Gradient models and the Hessian

Richard Kenyon (Brown University)
Istvan Prause (Helsinki)

Plan:

1. Gradient variational problems. (Darboux?)
2. New examples

Minimize area: $h : \mathbb{R}^2 \to \mathbb{R}$

$$\min_h \iint_U \sqrt{1 + h_x^2 + h_y^2}\, dx\, dy$$

Here $\sigma(s, t) = \sqrt{1 + s^2 + t^2}$ is the "surface tension".

This is a gradient variational problem or gradient model since σ depends only on ∇h.

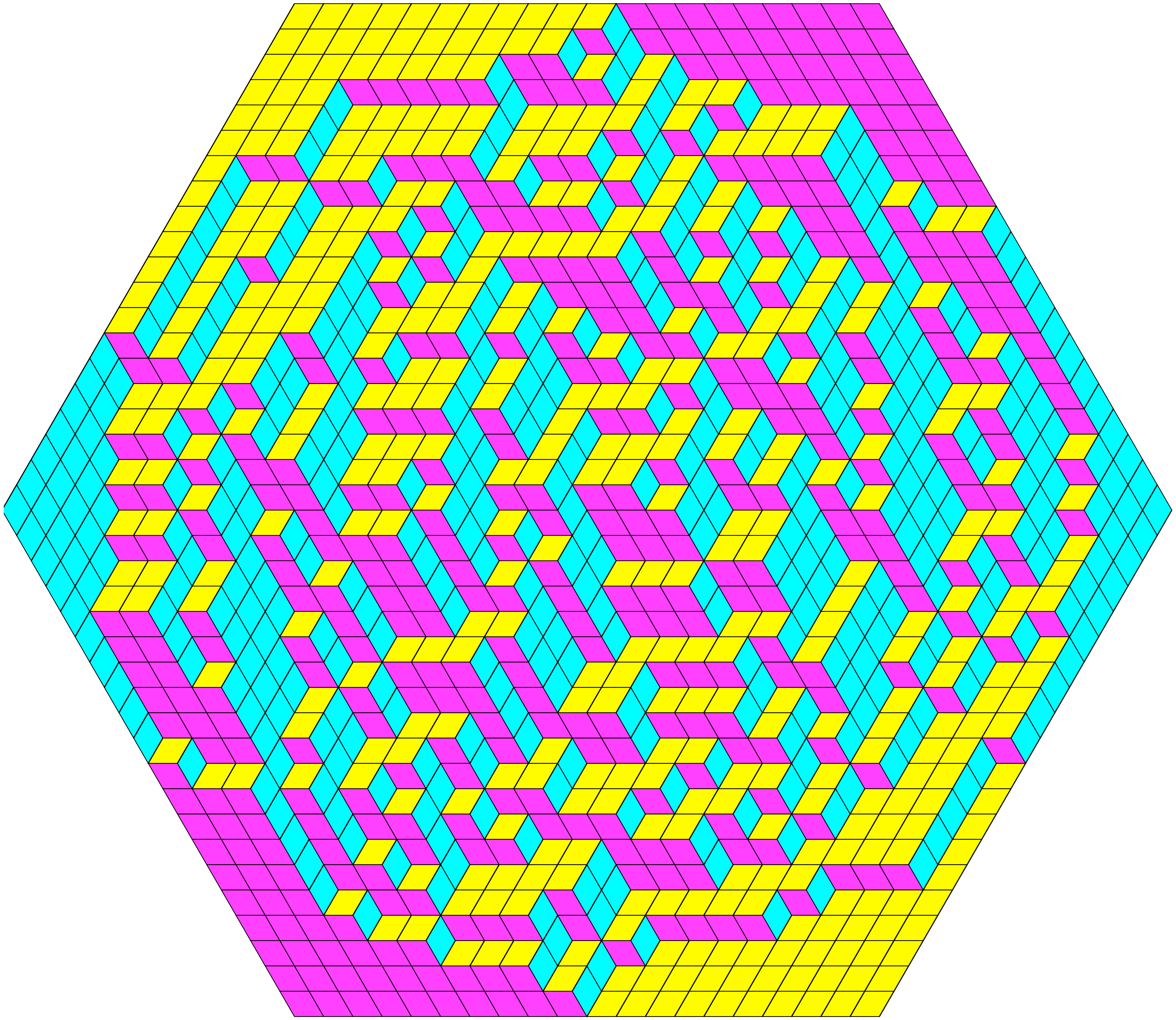
...

Weierstrass-Enneper parameterization of minimal surfaces

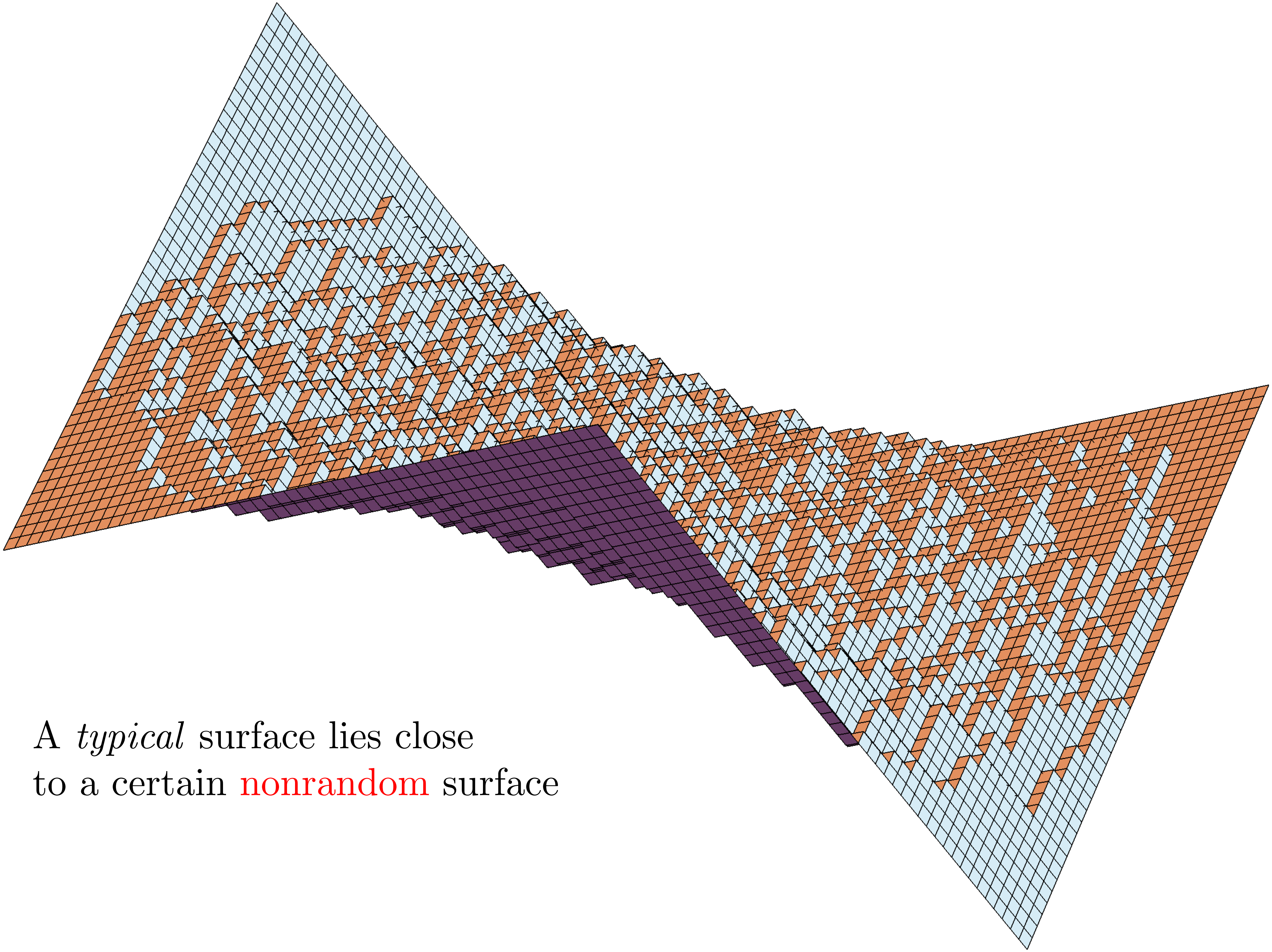
Let f, g be (arbitrary) analytic functions, then

$$Re\left(\int f(z)(1 - g(z)^2)\, dz, i\int f(z)(1 + g(z)^2)\, dz, \int f(z)g(z)dz\right)$$

parameterizes a minimal surface in $\mathbb{R}^3$.



“uniform lozenge tilings” also satisfy a variational principle

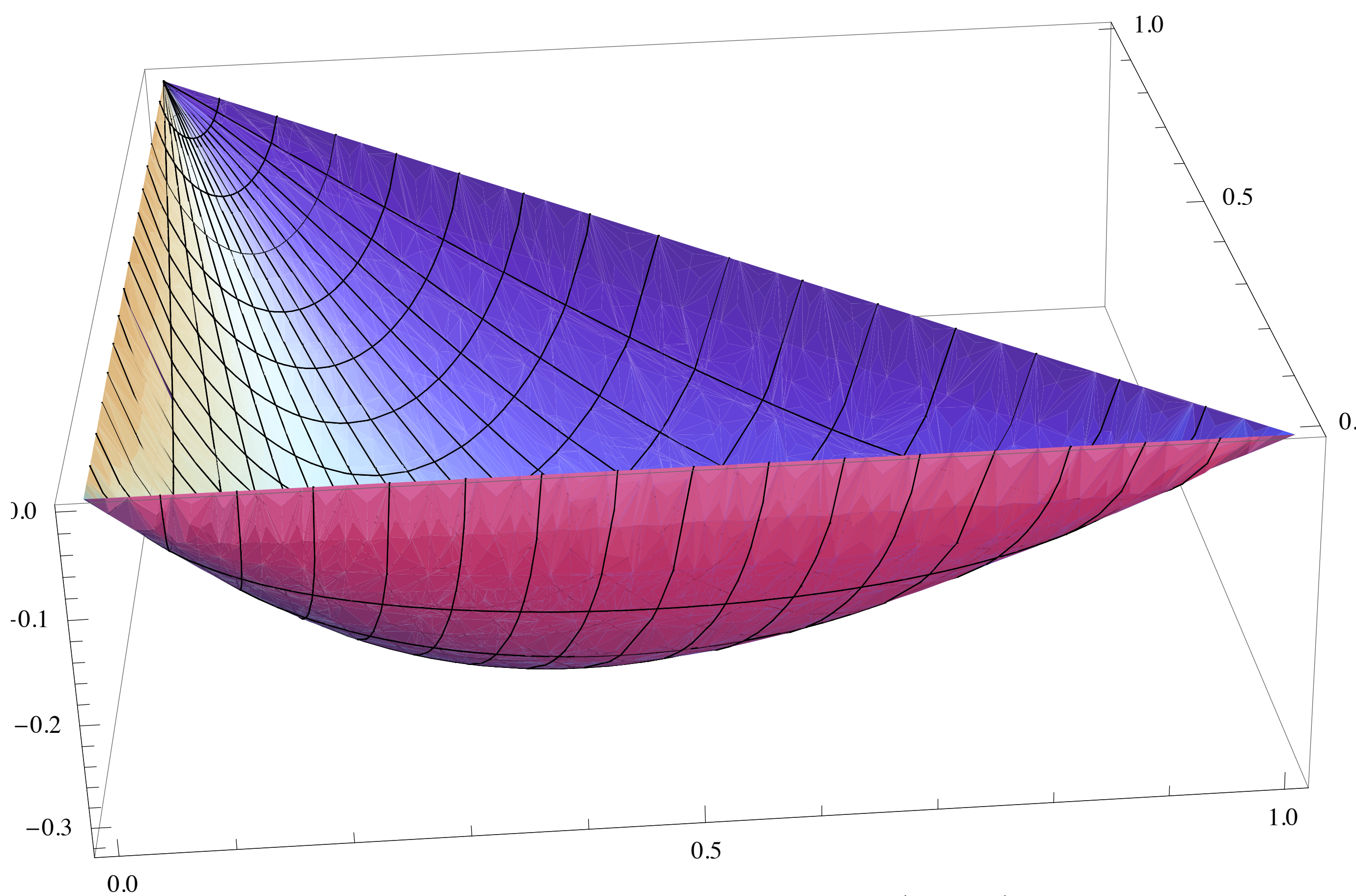


A *typical* surface lies close
to a certain nonrandom surface

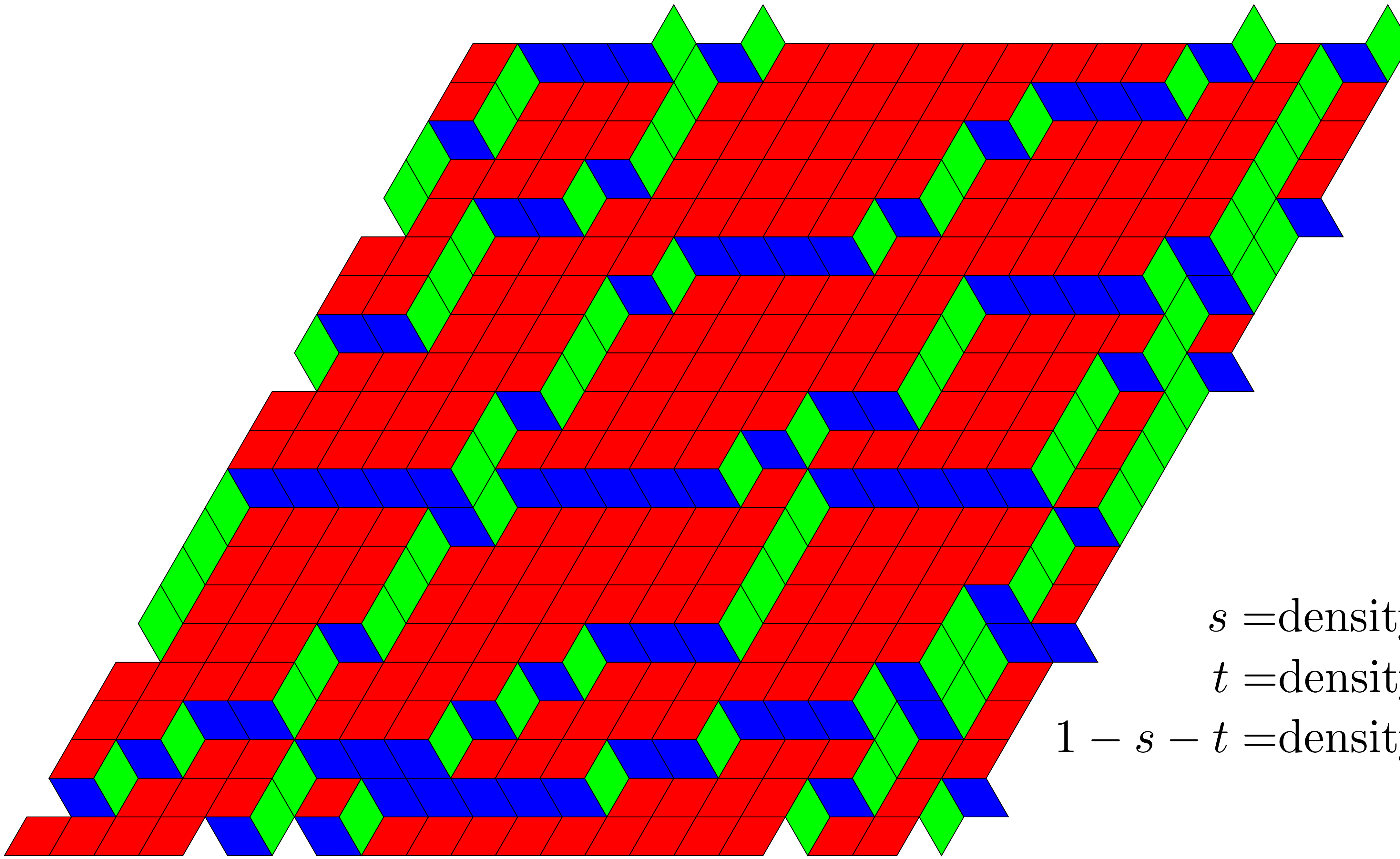
Lozenge tiling limit shape

Thm[Cohn,K,Propp (2000)] The function $h : R \to \mathbb{R}$ describing the limit shape is the unique minimizer of the surface tension integral

$$\min_h \iint_R \sigma(h_x, h_y)\, dx\, dy.$$



The surface tension $\sigma(s,t)$

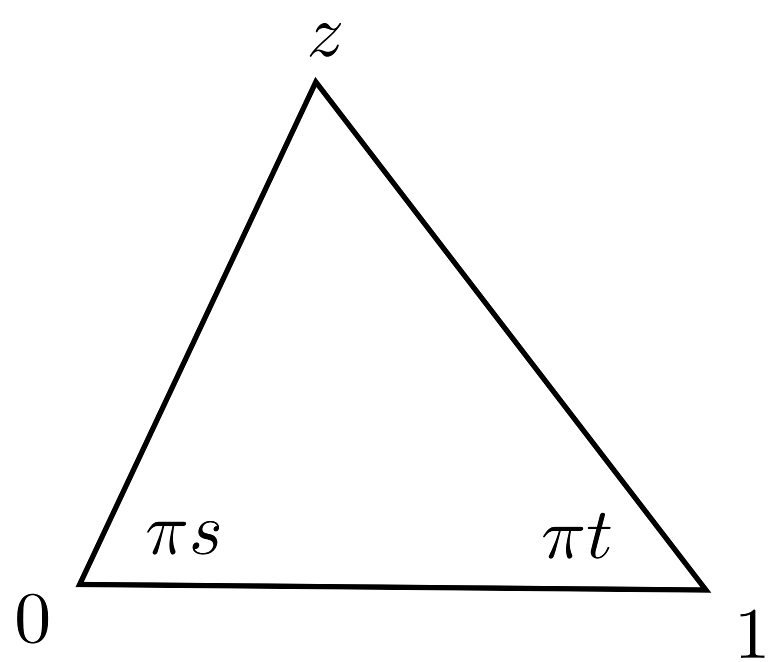


s =density of green
t =density of blue
$1 - s - t$ =density of red

for each slope (s, t) there is an associated growth rate (entropy) $-\sigma(s, t)$:

$$(\text{Number of tilings}) = e^{-\text{Area}\cdot\sigma(s,t)(1+o(1))}$$

$\sigma(s, t)$ is the *Legendre dual* of the free energy $F(X, Y)$, where tile weights are $\{1, e^X, e^Y\}$.



In terms of z,

$$\sigma(s,t) = D(z),$$

the Bloch-Wigner dilogarithm:

$$D(z) = \arg(1-z)\log|z| + \mathrm{Im}(\mathrm{Li}_2(z))$$

How to solve the variational problem?

The Euler-Lagrange equation for a gradient model is

$$\mathrm{div}_{x,y}(\nabla_{s,t}\sigma(\nabla_{x,y}h)) = 0$$

or, in terms of X, Y (recall $X = \sigma_s,\ Y = \sigma_t$)

$$X_x + Y_y = 0.$$

We can (magically) combine this with the "mixed partials" equation $h_{xy} = h_{yx}$, or

$$s_y = t_x$$

as follows:

Associated to σ is a Riemannian metric on N determined by its Hessian

domain of σ

$$g = \sigma_{ss}\, ds^2 + 2\sigma_{st}\, ds\, dt + \sigma_{tt}\, dt^2$$

Let $z = u + iv$ be a conformal coordinate for g:

$$g = e^{\phi}(du^2 + dv^2).$$

This z is the conformal coordinate for the model

Thm(Ampère) In terms of z the Euler-Lagrange equation and mixed-partials equation can be combined into a single equation

$$X_z z_x + Y_z z_y = 0.$$

Equivalently, thinking of $x = x(z, \bar{z}), y = y(z, \bar{z})$,

$$y_{\bar{z}} = \frac{Y_z}{X_z} x_{\bar{z}}.$$

For the dimer model, Y_z/X_z is analytic in z, so this equation can be solved by integrating wrt $\bar{z}$:

$$y = \frac{Y_z}{X_z}x + f(z)$$

e.g. lozenges $\frac{Y_z}{X_z} = \frac{z}{z-1}$

for an arbitrary analytic f.

What about other surface tension functions?

Thm[K-Prause] If the Hessian is the fourth power of a harmonic function of z, the equation can be similarly integrated.

(For dimers the Hessian is constant)

Pf: Let $\psi^4 = \text{Hess}(\sigma)$ and $t = \phi/\psi, \; s = \phi^*/\psi$.

$$X_z + i\psi^2 t_z = 0 = Y_z - i\psi^2 s_z$$

← Ampère

$$X_z + i(\psi\phi_z - \phi\psi_z) = 0$$

$$\Delta X + i(\psi\Delta\phi - \phi\Delta\psi + \underbrace{\psi_{\bar z}\phi_z - \phi_{\bar z}\psi_z}_{\in i\mathbb{R}}) = 0$$

$$\frac{z_x}{z_y} = -\frac{Y_z}{X_z} = \frac{s_z}{t_z} = \frac{\frac{1}{\psi}(\phi_z^* - s\psi_z)}{\frac{1}{\psi}(\phi_z - t\psi_z)} = \frac{-\frac{\phi_z^*}{\psi_z} + s}{-\frac{\phi_z}{\psi_z} + t}$$

$$\frac{y_{\bar z}}{x_{\bar z}} = -\frac{z_x}{z_y} = -\frac{A^*(z) + s}{A(z) + t}$$

← analytic if ψ harmonic

$$A^*(z)x_{\bar z} + A(z)y_{\bar z} + sx_{\bar z} + ty_{\bar z} = 0$$
$$A^*(z)x_{\bar z} + A(z)y_{\bar z} + h_{\bar z} = 0$$
$$A^*(z)x + A(z)y + h + F(z) = 0 \qquad \square$$

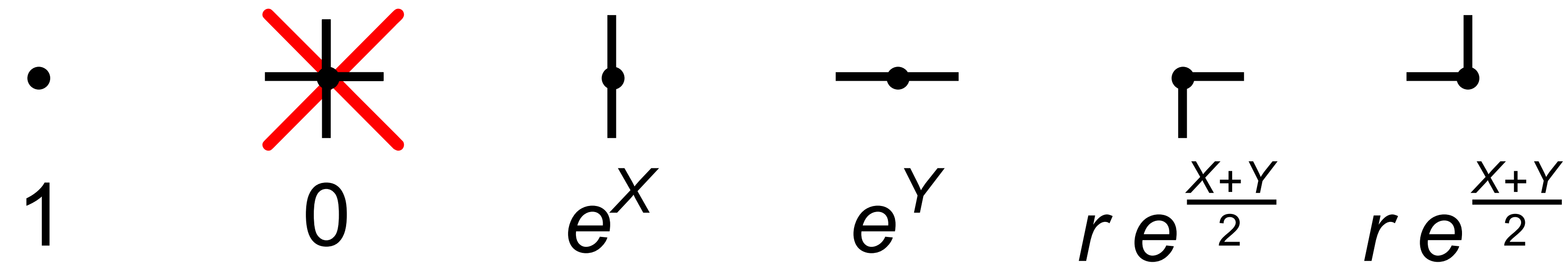
← arbitrary analytic function

Example(s)

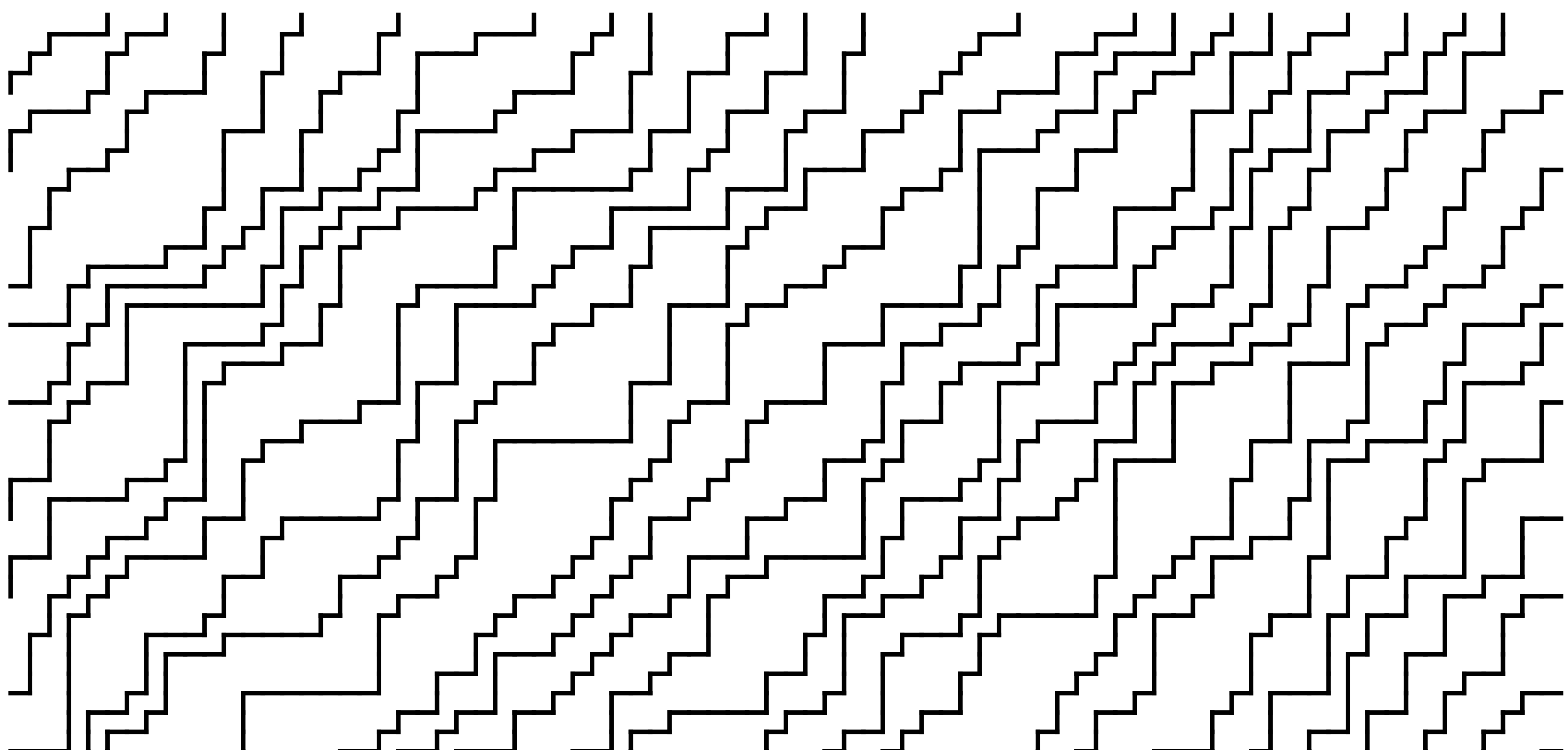
The five vertex model: a generalization of the lozenge tiling model
a special case of the six-vertex model $(\Delta \to \infty)$

(joint with J. de Gier, S. Watson)

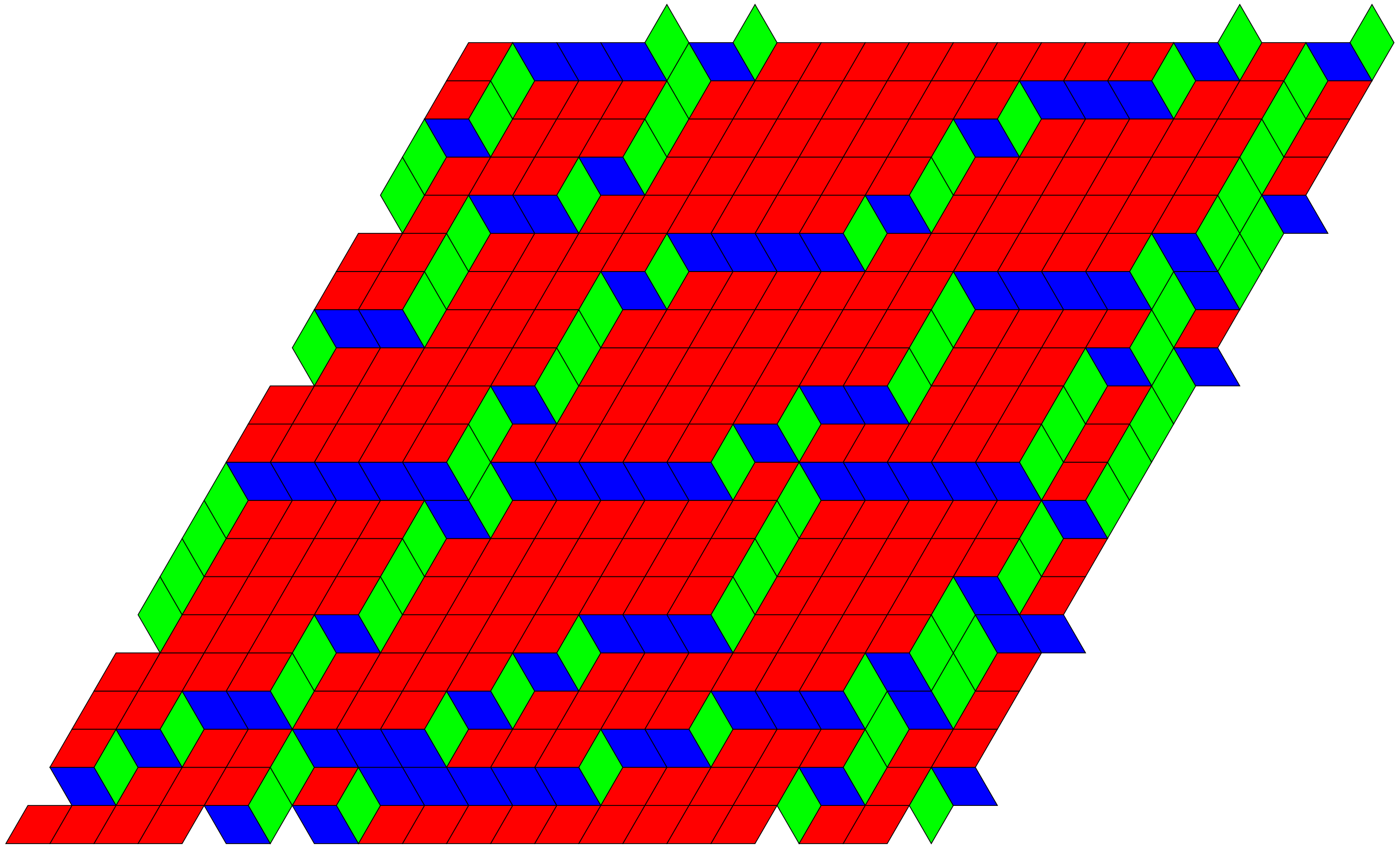
The five-vertex model

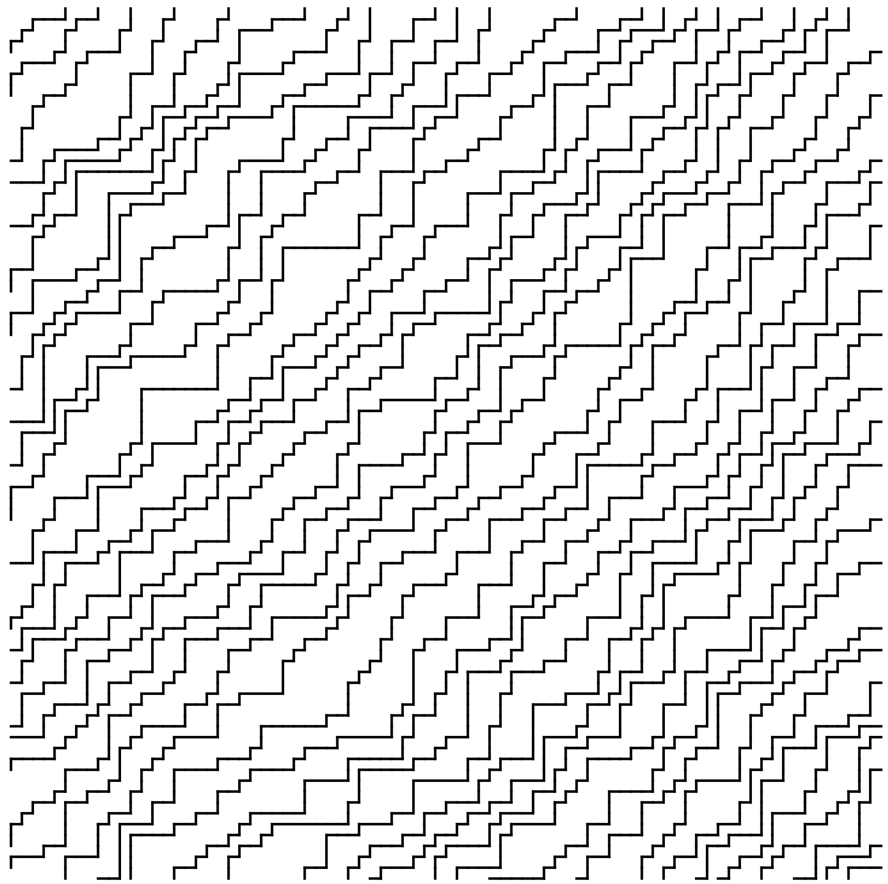


A configuration has probability $\frac{1}{Z}e^{vX+hY}r^c$ where r is the number of corners, v is the number of vertical edges, h is the number of horizontal edges.



lozenge tilings and the 5-vertex model

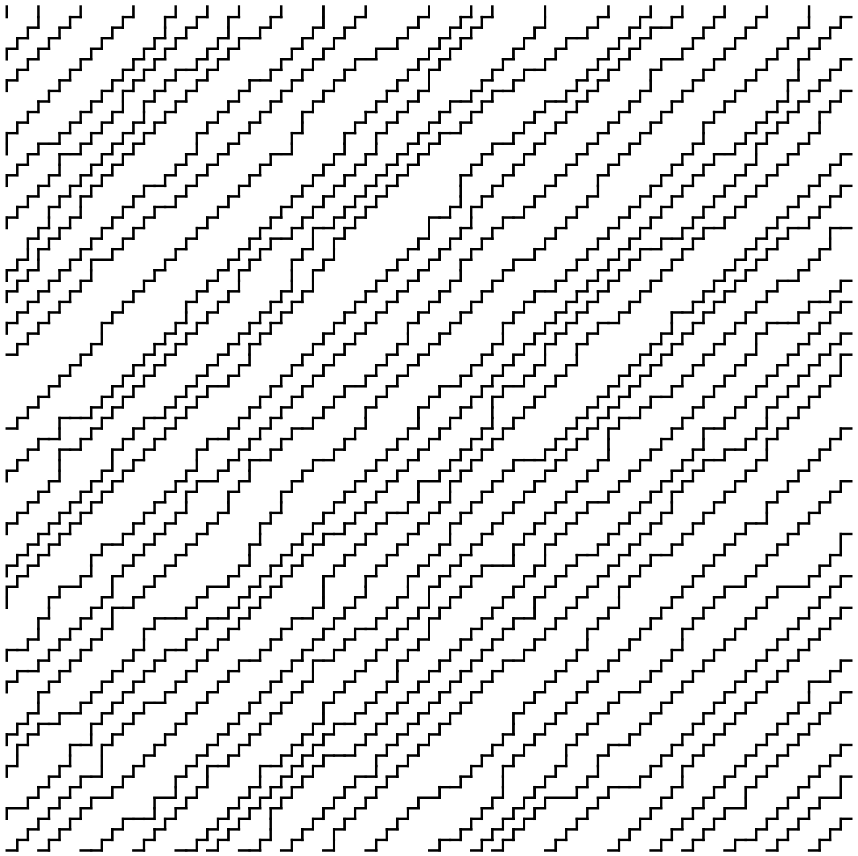




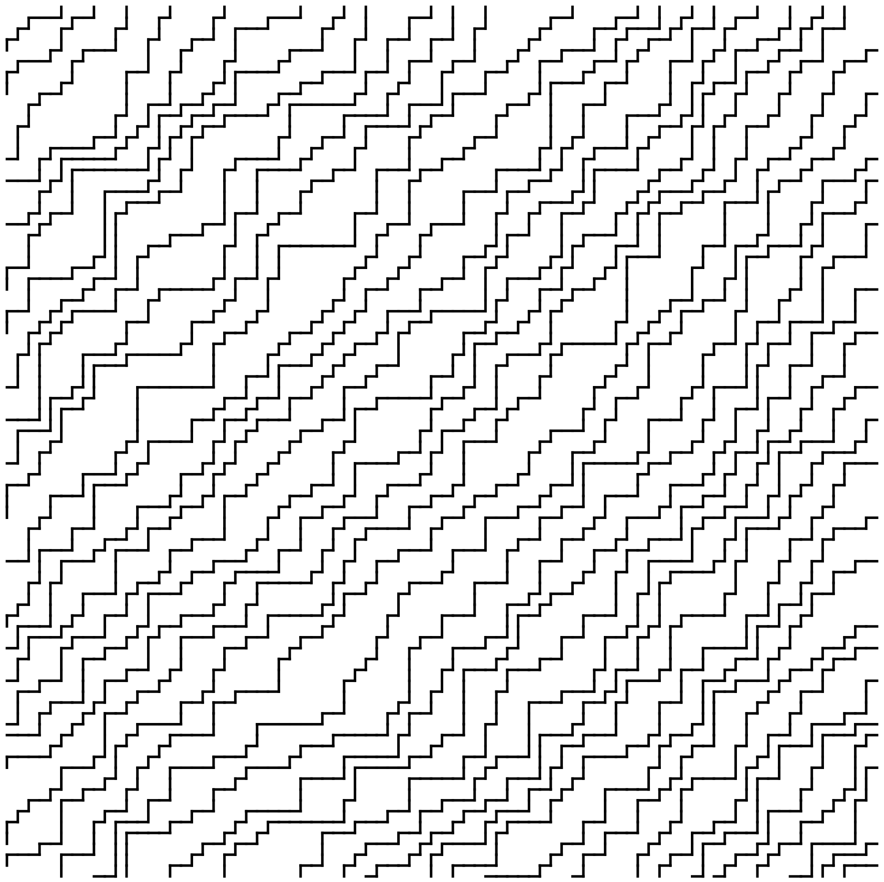
The 5 vertex model with $r = 1$ is the lozenge tiling model.

$r \neq 1$ means blue and green lozenges “interact”.

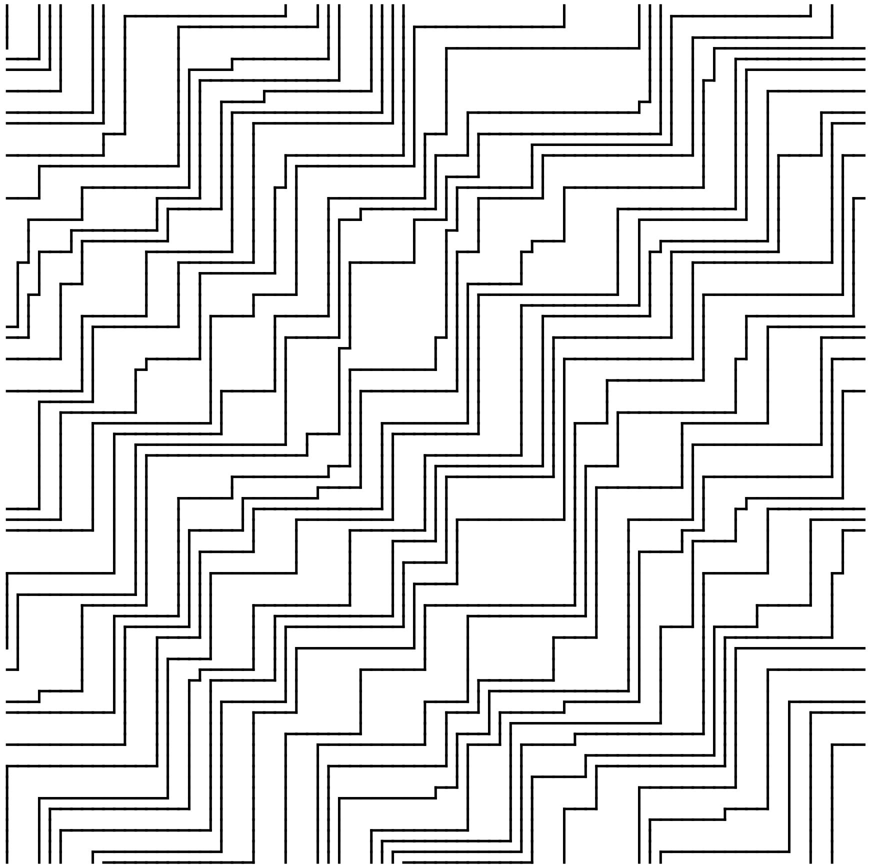
Simulations



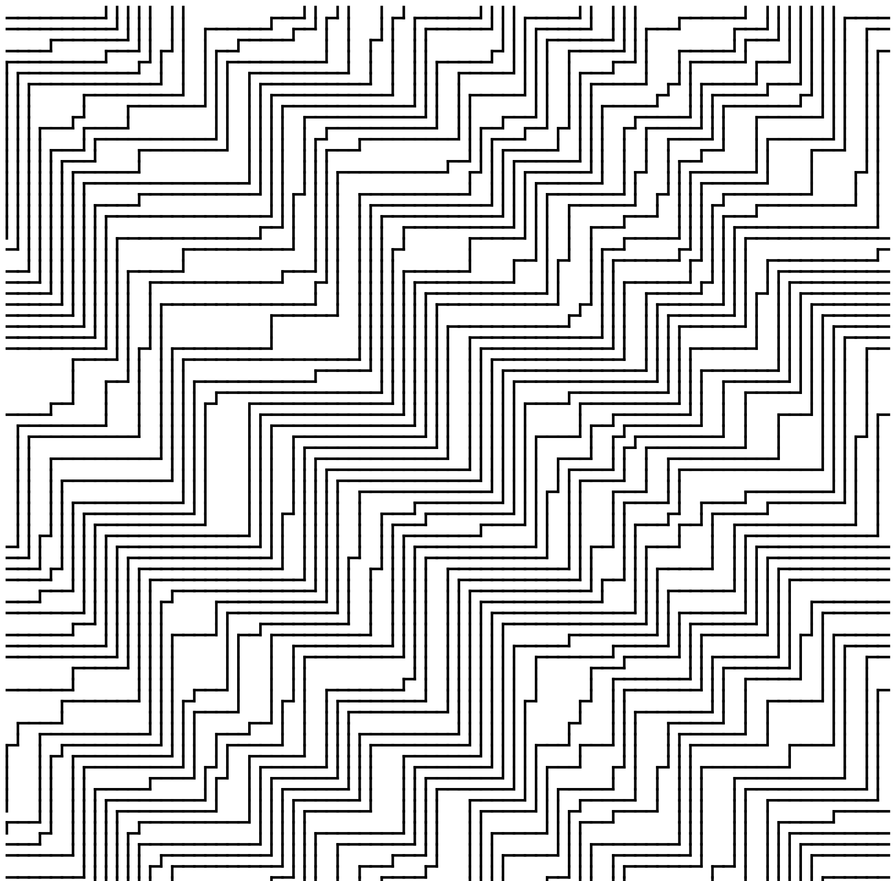
$r = 10$



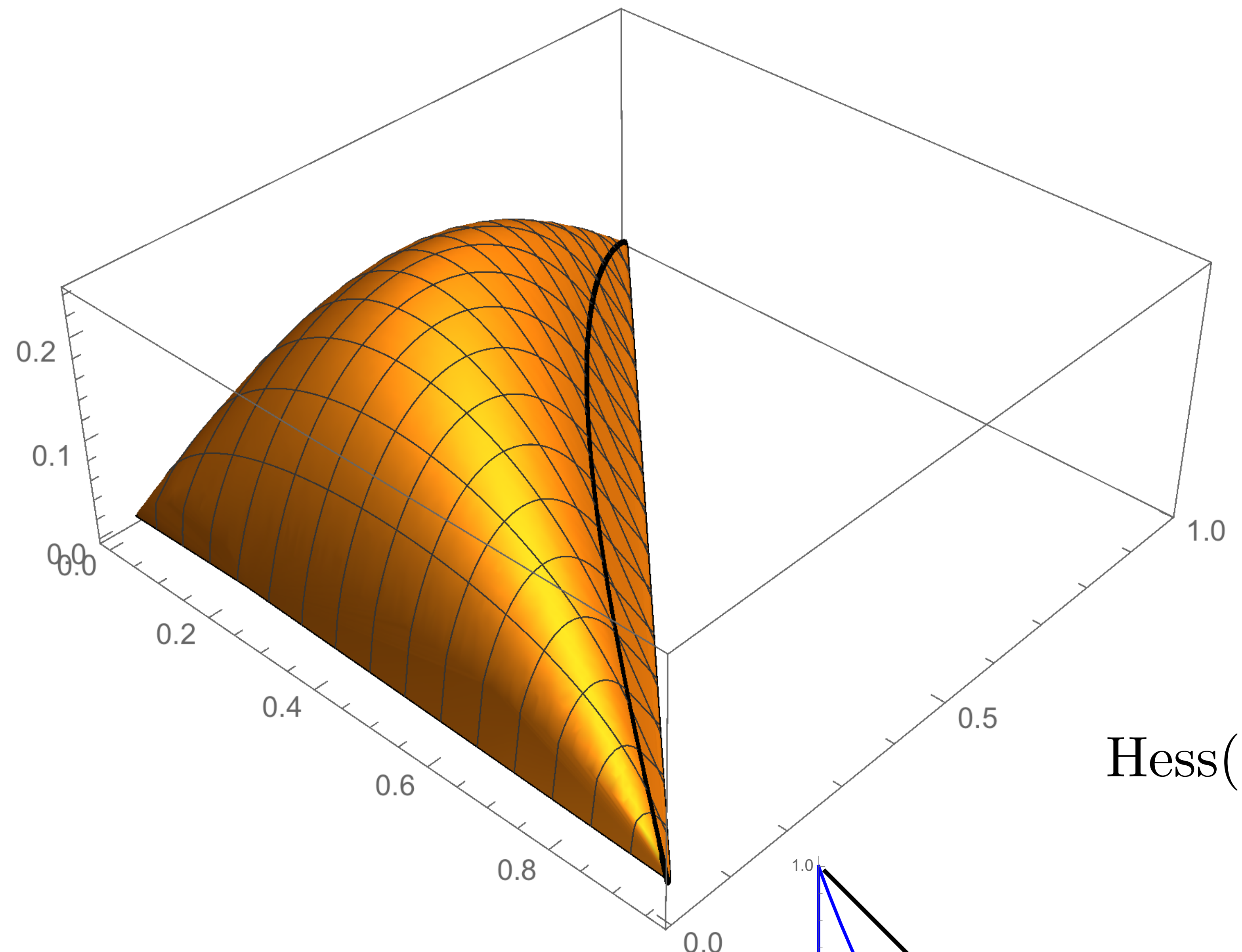
$r = 1$



$r = .1$



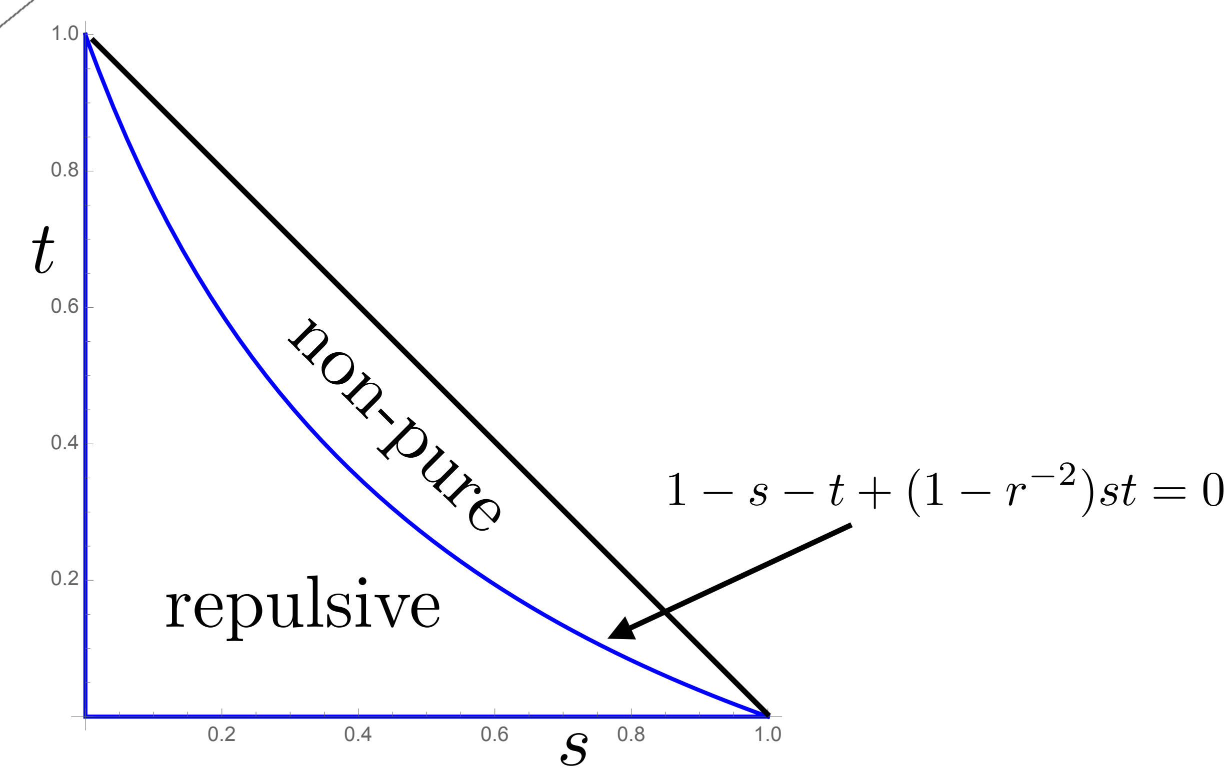
There is an explicit formula for $\sigma_r(s,t)$ in terms of the dilogarithm.

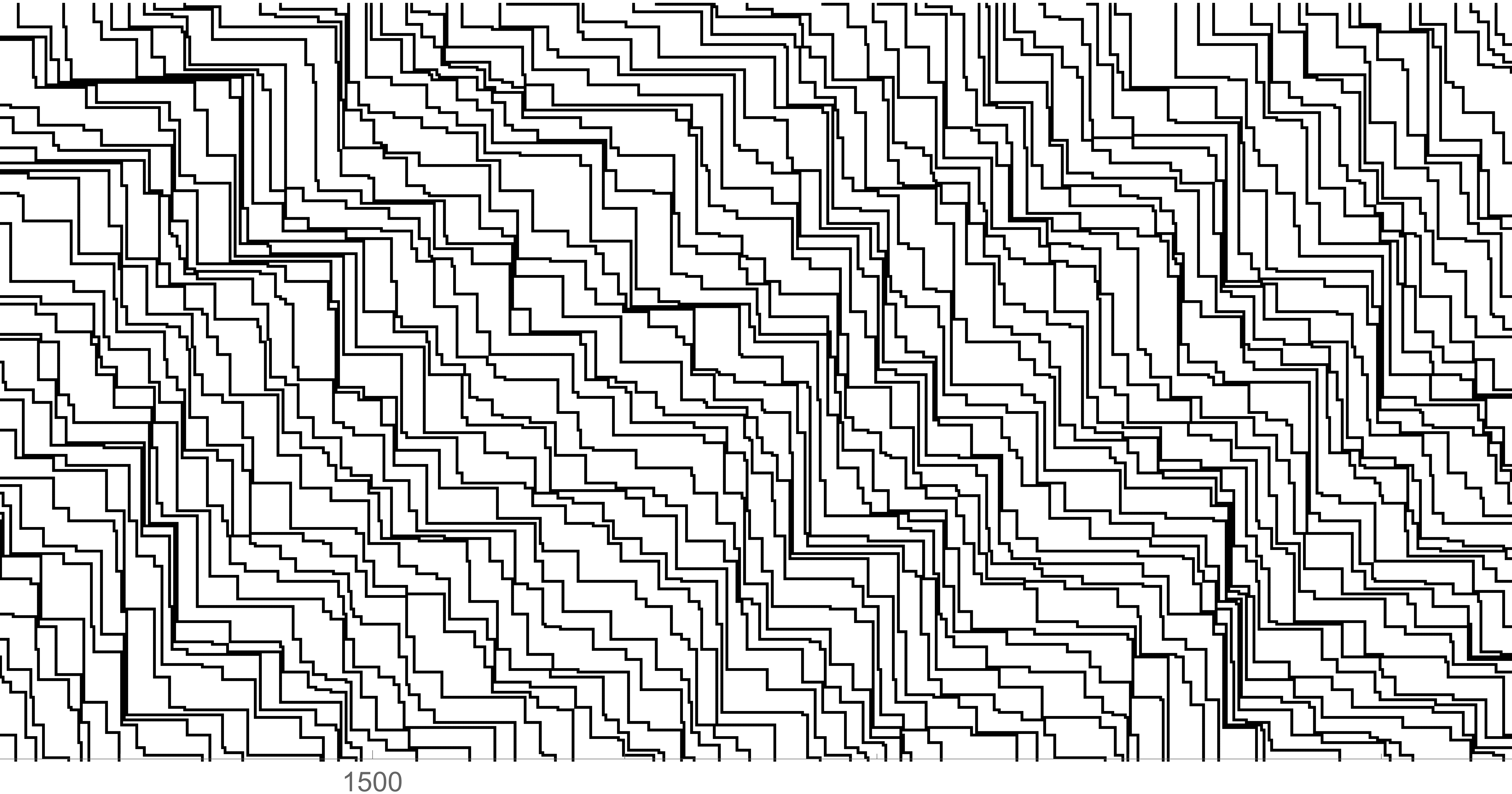


plot of $-\sigma$

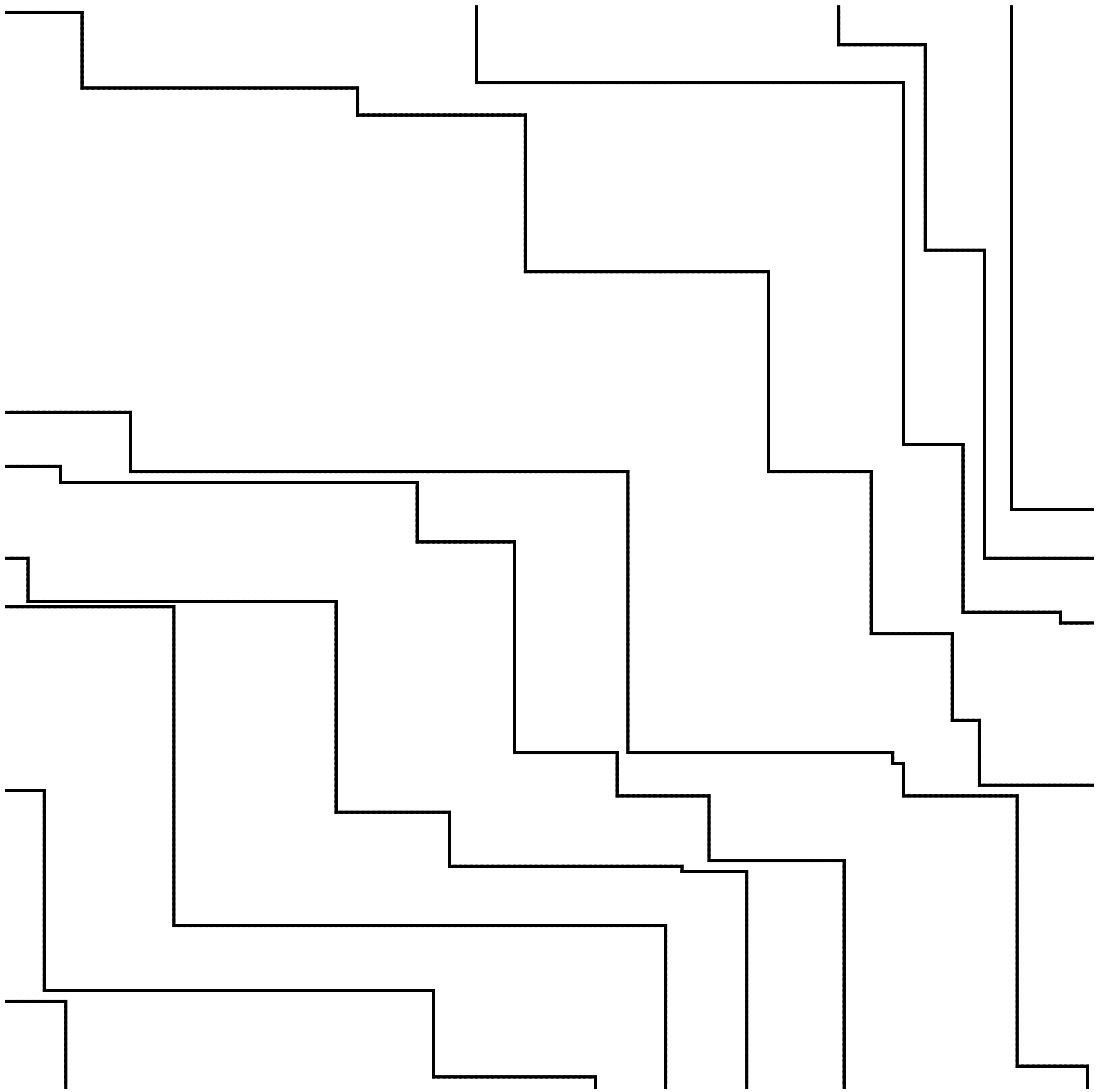
$$\mathrm{Hess}(\sigma) = (\arg \frac{z}{1-z})^4$$

when $r < 1$, $\sigma(s,t)$ is piecewise analytic:

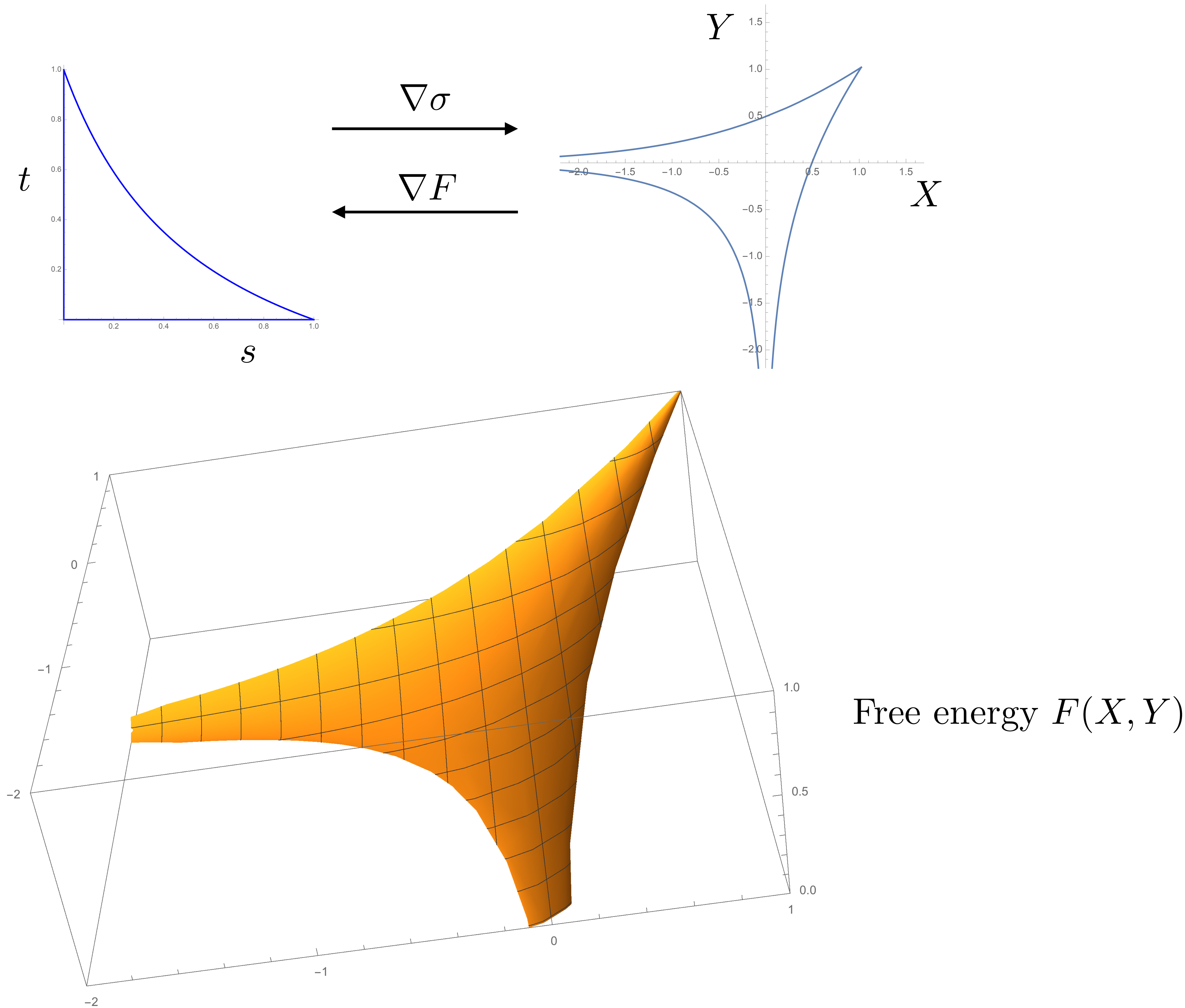




sample from a stochastic state



$r = .03$



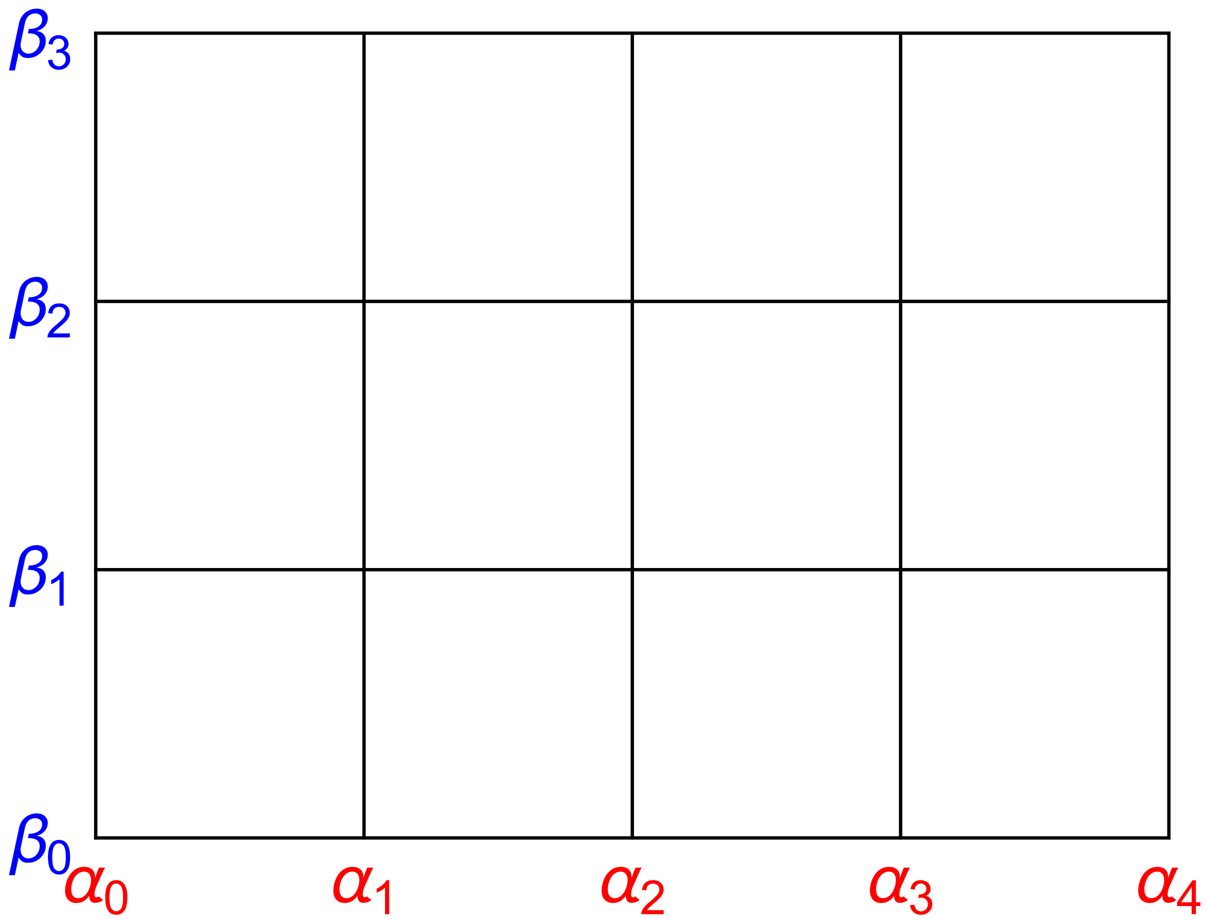
t
s
∇σ
∇F
Y
X
Free energy F(X, Y)

Are there generalizations of this model to which our harmonicity result applies?

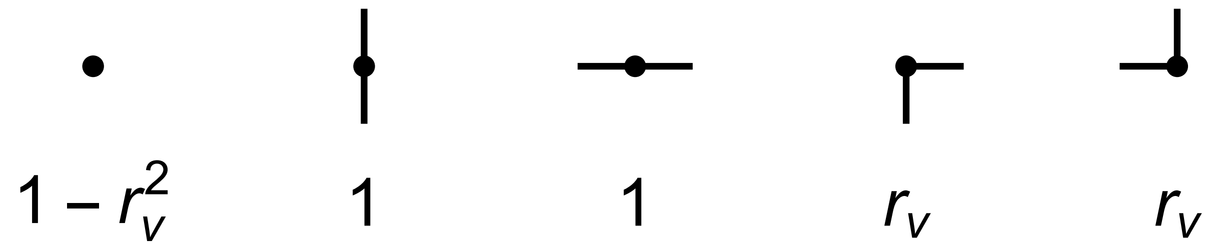
6 vertex model? no, but the proof method may still work.

"staggered-weight" 5-vertex model? yes! in certain cases...

The five-vertex model is a case when the Hessian is the fourth power of a harmonic function in z. In fact this property holds for a certain "periodically weighted" 5-vertex model:

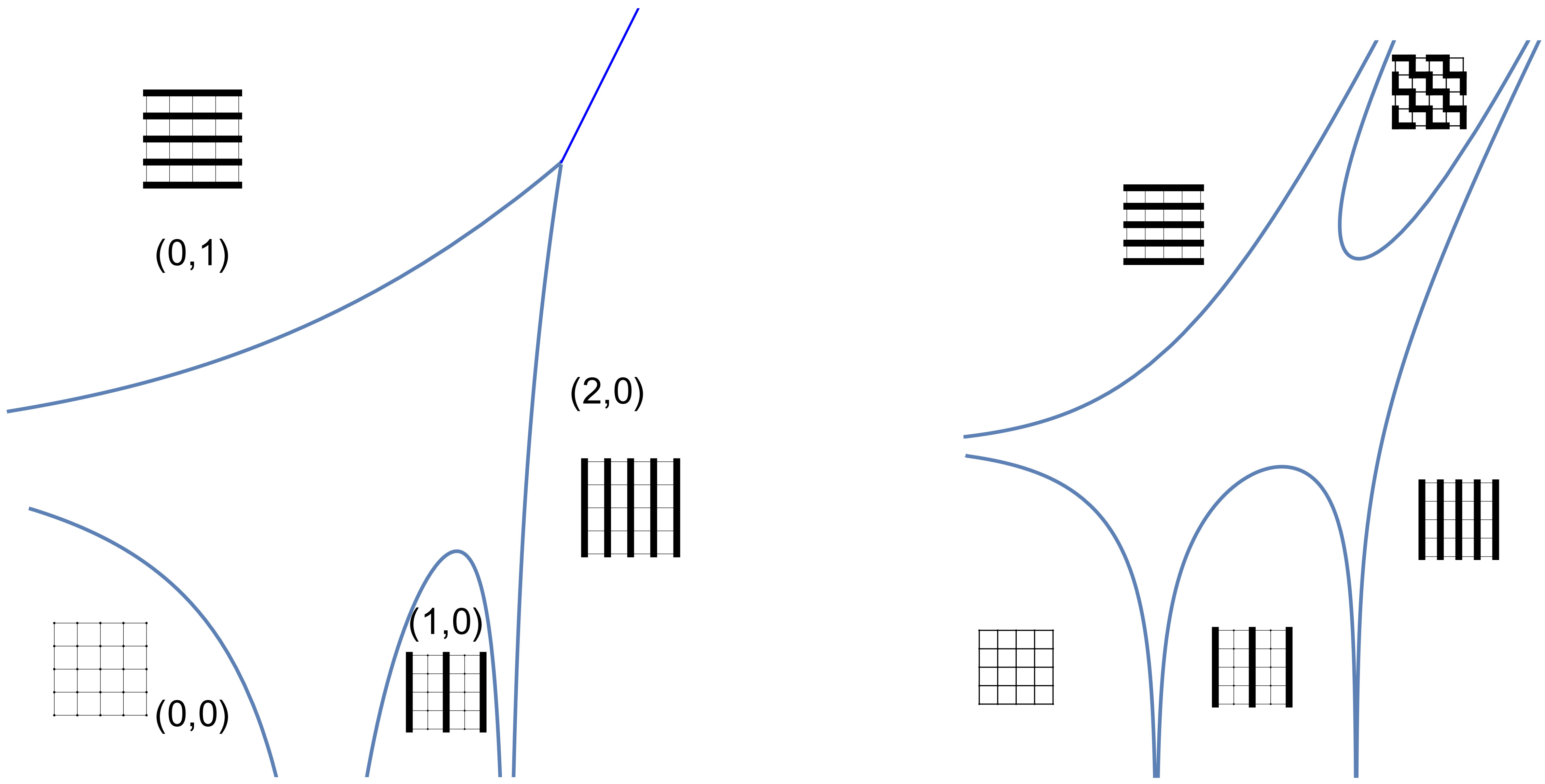


corner weights $r_{ij} = \alpha_i \beta_j$



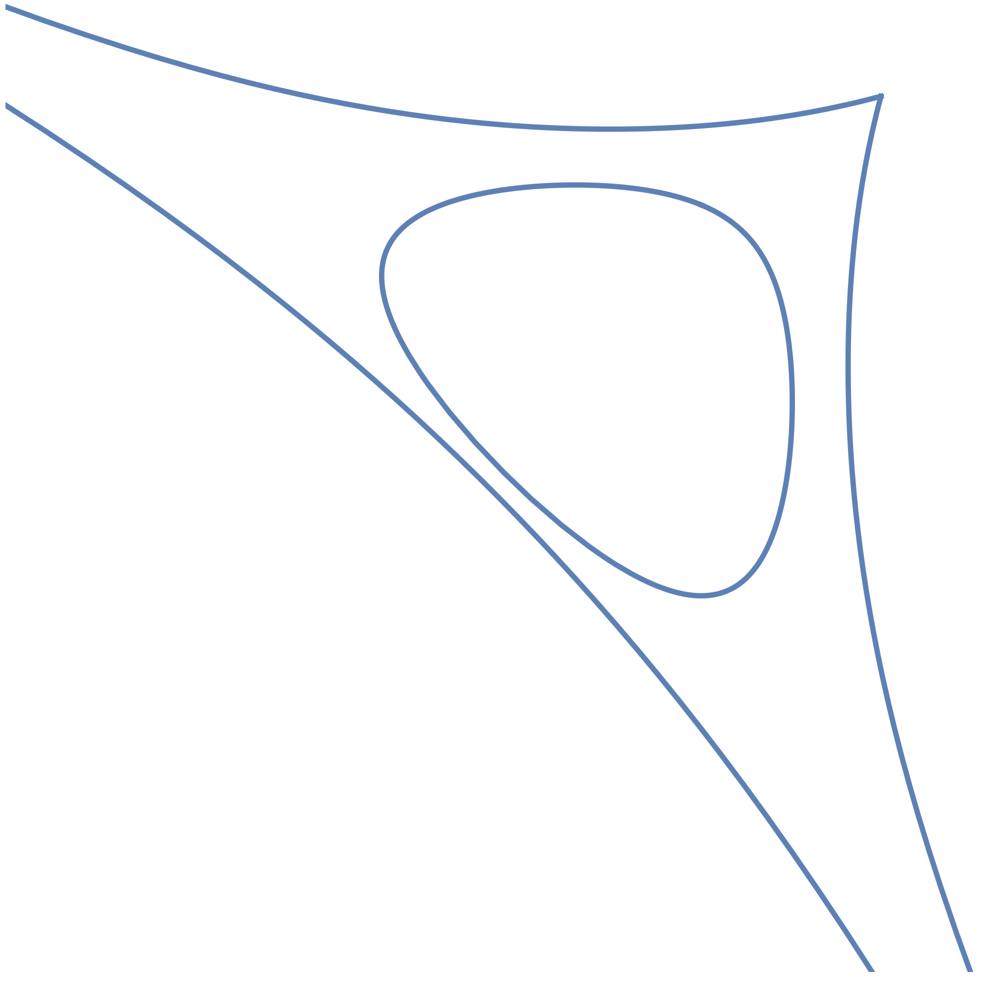
The proof is based on commuting transfer matrices.

Example: "Amoeba" for a 2×1 fundamental domain:



$r < 1$ $\qquad\qquad$ $r > 1$

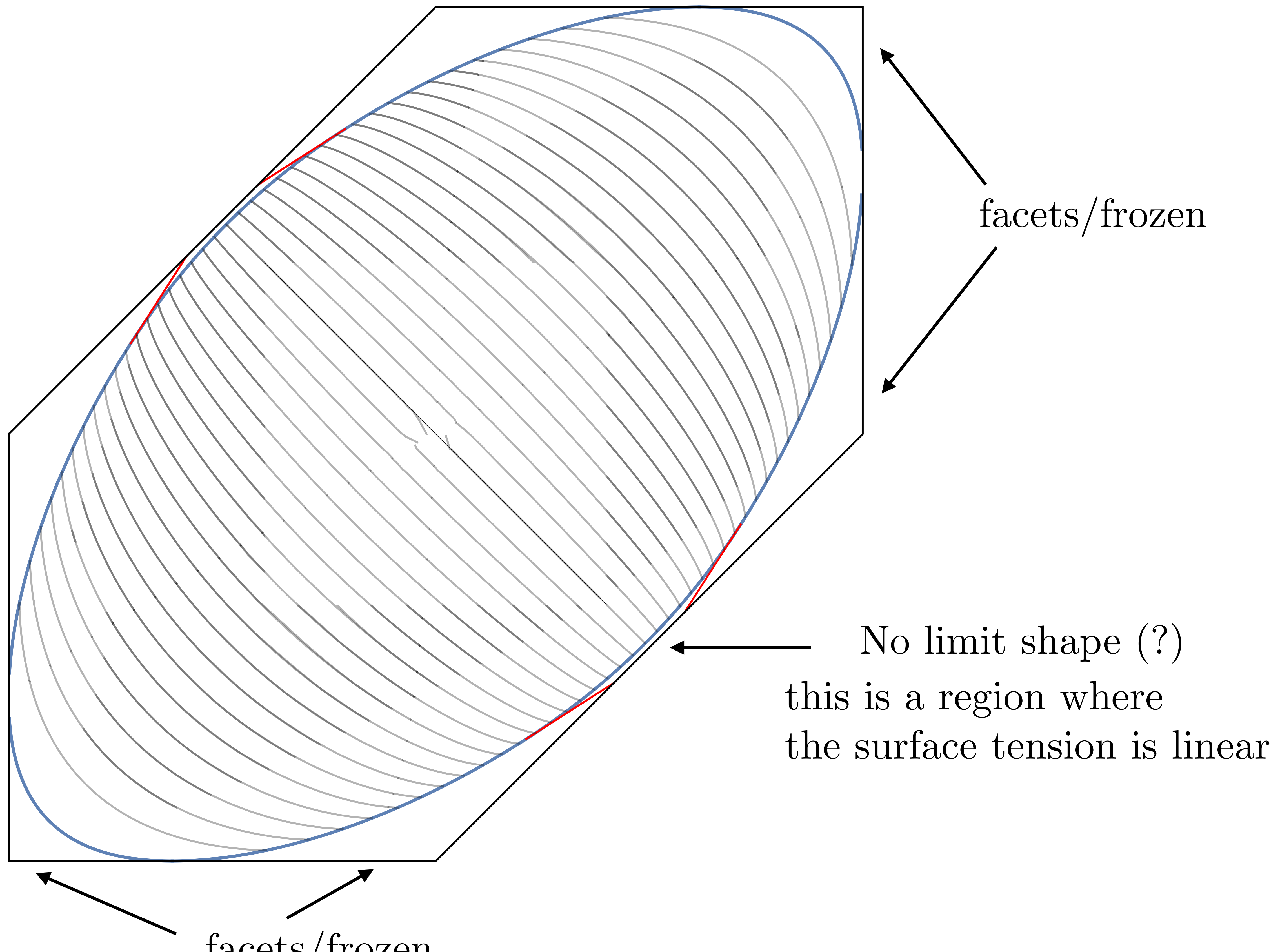
There are “higher-genus surface tensions” satisfying the harmonicity property but for which we don’t know if there is a corresponding probability model:

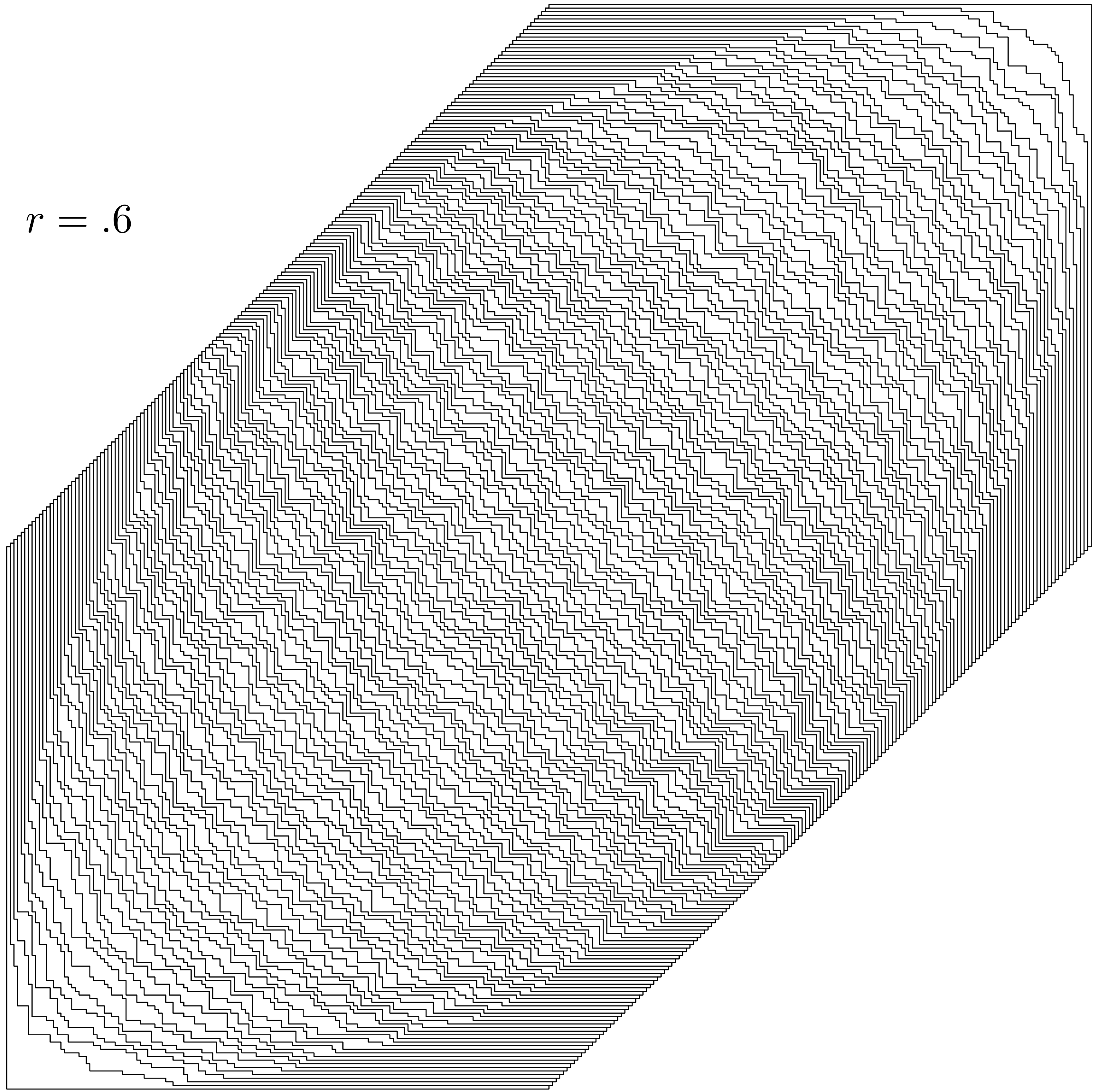


Genus 1 example?

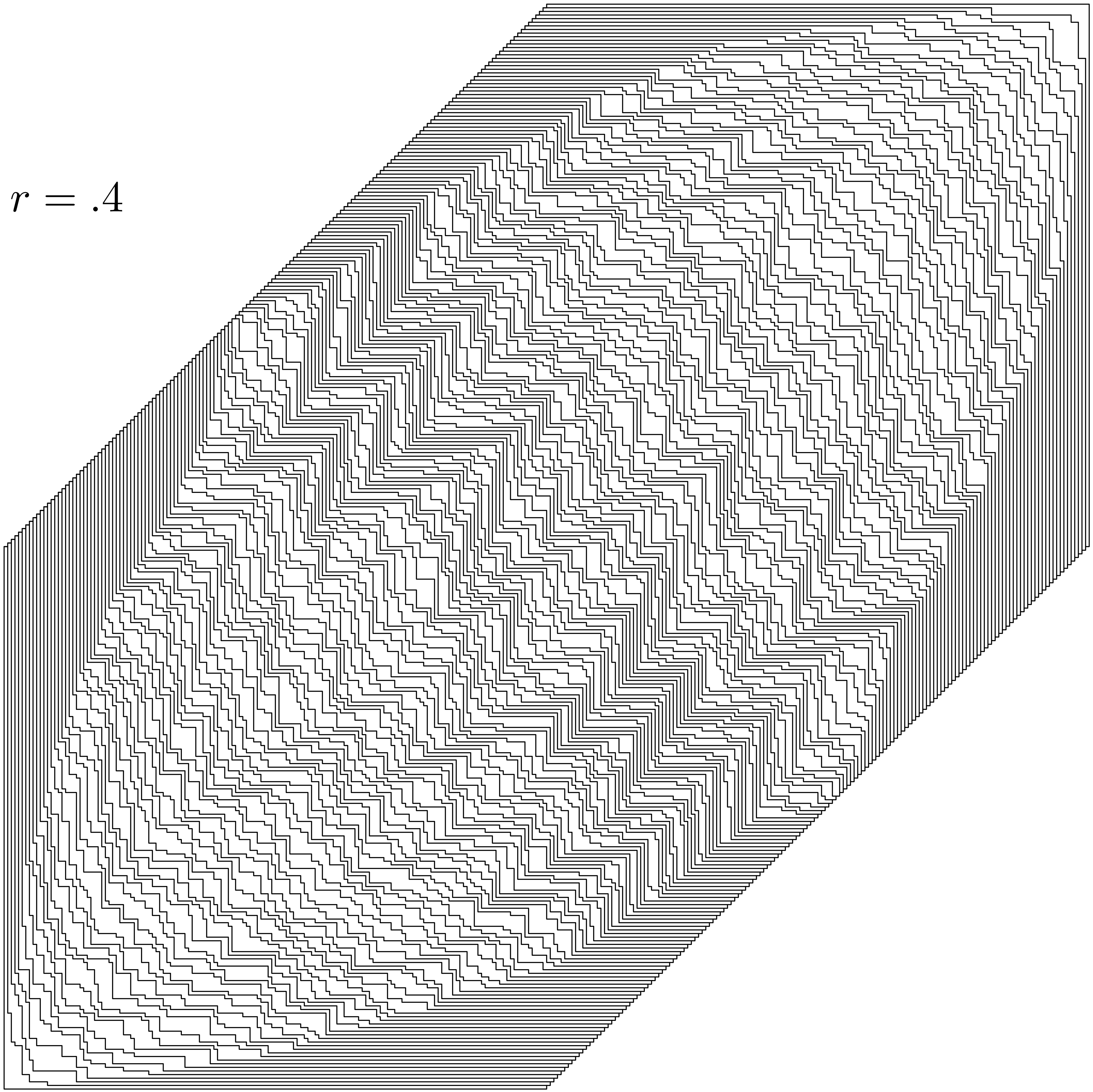
HAPPY BIRTHDAY!

THANK YOU

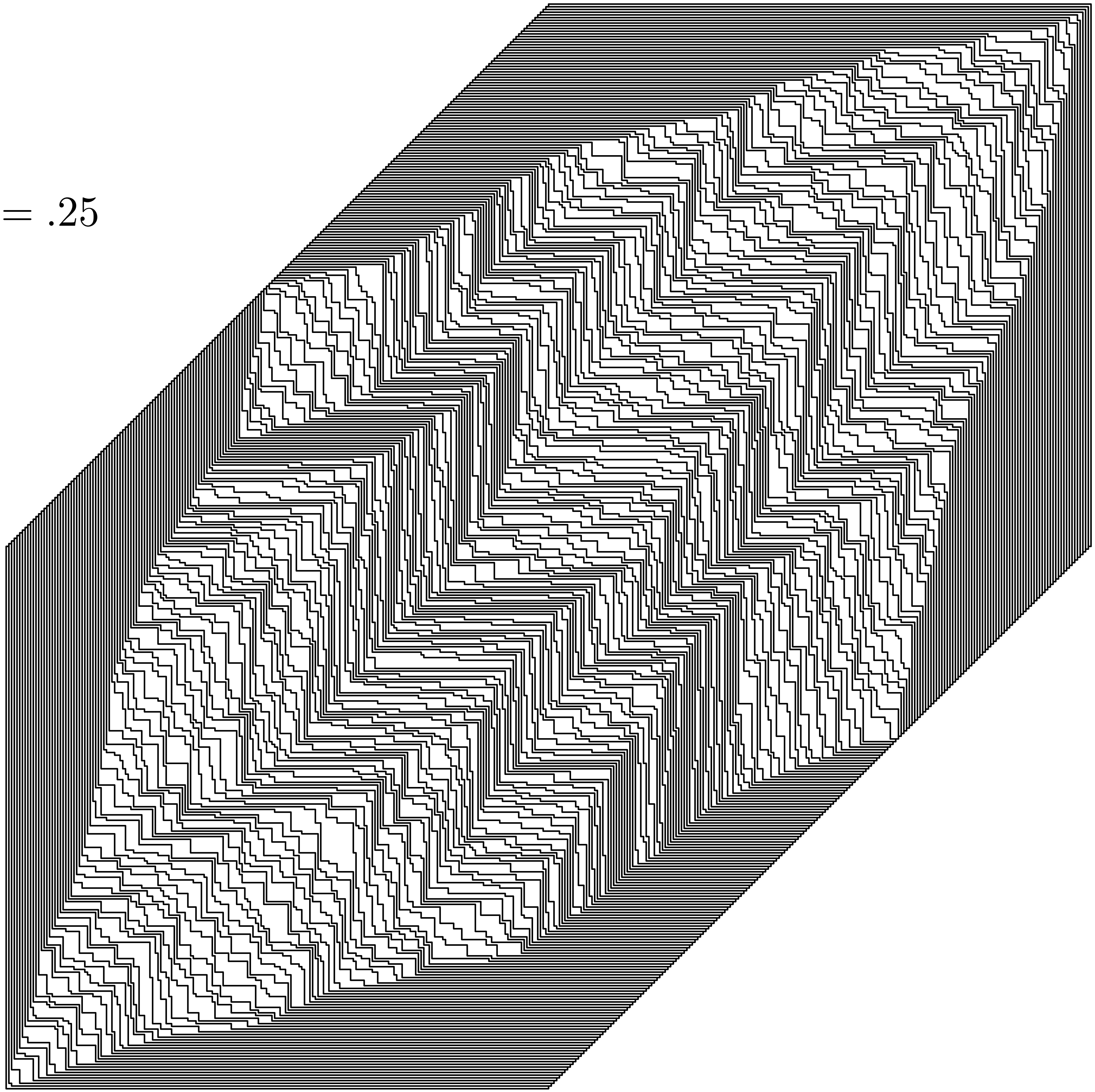

facets/frozen
No limit shape (?)
this is a region where
the surface tension is linear
facets/frozen

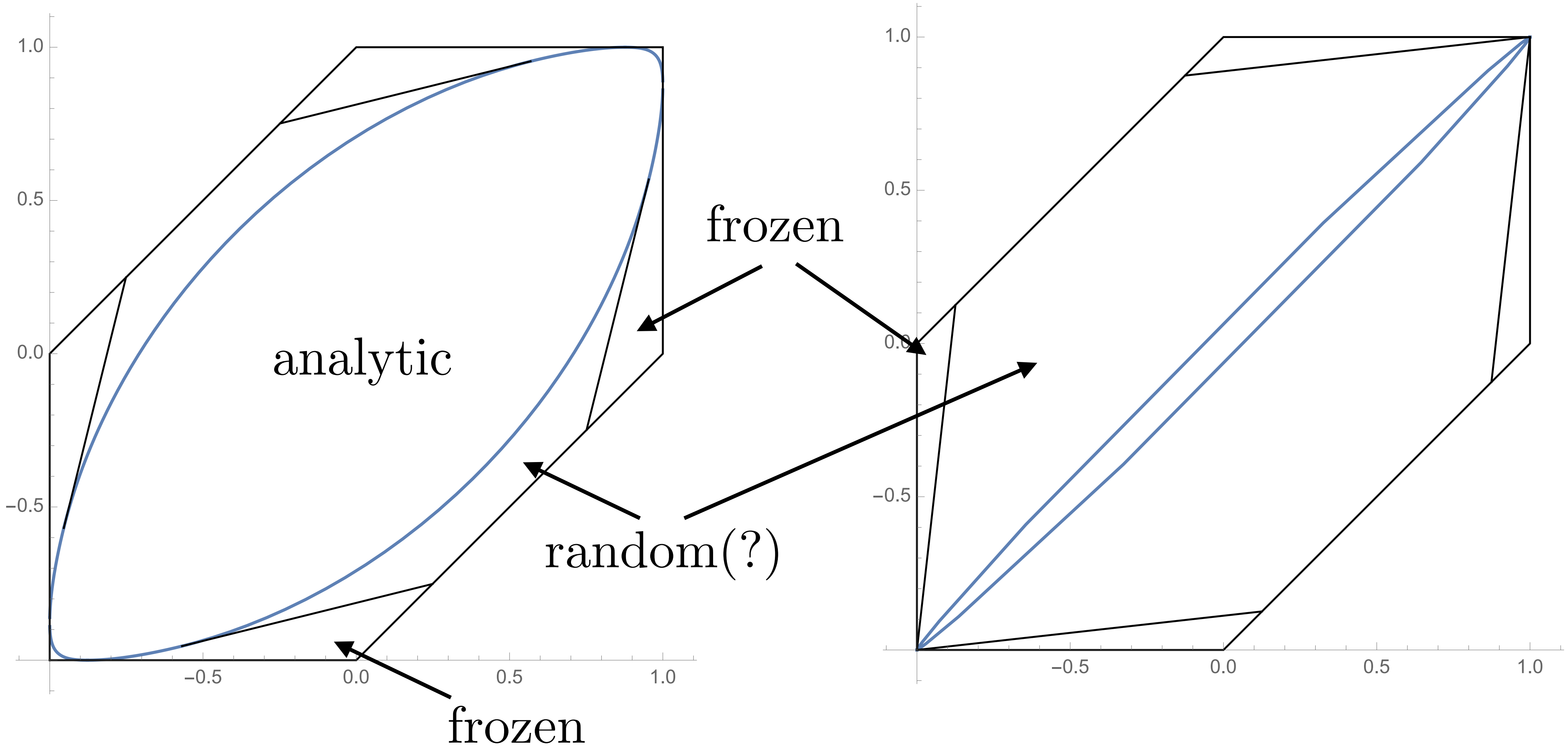

$r = .6$

$r = .4$



$r = .25$





$r = .5$ $\qquad$ $r = .35$

when $r \leq 1/3$ there is no limit shape.