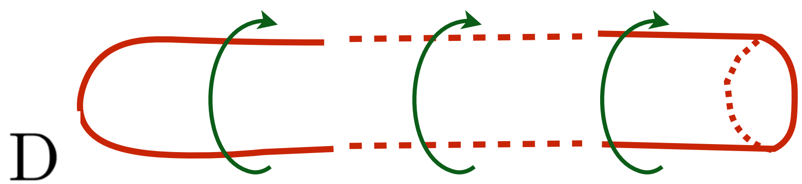


have a geometric and gauge theory interpretation.

The step

$$p = \hbar^{-\kappa}$$

of the qKZ equation is the parameter by which D rotates,



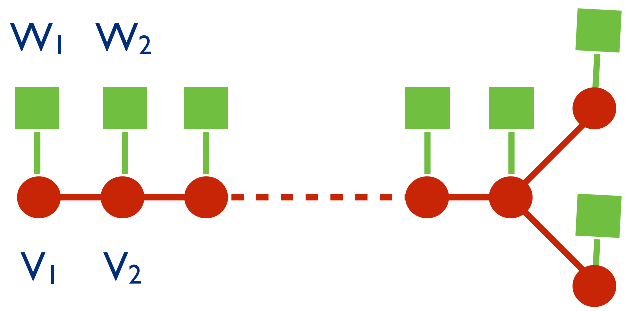
as we go around the S^1 in $\mathrm{D} \times S^1$

The positions of vertex operators,

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \, \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n) \, | \mu' \rangle$$

are equivariant parameters preserving the holomorphic symplectic form of

$$X = T^* \mathrm{Rep}\, \mathcal{Q} /\!/\!/\!/ G_{\mathcal{Q}}$$



$\mathcal{Q}$

while the parameter $\hbar$ in

scales it.

The highest weight vector of Verma module $\langle\mu|$ in

$$\Psi(a_1,\ldots,a_n) = \langle\mu|\ \Phi_{{}^L\rho_1}(a_1)\cdots\Phi_{{}^L\rho_n}(a_n)\ |\mu'\rangle$$

is related to the degree counting parameter

$$z \in \left(\mathbb{C}^\times\right)^{\mathrm{rk}({}^{\mathrm{L}}\mathfrak{g})}$$

by

$$z = \hbar^{\mu}$$

The quantum affine algebra

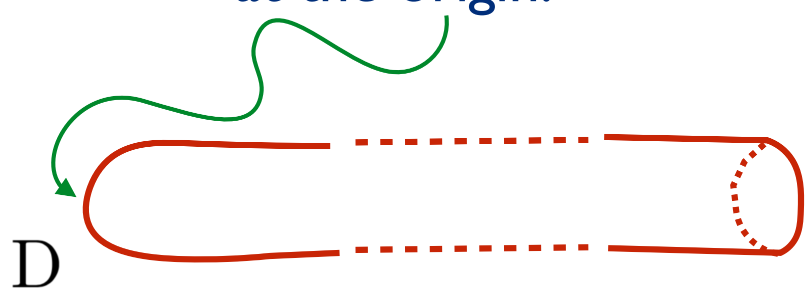
$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

acts on

$$\mathsf{Vertex}^K(X)$$

through insertions of equivariant K-theory classes,

at the origin:

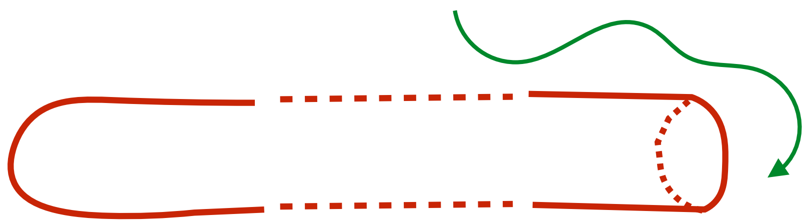


Which solution of the qKZ equation

$$\mathsf{Vertex}^K(X)$$

computes depends on the choice of data

at infinity of



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Physically, this choice means vertex functions should be thought of as valued in

$$\mathsf{Vertex}^K(X) \;\in Ell_{\mathrm{T}}(X)$$

While

$$\Psi = \mathsf{Vertex}^K(X)$$

solve the quantum Knizhnik-Zamolodchikov equation,

$$\Psi(a_1, \ldots pa_I, \ldots a_n) = R_{II-1}(pa_I/a_{I-1}) \cdots R_{I1}(pa_I/a_{I-1})(\hbar^\rho)_I$$
$$\times R_{In}(a_I/a_n) \ldots R_{II+1}(a_I/a_{I+1}) \Psi(a_1, \ldots a_I, \ldots a_n)$$

they are not the q-conformal blocks of $U_\hbar(\widehat{{}^L\mathfrak{g}})$

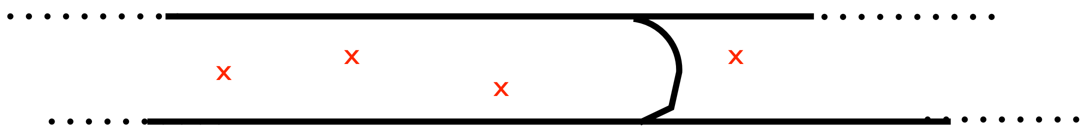
q-conformal blocks are the solutions of the qKZ equation

which are holomorphic in a chamber such as

$$\mathfrak{C}: \qquad |a_5| > |a_2| > |a_7| > \dots .$$

corresponding to choice of ordering of vertex operators in

$\langle\mu|$



$|\mu'\rangle$

This is a choice of equivariant parameters of

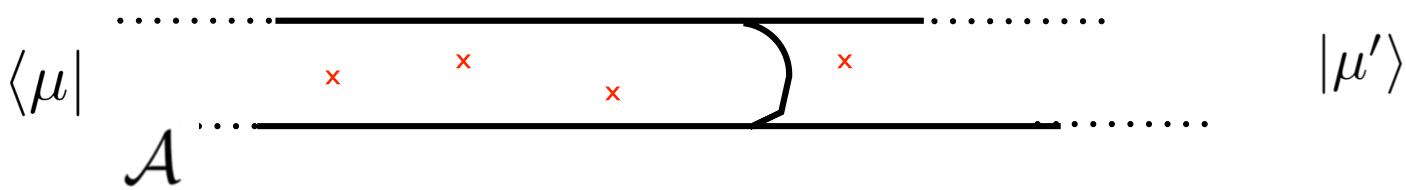
$$X = T^*\mathrm{Rep}\,\mathcal{Q}////G_{\mathcal{Q}}$$

Instead,

$$\Psi = \mathsf{Vertex}^{K}(X)$$

are holomorphic in a chamber of Kahler moduli of X

$$z = \hbar^{\mu}$$



corresponding to the choice of Verma module $\langle \mu |$.

So, this does not give an answer to the
question we are after,
namely to find a geometric interpretation of conformal blocks,
even after q-deformation.

It turns out that

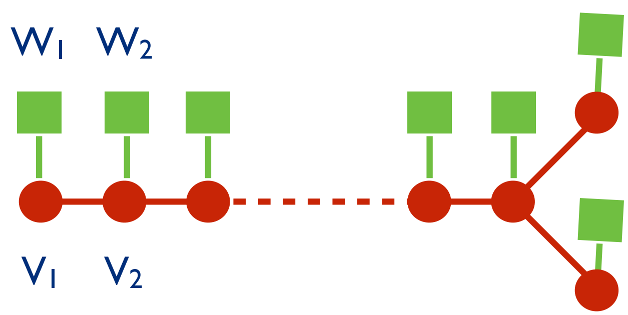
$$X = T^* \mathrm{Rep}\, \mathcal{Q} /\!\!/\!\!/\!\!/ G_{\mathcal{Q}}$$

is not the only geometry that underlies solutions to the qKZ equation corresponding to our problem.

There is a second one,
which turns out to the relevant one.

There are two natural holomorphic symplectic varieties one can associate to the 3d quiver gauge theory with quiver

$\mathcal{Q}$



and N=4 supersymmetry.

One such variety is
the Nakajima quiver variety

$$X = T^* \mathrm{Rep}\, \mathcal{Q} /\!\!/\!\!/\!\!/ G_{\mathcal{Q}}$$

This is the
Higgs branch of vacua
of the 3d gauge theory.

The other is the Coulomb branch,

which we will denote by

$$X^\vee$$

The Coulomb branch

$$X^\vee = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu}$$

of our gauge theory is a certain intersection of slices

$$\mathrm{Gr}^{\overline{\lambda}}{}_{\nu} = \overline{\mathrm{Gr}}^{\lambda} \cap \mathrm{Gr}_{\nu} \qquad\qquad \lambda \geq \nu \geq 0$$

in the (thick) affine Grassmanian of G.

$$\mathrm{Gr}_G = G((z))/G[z]$$

Here, G is the adjoint form of a Lie group with Lie algebra $\mathfrak{g}$

Hanany, Witten,….

Braverman, Finkelberg, Nakajima

Bullimore, Dimofte, Gaiotto

Another way to think about

$$X^\vee = \overline{\mathrm{Gr}^{\lambda}}_{\nu}$$

is as the moduli space of

G -monopoles,

on

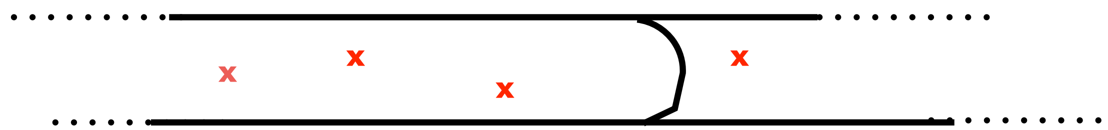
$\mathbb{R} \times \mathbb{C}_\hbar$

where λ is the charge of singular monopoles,

and ν the total monopole charge.

Hanany, Witten

The positions of vertex operators on



$\mathcal{A}$

are equivariant parameters of

$$X$$

and the Kahler parameters of

$$X^{\vee}$$

We prove that, whenever it is defined,

the K-theoretic vertex function of $X^\vee$

$$\mathsf{Vertex}^K(X^\vee)$$

solves the same qKZ equation

as

$$\mathsf{Vertex}^K(X)$$

the K-theoretic vertex function of X

This is a consequence or

three dimensional mirror symmetry

which says that,

with suitable identifications of parameters and boundary conditions,

the theories based on

X and $X^\vee$

are indistinguishable.

From perspective of

$$X^\vee$$

the qKZ equation

$$\Psi(a_1, \dots pa_I, \dots a_n) = R_{I I-1}(pa_I/a_{I-1}) \cdots R_{I1}(pa_I/a_{I-1})(\hbar^\rho)_I$$
$$\times R_{In}(a_I/a_n) \dots R_{I I+1}(a_I/a_{I+1}) \Psi(a_1, \dots a_I, \dots a_n)$$

is the quantum difference equation since the a -variables are the Kahler variables of $X^\vee$.

The “quantum difference equation"

is the K-theory analogue of

of the quantum differential equation of Gromov-Witten theory.

Here “quantum” refers to the

quantum cohomology cup product on

$$H^*(X^\vee)$$

used to define it.

While

$\mathsf{Vertex}^K(X)$ and $\mathsf{Vertex}^K(X^\vee)$

solve the same qKZ equation,

they provide two different basis of its solutions.

While

$\mathsf{Vertex}^K(X)$

leads to solutions of qKZ which are analytic in

z -variables, but not in a -variables,

$\mathsf{Vertex}^K(X^\vee)$

does the opposite.

Kahler for X

and

equivariant for $X^\vee$

Kahler for $X^\vee$

and

equivariant for X

Now we can return to

our main interest,

which is obtaining a geometric realization of

$$\widehat{{}^L\mathfrak{g}}$$

conformal blocks.

The conformal limit is the limit which takes

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

and the qKZ equation to the corresponding KZ equation.

It is amounts to

$$\begin{aligned} \hbar &\to 1 \\ p = \hbar^{-\kappa} &\to 1 \\ z = \hbar^{\mu} &\to 1 \end{aligned} \qquad \kappa, a,\ \mu \text{ fixed}$$

This corresponds to keeping the data of the conformal block fixed.

The conformal limit treats

X and $X^\vee$

very differently,

since it treats the

z- and the a-variables,

differently:

$$z \to 1, \qquad a \text{ fixed}$$

Kahler for X

Kahler for $X^\vee$

The conformal limit,

is not a geometric limit from perspective of the Higgs branch

$$X = T^*\mathrm{Rep}\,\mathcal{Q}/\!\!/\!\!/\!\!/G_{\mathcal{Q}}$$

The limit results in a badly singular space,

since

$$z \;\to\; 1$$

is a limit in its Kahler variables.

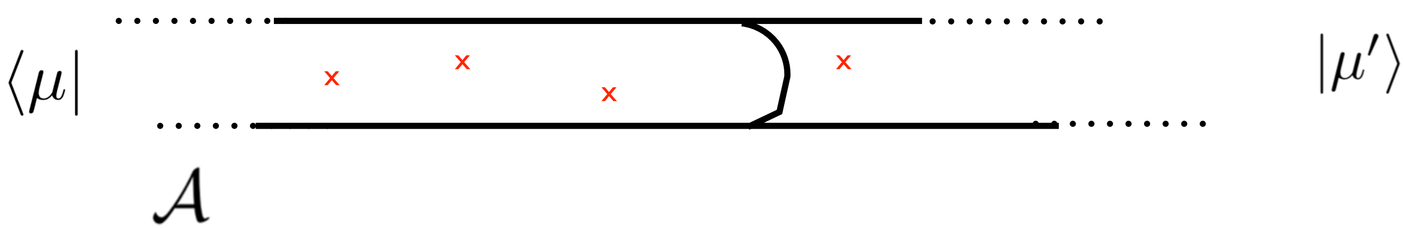
By contrast, from perspective of the Coulomb branch,

$$X^\vee = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu}$$

the limit is perfectly geometric.

Its Kahler variables are the a -variables,

the positions of vertex operators,



which are kept fixed.

From perspective of $X^\vee$,

the conformal limit,

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

is the cohomological limit

taking:

$$\text{quantum K-theory of } X^\vee \to \text{quantum cohomology of } X^\vee$$

The Knizhnik-Zamolodchikov equation

we get in the conformal limit

$$\kappa\, a_I \frac{\partial}{\partial a_I}\,\Psi = \sum_{J\neq I} r_{IJ}(a_I/a_J)\,\Psi$$

becomes the quantum differential equation

of $X^\vee$

It follows that conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

have a geometric interpretation as

cohomological vertex functions

$$\Psi = \mathsf{Vertex}(X^\vee)$$

computed by equivariant Gromov-Witten theory of

$$X^\vee$$

The cohomological vertex function counts holomorphic maps

$$\mathrm{D} \dashrightarrow X^{\vee}$$

equivariantly with respect to

$$\mathrm{T}^{\vee} = \Lambda \times \mathbb{C}^{\times}_{\mathfrak{q}}$$

where one scales the holomorphic symplectic form of $X^{\vee}$ by

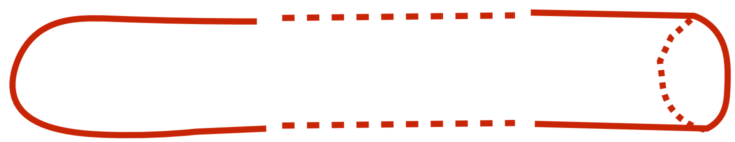
$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

choice of Verma module

The domain curve

$$\mathrm{D}$$

is an infinite cigar with an S^1 boundary at infinity.



The boundary data is a choice of a K-theory class

$$[\mathcal{F}] \in K_{\mathrm{T}^\vee}(X^\vee)$$

The geometric interpretation of

conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

in terms of

$$X^\vee$$

has far more information than the conformal blocks themselves.

Underlying

the Gromov-Witten theory of

$$X^\vee$$

is a two-dimensional supersymmetric sigma model with

$X^\vee$ as a target space.

The physical meaning of

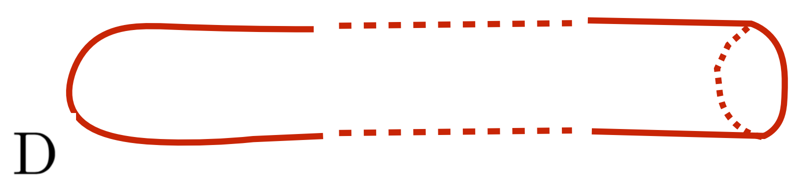
Gromov-Witten vertex function

$$\mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

is the partition function of the supersymmetric sigma model

with target $X^\vee$

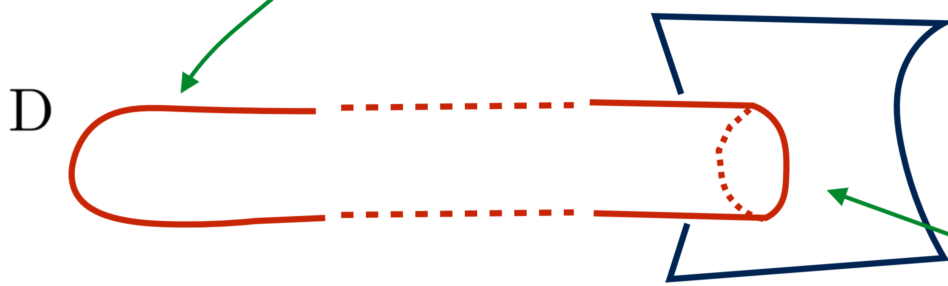
on D



To get

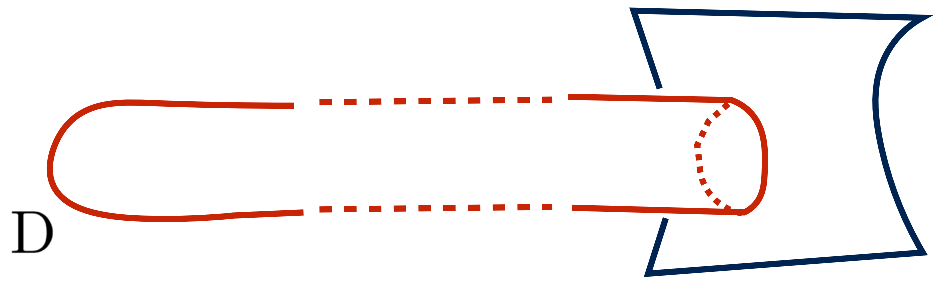
$$\mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

one has, in the interior of D , an A-type twist



and at infinity, one places a B-type boundary condition.

The B-type boundary condition



is an object

$$\mathcal{F} \in D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

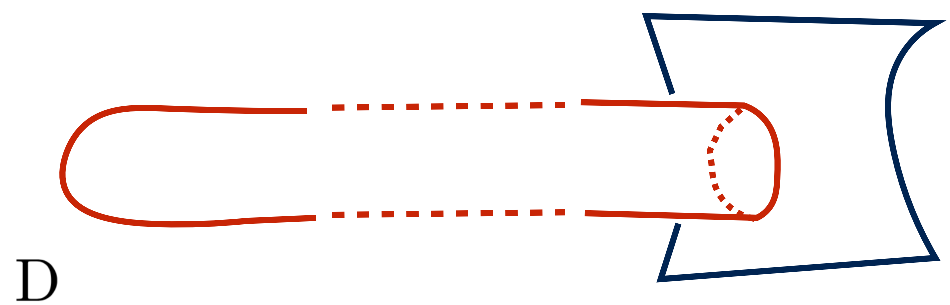
of the derived category of $\mathrm{T}^\vee$ equivariant coherent sheaves on

$X^\vee$

The choice of a B-type brane

$$\mathcal{F} \in D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

at infinity of D determines which



conformal block of $\widehat{{}^L\mathfrak{g}}$

$$\mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

computes.

Since the Knizhnik-Zamolodchikov equation,

solved by conformal blocks of

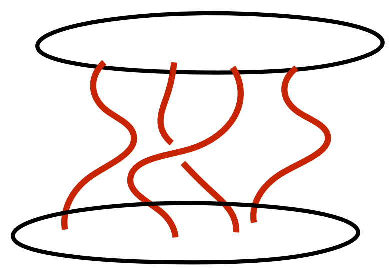
$$\widehat{L\mathfrak{g}}$$

is the quantum differential equation of $X^\vee$ …

the action of

$$U_{\mathfrak{q}}({}^L\mathfrak{g})$$

corresponding to a braid B on the space of conformal blocks



is the monodromy of the quantum differential equation of $X^\vee$,

along the path B in its Kahler moduli.

Monodromy of the quantum differential equation

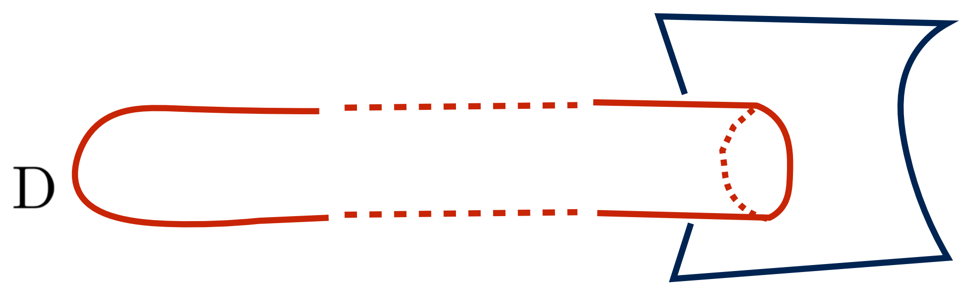
acts on

$$\Psi_{\mathcal{F}} = \mathsf{Vertex}(X^\vee)[\mathcal{F}]$$

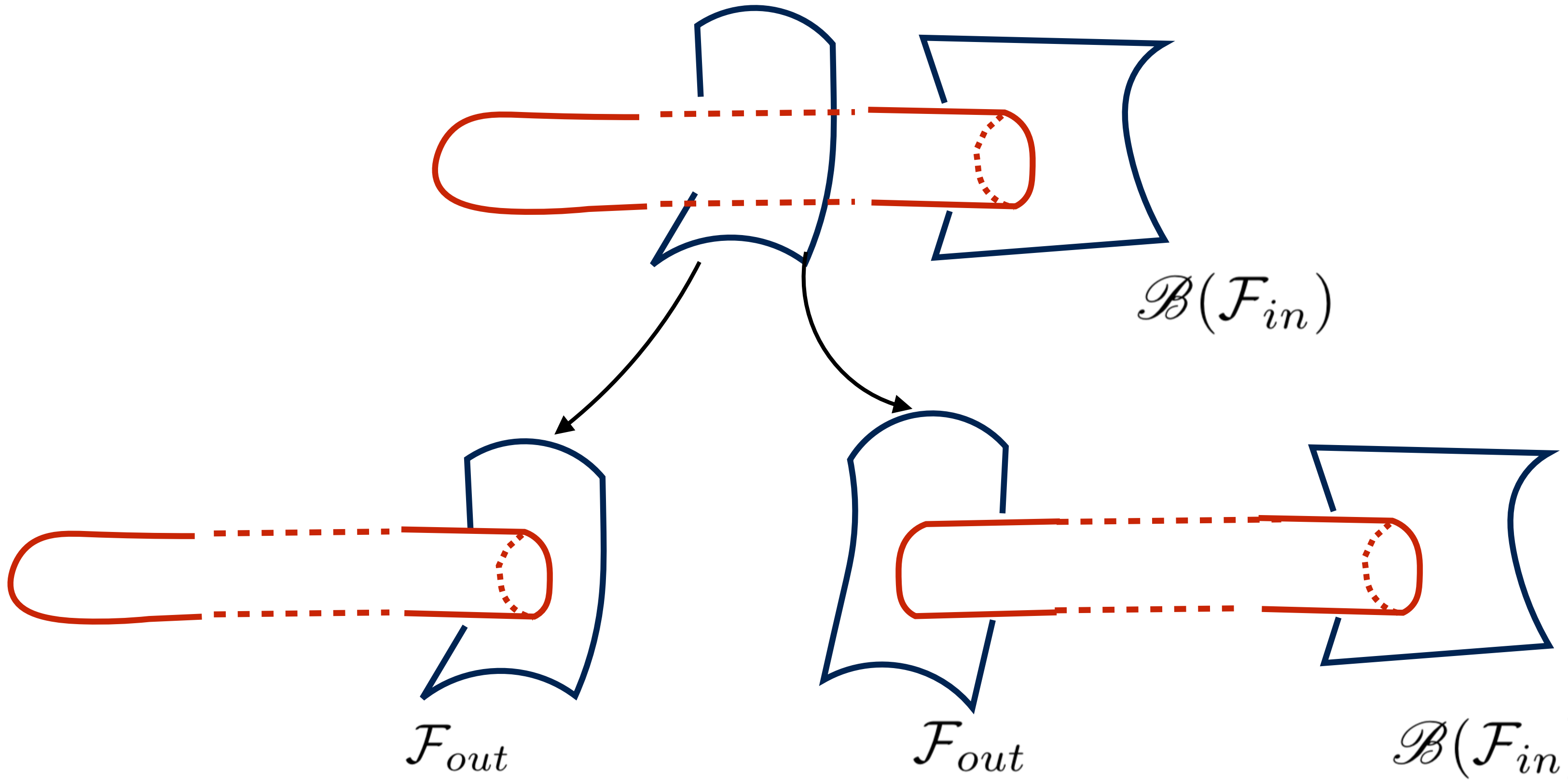
via its action on K-theory classes

$$[\mathcal{F}] \in K_{\mathrm{T}^\vee}(X^\vee)$$

inserted at the boundary at infinity of



By cutting and gluing,

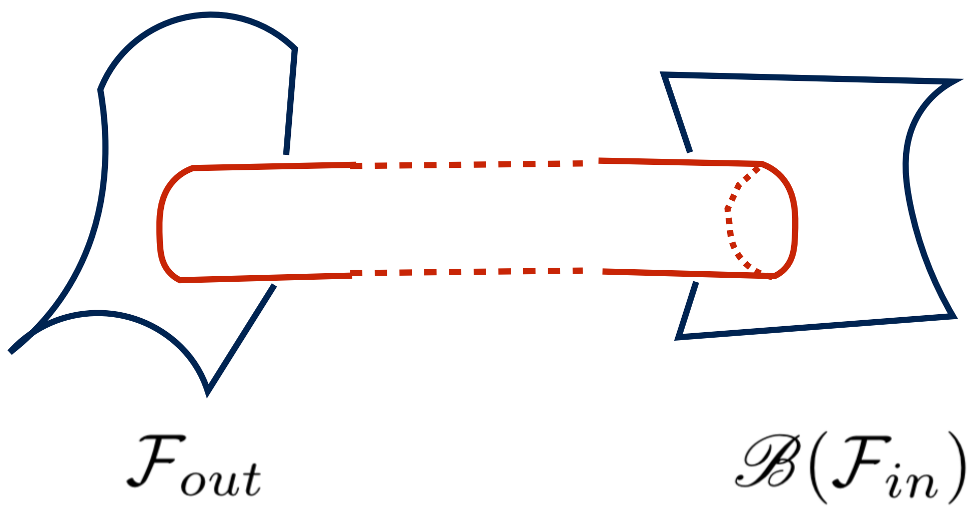


it follows……

the matrix element of the monodromy matrix

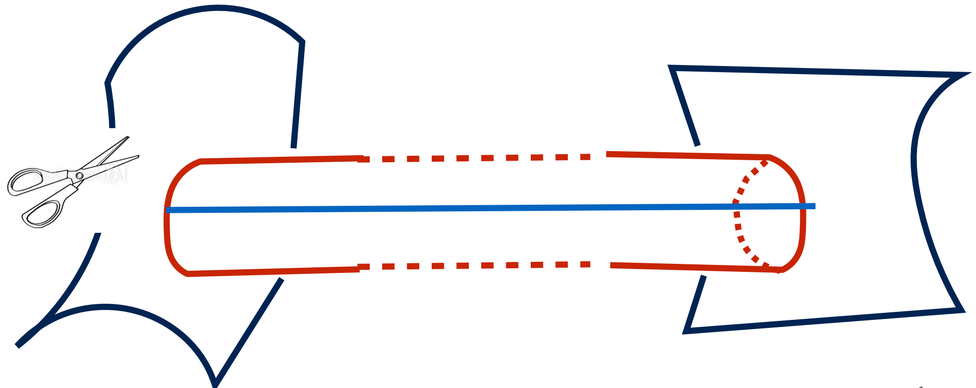
$$\left(\Psi_{\mathcal{F}_{out}} \,\middle|\, \mathfrak{B}\, \Psi_{\mathcal{F}_{in}} \right)$$

is the annulus amplitude of the B-twisted sigma model to $X^{\vee}$



with the pair of B-branes at the boundary.

The B-model annulus amplitude, is essentially per definition,



$$\mathcal{F}_{out} \qquad\qquad \mathscr{B}(\mathcal{F}_{in})$$

the supertrace, over the graded Hom space between the branes

$$H^{*,*}(K) = \mathrm{Ext}^*_{\mathrm{T}^\vee}(\mathcal{F}_{out}, \mathscr{B}(\mathcal{F}_{in}))$$

computed in

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

the derived category of $\mathrm{T}^\vee$ -equivariant coherent sheaves on $X^\vee$

This is the statement of the

theorem of Roman Bezrukavnikov and Andrei which says that,

the action of braiding matrix on

$$K_{\mathrm{T}^\vee}(X^\vee)$$

via the monodromy of the quantum differential equation

lifts to

a derived auto-equivalence functor of

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

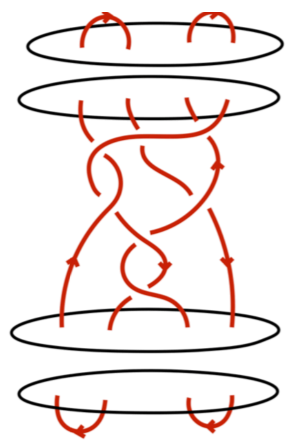
for any smooth holomorphic symplectic variety $X^\vee$.

This also implies that

categorification of quantum invariants of links

comes from

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$



since they can be expressed as matrix elements of the braiding matrix

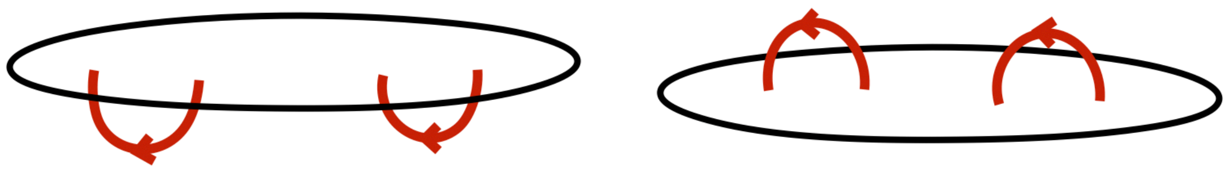
$$(\Psi_{\mathcal{F}_{out}} | \mathfrak{B}\, \Psi_{\mathcal{F}_{in}})$$

between pairs of conformal blocks.

Denote by

$$\mathcal{F}_{in}, \mathcal{F}_{out} \in D^b Coh_{\mathrm{T}}(X)$$

the branes that give rise to conformal blocks



and by

$$\mathscr{B}(\mathcal{F}_{in}) \in D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

the image of $\mathcal{F}_{in}$ under the braiding functor.

The corresponding categorified link invariant

is the graded Hom between the branes

$$H^{*,*}(K) = \mathrm{Ext}^*_{\mathrm{T}^\vee}(\mathcal{F}_{out}, \mathscr{B}(\mathcal{F}_{in}))$$

computed in

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

In addition to the homological grade, there is a second grade,

coming from the $\mathbb{C}^\times_{\mathfrak{q}} \in \mathrm{T}^\vee$ -action,

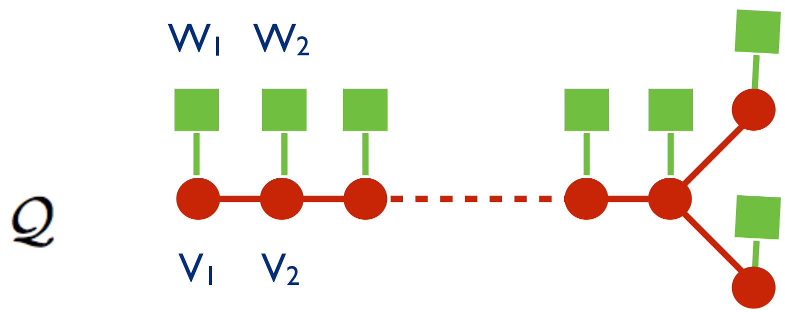
that scales the holomorphic symplectic form on $X^\vee$,

with weight

$$\mathfrak{q} = e^{\frac{2\pi i}{\kappa}}$$

The three dimensional gauge theory

we started with



in addition, leads to a

second description

of the categorified knot invariants.

It leads to a description in terms of a

two-dimensional equivariant mirror of

$$X^\vee = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu}$$

The equivariant mirror description of $X^\vee$ is also a new result.

The equivariant mirror

is a Landau-Ginzburg theory with target Y ,

and potential

$$W \, .$$

The Landau-Ginzburg potential

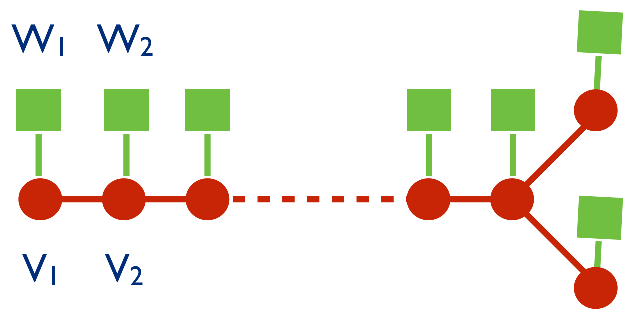
$$W$$

and the target

$$Y = \mathcal{A}^{\rm rk}/\text{Weyl}$$

can be derived from the 3d gauge theory.

$\mathcal{Q}$



The potential is a limit of the three dimensional effective superpotential, given as a sum of contributions associated to its nodes and its arrows.

One instructive, if roundabout, way to

discover the mirror description,

is as follows.

Recall that, in the conformal limit,

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

the K-theoretic vertex function of X

$$\mathsf{Vertex}^K(X)$$

has no geometric interpretation in terms of X

While it is not given in terms of X

the conformal limit of

$$\mathsf{Vertex}^{K}(X)$$

must exist.

To find the limit, one uses the integral formulation of

$$\mathsf{Vertex}^{K}(X)$$

discovered jointly with E. Fenkel.

The integrals

come from studying quasi-maps to

$$X = T^*\mathrm{Rep}\,\mathcal{Q}/\!\!/\!\!/\!\!/G_{\mathcal{Q}}$$

in geometric-invariant theory terms.

One views them as maps to the pre-quotient

and, projecting to gauge invariant configurations,

we end up integrating over the maximal torus of

$$G_{\mathcal{Q}} = \prod_a GL(V_a)$$

The conformal limit of $\text{Vertex}^K(X)$ has the form:

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \; e^{W/\kappa}$$

It gives integral solutions to the Knizhnik-Zamolodchikov equation corresponding to the conformal blocks

$$\Psi(a_1, \ldots, a_n) = \langle \mu | \; \Phi_{{}^L\rho_1}(a_1) \cdots \Phi_{{}^L\rho_n}(a_n) \; | \mu' \rangle$$

of $\widehat{{}^L\mathfrak{g}}_\kappa$

The function W that enters

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \, e^{W/\kappa}$$

is the Landau-Ginzburg potential,
and Ω is a top holomorphic form on Y.

The potential is a sum over three types of terms

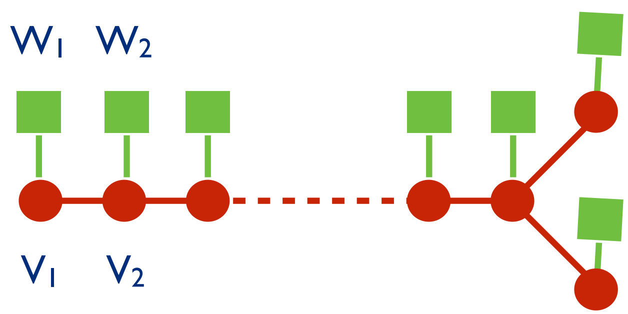
$$W = W_1 + W_2 + W_3$$

one of which comes from the nodes

$$W_1 = \sum_a \sum_\alpha \ln(x_{\alpha,a})^{({}^L e_a, \mu)}$$

and two from the arrows.

$$W_2 = \sum_{a,\alpha} \sum_I \ln(x_{\alpha,a} - a_I)^{({}^L e_a, \lambda_I)} \qquad W_3 = -\sum_{a,b} \sum_{\alpha<\beta} \ln(x_{\alpha,a} - x_{\beta,b})^{({}^L e_a, {}^L e_b)}$$



The integration in

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \ e^{W/\kappa}$$

is over a Lagrangian $\mathcal{L}$ in

$$Y = \mathcal{A}^{\mathrm{rk}}/\mathrm{Weyl}$$

the target space of the Landau-Ginzburg model.

We are re-discovering

from

geometry and supersymmetric gauge theory,

the integral representations of conformal blocks of

$$\widehat{{}^L\mathfrak{g}}$$

They are very well known,

and go back to work of Feigin and E.Frenkel in the ’80’s

and Schechtman and Varchenko.

The fact that the Knizhnik-Zamolodchikov equation which the Landau-Ginzburg integral solves

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \; e^{W/\kappa}$$

is also the quantum differential equation of $X^{\vee}$ ....

…..gives a Givental type proof of 2d mirror symmetry

at genus zero,

relating

equivariant A-model on $X^\vee$

to

B-model on Y with superpotential W.

The Landau-Ginzburg origin of

conformal blocks

automatically

leads to categorification of the corresponding

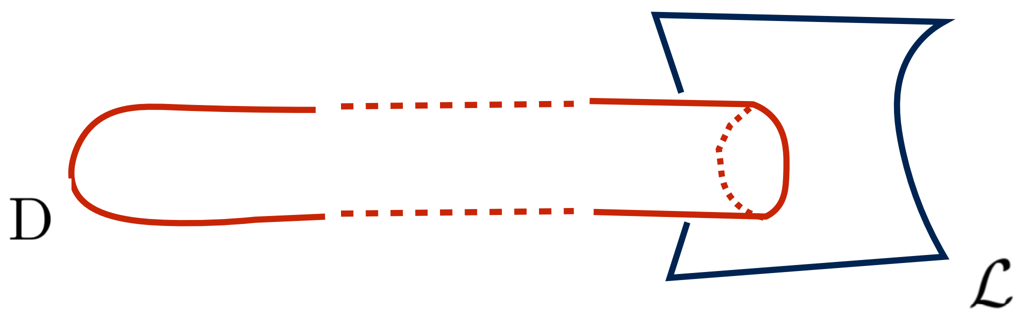
braid and link invariants.

From the Landau-Ginzburg perspective

the conformal block

$$\Psi_{\mathcal{L}} = \int_{\mathcal{L}} \Omega \; e^{W/\kappa}$$

is the partition function of the B-twisted theory on D ,

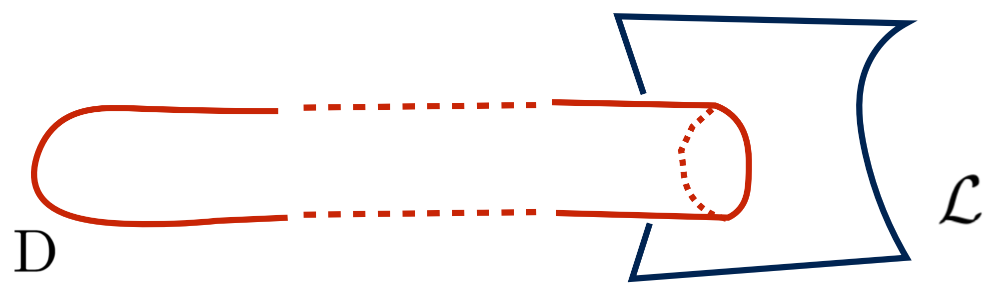


with A-type boundary condition at infinity, corresponding to the Lagrangian $\mathcal{L}$ in Y .

Thus, corresponding to a solution to the

Knizhnik-Zamolodchikov equation

is an A-brane at the boundary of D at infinity,



The brane is an object of

$$\mathcal{FS}(Y, W)$$

the Fukaya-Seidel category of A-branes on Y with potential W

The categorified link invariant arizes as the Floer cohomology group

$$H^{*,*}(K) = HF^{*,*}(\mathcal{L}_{out}, \mathscr{B}\mathcal{L}_{in})$$

where the second grade is

is the winding number,

associated to the

non-single valued potential.

We get a 2d equivariant mirror description of categorified knot invariants

based on

$$\mathcal{FS}(Y,W)$$

the Fukaya-Seidel category of A-branes on Y ,

the target of the Landau-Ginzburg model,

with potential W .

It turns out that there is

a third approach to categorification

which is related to the other two,

though less tractable.

It is important to understand the connection,

to get a unified picture of

the knot categorification problem,

and its solutions.

This will also demystify an aspect of the story so far which seems strange:

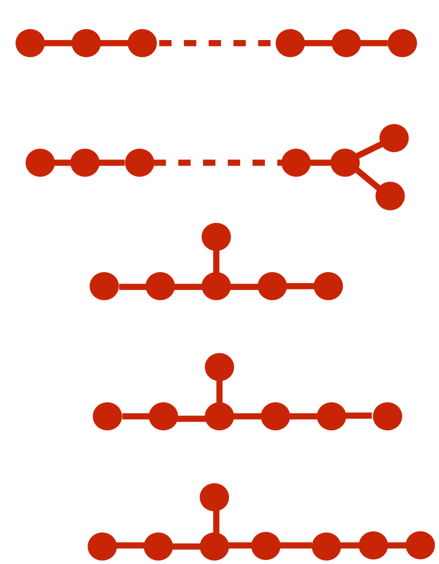
What do three dimensional supersymmetric gauge theories have to do with knot invariants?

The explanation comes from string theory.

More precisely, it comes from the six dimensional

little string theory

labeled by a simply laced Lie algebra $\mathfrak{g}$



$\mathfrak{g} = A_n$

$\mathfrak{g} = D_n$

$\mathfrak{g} = E_6$

$\mathfrak{g} = E_7$

$\mathfrak{g} = E_8$

with (2,0) supersymmetry.

The six dimensional string theory is obtained by taking a limit of IIB string theory on an ADE surface singularity of type

$$\mathfrak{g}$$

.

In the limit, one keeps only the degrees of freedom supported at the singularity and decouples the 10d bulk.

The q-conformal blocks of the

$$U_\hbar(\widehat{{}^L\mathfrak{g}})$$

can be understood

as the supersymmetric partition functions of

the $\mathfrak{g}$ -type little string theory,

with D-branes.

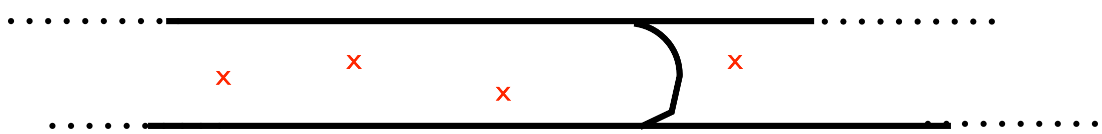
One wants to study the six dimensional (2,0) little string theory on

$$M_6 = \mathcal{A} \times \mathrm{D} \times \mathbb{C}_\hbar$$

where

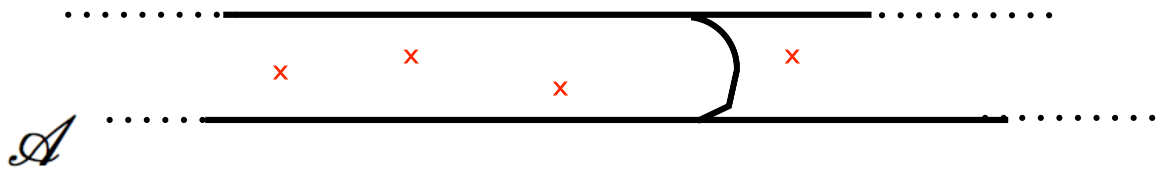
$$\mathscr{A}$$

is the Riemann surface where the conformal blocks live:



and D is the domain curve of the 2d theories we had so far.

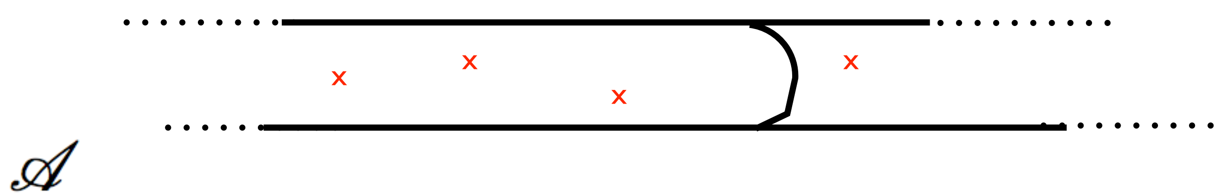
The vertex operators on the Riemann surface



come from a collection of defects in the little string theory,

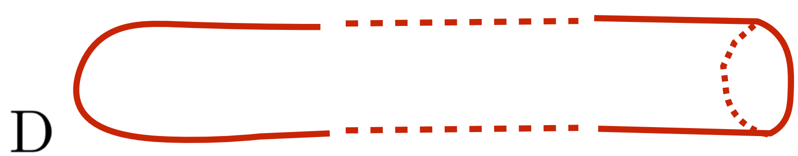
which are inherited from D-branes of the ten dimensional string.

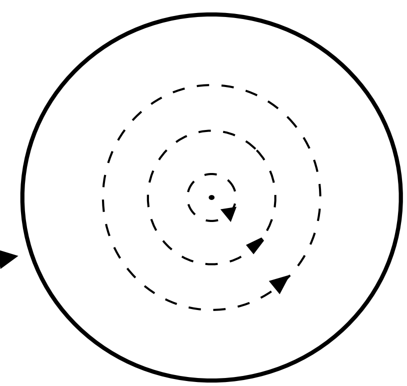
The D-branes needed are



two dimensional defects of the six dimensional theory on

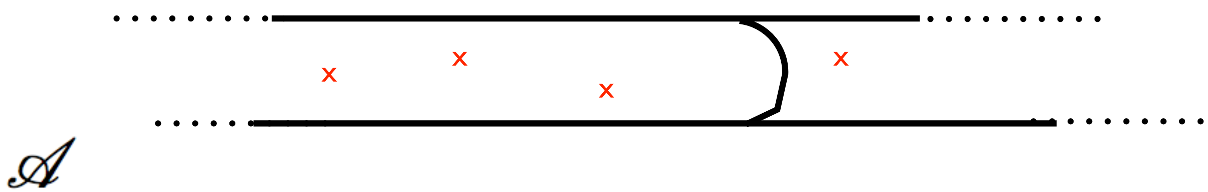
$$M_6 = \mathcal{A} \times \mathrm{D} \times \mathbb{C}_\hbar$$



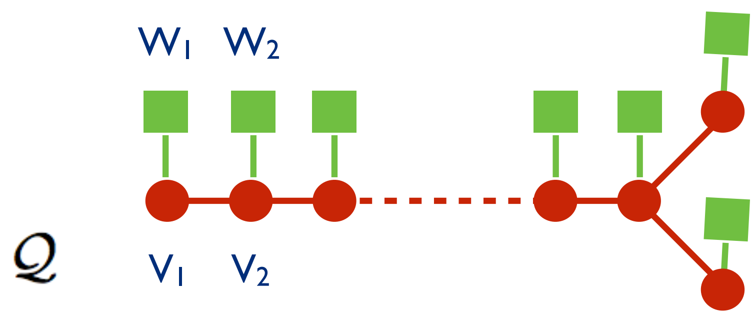


supported on D and the origin of $\mathbb{C}_\hbar$

The choice of which conformal blocks we want to study translates into choices of D-branes



The theory on the D-branes is the quiver gauge theory



that arose earlier in the talk.

This is a consequence of the familiar description of

D-branes on ADE singularities

due to Douglas and Moore in '96.

The theory on the D-branes supported on D

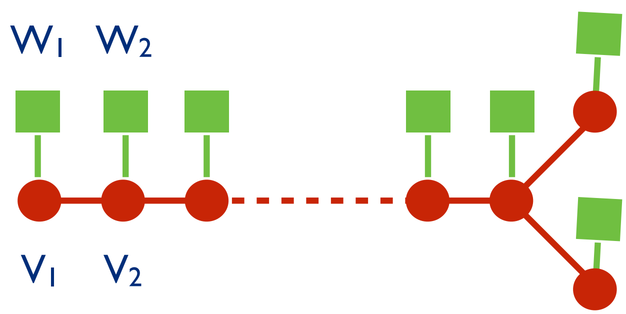
is a three dimensional quiver gauge theory on

$$\mathrm{D} \times S^1$$

rather than a two dimensional theory on D ,

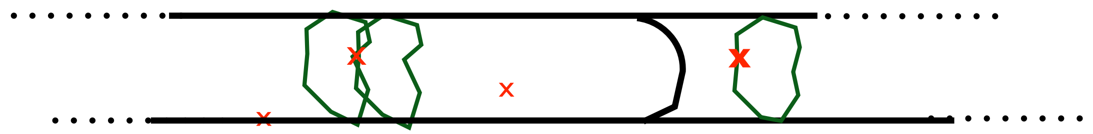
due to a stringy effect.

$\mathcal{Q}$



In a string theory,

one has to include the winding modes of strings around $\mathscr{A}$



$\mathscr{A}$

These turn the theory on the defects supported on D ,

to a three dimensional quiver gauge theory on

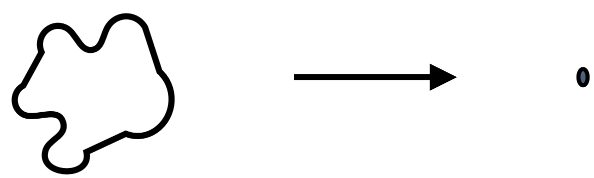
$$\mathrm{D} \times S^1$$

where the S^1 is the dual of the circle in $\mathscr{A}$

The conformal limit of the algebras

$$U_\hbar(\widehat{{}^L\mathfrak{g}}) \longrightarrow \widehat{{}^L\mathfrak{g}}$$

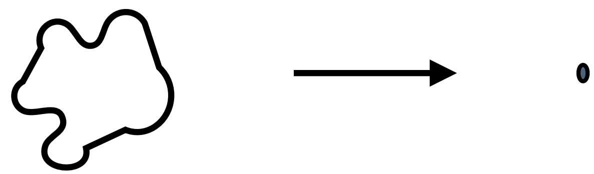
coincides with the conformal, point particle, limit of little string theory



in which it becomes the six dimensional conformal field theory

of type $\mathfrak{g}$ (with (2,0) supersymmetry)

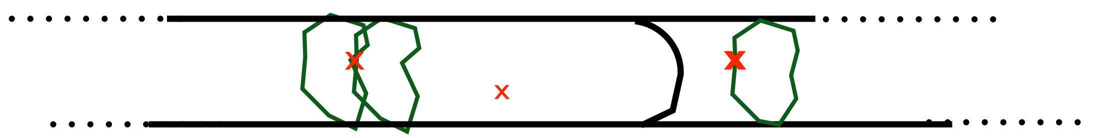
In the point particle limit,



the winding modes that made the theory

on the defects three dimensional, instead of two,

become infinitely heavy.



As a result, in the conformal limit,

the theory on the defects

becomes a two dimensional theory on

D

The two dimensional theory on the defects
of the six dimensional (2,0) theory was sought previously.

It is not a gauge theory,
but it has two other descriptions,
I described earlier in the talk.

One description

is based on the supersymmetric sigma model

describing maps

$$\mathrm{D} \dashrightarrow X^{\vee}$$

with equivariant mass deformation.

The other is in terms of the mirror

Landau-Ginzburg model on D with potential W

These approaches to categorification

started with the theories on the knots.

There is a third description,

due to Witten.

It is obtained from

the perspective of the 6d theory in the bulk.

Compactified on a very small circle,

the six dimensional $\mathfrak{g}$ -type (2,0) conformal theory

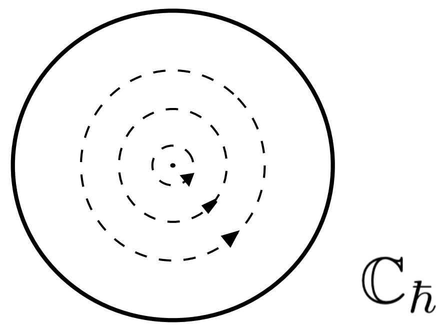
with no classical description,

becomes a $\mathfrak{g}$ -type gauge theory

in one dimension less.

To get a good 5d gauge theory description of the problem,

the circle one shrinks corresponds to S^1 in



so from a six dimensional theory on

$$M_6 = \mathcal{A} \times \mathrm{C} \times \mathbb{C}_\hbar$$

one gets a five-dimensional gauge theory on a manifold with a boundary

The five dimensional gauge theory is supported on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D} \quad \text{where} \quad \widetilde{M}_3 = \mathscr{A} \times \mathbb{R}_{\geq 0}$$

It has gauge group

$$G$$

which is the adjoint form of a Lie group with lie algebra $\mathfrak{g}$.

The two dimensional defects are monopoles

of the 5d gauge theory on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D}$$

supported on D and at points on,

$$\widetilde{M}_3 = \mathcal{A} \times \mathbb{R}_{\geq 0} \ ,$$

along its boundary.

Witten shows that the five dimensional theory on

$$\widetilde{M}_5 = \widetilde{M}_3 \times \mathrm{D}$$

can be viewed as a gauged

Landau-Ginzburg model on D with potential

$$\mathcal{W}_{\mathrm{CS}} = \int_{\widetilde{M}_3} \mathrm{Tr}(A \wedge dA + A \wedge A \wedge A)$$

on an infinite dimensional target space $\mathcal{Y}_{\mathrm{CS}}$

corresponding to $\mathfrak{g}_{\mathbb{C}}$ connections on $\widetilde{M}_3 = \mathcal{A} \times \mathbb{R}_{\geq 0}$

with suitable boundary conditions (depending on the knots).

To obtain knot homology groups in this approach,
one ends up counting solutions to
certain five dimensional equations.

The equations arise in
constructing the Floer cohomology groups
of the five dimensional Landau-Ginzburg theory.

Thus, we end up with three different approaches
to the knot categorification problem,
all of which have the same
six dimensional origin.

They all describe the same physics
starting in six dimensions.

The two geometric approaches,
describe the physics from perspective of the defects.

The approach based on the 5d gauge theory,
describe it from perspective of the bulk.

In general,
theories on defects
capture only the local physics of the defect.

In this case,
they capture all of the relevant physics,
due to a version of supersymmetric localization:
in the absence of defects,
the bulk theory is trivial.

The approach based on

$$D^b Coh_{\mathrm{T}^\vee}(X^\vee)$$

is equivalent to that
of Kamnitzer and Cautis in type A.

The approach based on

$$\mathcal{FS}(Y, W)$$

is related to the approach of

Seidel and Smith

for

$${}^L\mathfrak{g} = A_1$$

Seidel and Smith

use Fukaya category of the ordinary mirror of

$$X^\vee$$

to construct

"symplectic Khovanov homology."

The “symplectic Khovanov homology”

is a singly-graded homology theory

which is a specialization of the ordinary Khovanov homology to

$$\mathfrak{q} = 1$$

The approach based on

$$\mathcal{FS}(Y, W)$$

includes the grading from the outset,

and works for any

$${}^{L}\mathfrak{g}$$

The equivariant mirror symmetry relating

the (Y, W) Landau-Ginsburg model

and

$X^\vee$

should become a very useful tool

in both geometry and representation theory.

The question which categories

one should study:

$$D^b Coh_{\mathrm{T}}(X) \qquad \text{and} \qquad \mathcal{FS}(Y, W)$$

gets traded for more focused questions like

……understanding the geometry of objects in these categories that

map to themselves under derived functors corresponding to half twists,

up to degree shifts

and whose self-Homs categorify quantum dimensions.

This has an extension to non-simply laced Lie algebras,

for which

$$^{L}\mathfrak{g} \neq \mathfrak{g}$$

which is dictated by string theory.

One uses the fact that the non-simply laced Lie algebra

$$\mathfrak{g}$$

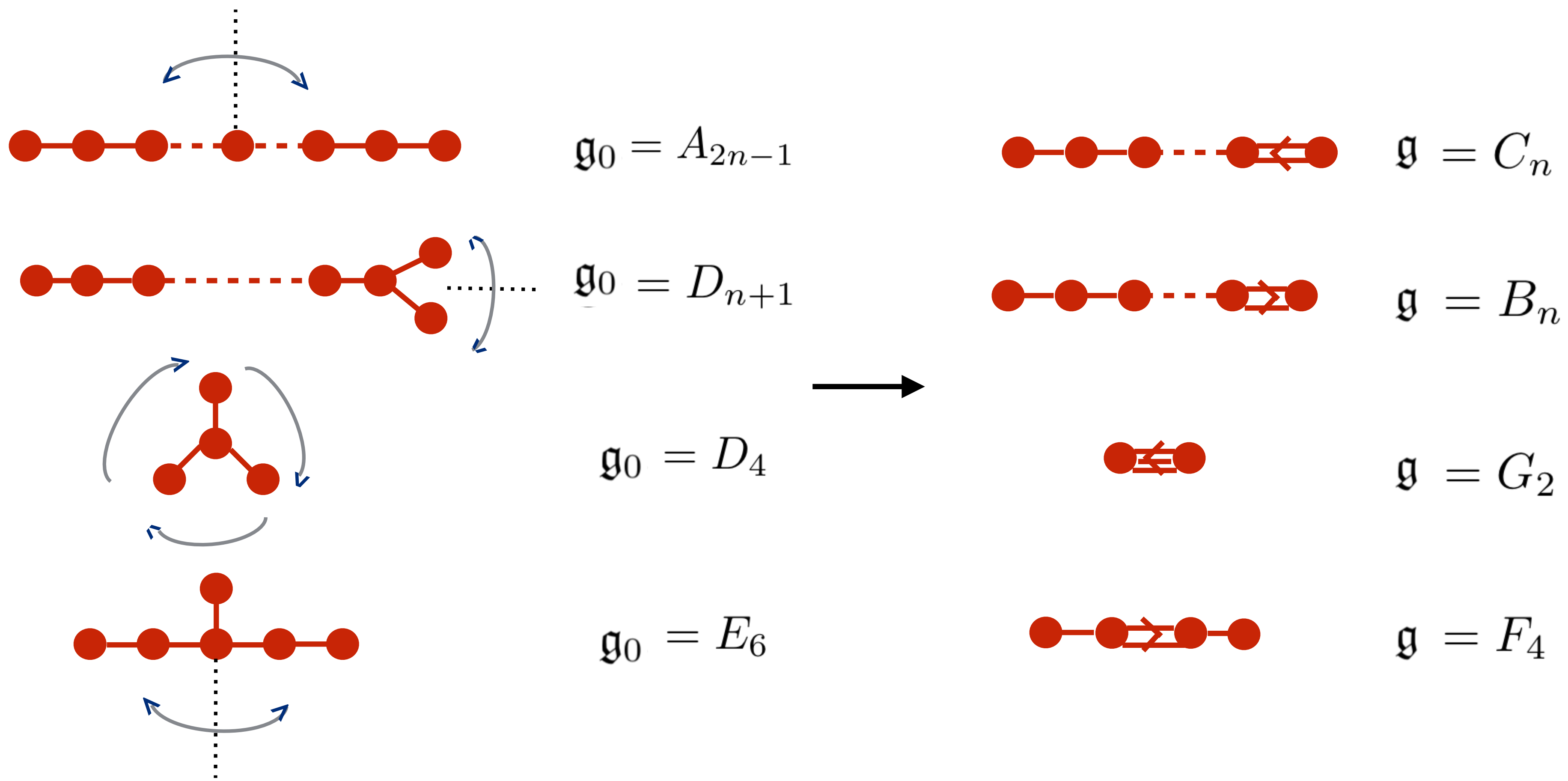
can be obtained from a simply laced Lie algebra

$$\mathfrak{g}_0$$

using an outer automorphism H of its Dynkin diagram

$$(\mathfrak{g}_0, H) \rightarrow \mathfrak{g}$$

H acts as an involution of the Dynkin diagram of $\mathfrak{g}_0$



To get knot invariants based on the Lie algebra

$$^{L}\mathfrak{g}$$

one studies little string theory of type

$$\mathfrak{g}_0$$

on

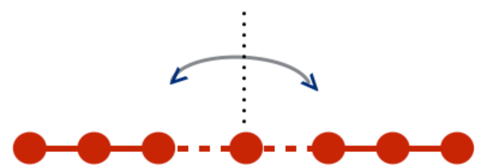
$$M_6 = \mathcal{A} \times \mathrm{D} \times \mathbb{C}_{\hbar}$$

with an H-twist

The twist

permutes the nodes of the Dynkin diagram of $\mathfrak{g}_0$

by a generator of H,



$\mathfrak{g}_0 = A_{2n-1}$

as we go once around the origin of the complex D -plane

which supports the defects.

D

The theory on D-branes is a three dimensional

$\mathfrak{g}_0$ -type quiver gauge theory

on

$$\mathrm{D} \times S^1$$

with an H- twist around D

The Coulomb branch

$$X^\vee$$

of this theory is the moduli space of singular

G -monopoles

or, equivalently, intersection of a pair of orbits

$$X^\vee = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu} \;= \overline{\mathrm{Gr}}^{\lambda} \cap \mathrm{Gr}_{\nu}$$

in the affine Grassmanian of

$$G$$

This follows since

D-branes which provide the gauge theory

are monopoles in the

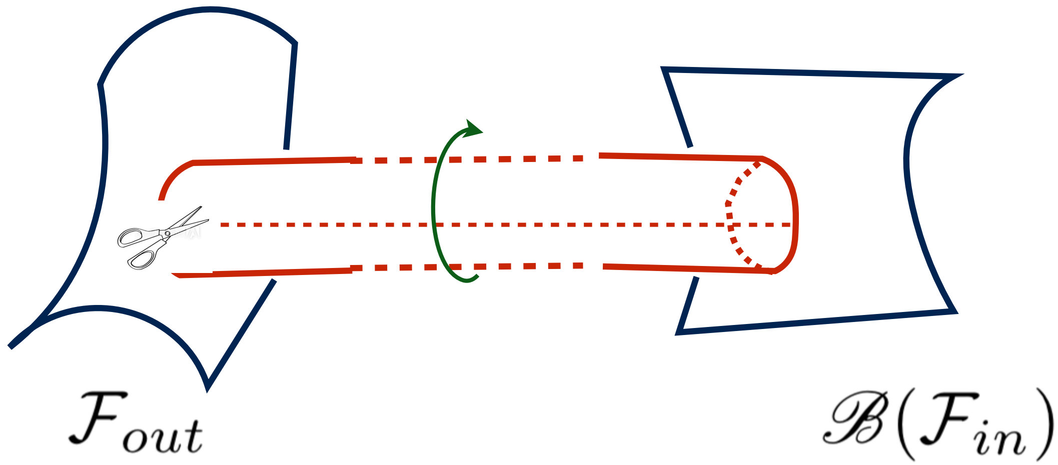
G -type gauge theory in the bulk

$$X^\vee = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu}$$

and is their moduli space.

G is the Lie group of adjoint form, with Lie algebra $\mathfrak{g}$.

In this case the relevant categories become



$D^b Coh_{\mathrm{T}}(X^\vee)$ and $\mathcal{FS}(Y,W)$

where $X^\vee = \mathrm{Gr}^{\overline{\lambda}}{}_{\nu}$ is the intersection of sliced in affine Grassmanian of G

and (Y,W) categorify conformal blocks of $\widehat{{}^L\mathfrak{g}}_{\kappa}$
and their braiding.